

A necessary condition for generic rigidity of bar-and-joint frameworks in d -space

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Abstract

A graph $G = (V, E)$ is d -sparse if each subset $X \subseteq V$ with $|X| \geq d$ induces at most $d|X| - \binom{d+1}{2}$ edges in G . Laman showed in 1970 that a necessary and sufficient condition for a realisation of G as a generic bar-and-joint framework in \mathbb{R}^2 to be rigid is that G should have a 2-sparse subgraph with $2|V| - 3$ edges. Although Laman's theorem does not hold when $d \geq 3$, Cheng and Sitharam recently showed that if G is generically rigid in \mathbb{R}^3 then every maximal 3-sparse subgraph of G must have $3|V| - 6$ edges. We extend their result to all d by showing that if G is generically rigid in \mathbb{R}^d then every maximal d -sparse subgraph of G must have $d|V| - \binom{d+1}{2}$ edges.

1 Introduction

A d -dimensional (*bar-and-joint*) framework is a pair (G, p) where $G = (V, E)$ is a graph and $p : V \rightarrow \mathbb{R}^d$. It is a long standing open problem to determine when a given bar-and-joint framework is *rigid* i.e. every continuous motion of the points $p(v)$ which preserves the distances $\|p(u) - p(v)\|$ for all $uv \in E$ must also preserve the distances $\|p(u) - p(v)\|$ for all $u, v \in V$. It is not difficult to see that a 1-dimensional framework (G, p) is rigid if and only if the graph G is connected. Abbot [1] has recently shown that the problem of determining rigidity is NP-hard for all $d \geq 2$.

This problem becomes more tractable, however, if we assume that the framework is *generic* i.e. there are no algebraic dependencies between the coordinates of the points $p(v)$, $v \in V$. It is known that the rigidity of a d -dimensional generic framework (G, p) depends only on the graph G . Indeed we can define an $|E| \times d|V|$ matrix, the d -dimensional rigidity matrix $R_d(G)$, whose entries are linear combinations of indeterminates representing the coordinates of the points $p(v)$, in such a way that (G, p) is rigid if and only if the rank of $R_d(G)$, $r_d(G)$, is equal to $d|V| - \binom{d+1}{2}$. This naturally gives rise to a matroid on E , the d -dimensional rigidity matroid $\mathcal{R}_d(G)$ in which a set of edges $F \subseteq E$ is *independent/dependent* if and only if the corresponding rows of $R_d(G)$ are linearly

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independent/dependent. We refer the reader to [7] for a precise definition of the rigidity matrix, the rigidity matroid, and other information on the topic of rigidity.

Laman [4] characterized when a 2-dimensional generic framework is rigid (see also Lovász and Yemini [5]). His characterization is based on the following concept. We say that a subgraph H of G is d -sparse if each subset X of at least d vertices of H induces at most $d|X| - \binom{d+1}{2}$ edges of H . Maxwell [6] showed that being d -sparse is a necessary condition for the rows of $R_d(G)$ labeled by the edges of H to be linearly independent. Laman showed that that this condition is also sufficient when $d = 2$ and deduced that a 2-dimensional generic framework (G, p) is rigid if and only if it has a 2-sparse subgraph with $2|V| - 3$ edges. Since every linearly independent set of rows of $R_2(G)$ can be extended to a basis for the row space of $R_2(G)$, Laman's theorem implies that every maximal 2-sparse subgraph of G has the same number of edges.

It is known that the condition that H is d -sparse is not sufficient for the rows of $R_d(G)$ labeled by the edges of H to be linearly independent when $d \geq 3$. Indeed it is not even true that all maximal d -sparse subgraphs of G have the same number of edges when $d \geq 3$. On the other hand Cheng and Sitharam [3] have recently shown that the number of edges in any maximal 3-sparse subgraph of G does at least give an upper bound on $r_3(G)$.

The purpose of this paper is to prove a result, Theorem 4.1 below, which extends the theorem of Cheng and Sitharam to all values of d .

2 Sparse subgraphs

Let $G = (V, E)$ be a graph and $d \geq 1$ be an integer. For $X \subseteq V$ we use $E_G(X)$ to denote the set, and $i_G(X)$ the number, of edges of G joining pairs of vertices of X . We simplify these to $E(X)$ and $i(X)$ when it is obvious to which graph we are referring. We may rewrite the condition for G to be d -sparse as $i(X) \leq d|X| - \binom{d+1}{2}$ for all $X \subseteq V$ with $|X| \geq d$. (Note that if $|X| \in \{d, d+1\}$ then we have $i(X) \leq \binom{|X|}{2} = d|X| - \binom{d+1}{2}$ and the inequality holds trivially.) A subgraph $H = (U, F)$ of a d -sparse graph G is d -critical if either $|U| = 2$ and $|F| = 1$, or $|U| \geq d+2$ and $|F| = d|U| - \binom{d+1}{2}$. The assumption that G is d -sparse implies that every d -critical subgraph of G is an induced subgraph. A d -critical component of G is a d -critical subgraph which is not properly contained in any other d -critical subgraph of G .

Lemma 2.1 *Let $G = (V, E)$ be a d -sparse graph and $H_1 = (U_1, F_1), H_2 = (U_2, F_2)$ be distinct critical components of G . Then $|U_1 \cap U_2| \leq d - 1$ and, if equality holds, then $i_G(U_1 \cap U_2) = \binom{d-1}{2}$.*

Proof: Suppose that $|U_1 \cap U_2| \geq d - 1$. When $|U_1 \cap U_2| \geq d$ we have $i(U_1 \cap U_2) \leq d|U_1 \cap U_2| - \binom{d+1}{2}$ since G is sparse. When $|U_1 \cap U_2| = d - 1$, we have $i(U_1 \cap U_2) \leq \binom{d-1}{2} = d|U_1 \cap U_2| - \binom{d+1}{2} + 1$ trivially. The maximality of H_1, H_2 and the definition of a critical component imply that $|U_1|, |U_2| \geq d + 2$, and $d(|U_1| + |U_2|) - 2\binom{d+1}{2} = i_G(U_1) + i_G(U_2) \leq i_G(U_1 \cup U_2) + i_G(U_1 \cap U_2) \leq$

$d|U_1 \cup U_2| - \binom{d+1}{2} - 1 + d|U_1 \cap U_2| - \binom{d+1}{2} + 1 = d(|U_1| + |U_2|) - 2\binom{d+1}{2}$. Equality must hold throughout. In particular we have $i_G(U_1 \cap U_2) = \binom{d+1}{2} + 1$. This implies that $|U_1 \cap U_2| = d - 1$ and $i_G(U_1 \cap U_2) = \binom{d-1}{2}$. •

3 Covers

Let k, t be nonnegative integers, $G = (V, E)$ be a graph and \mathcal{X} be a family of subsets of V . We say that \mathcal{X} is a *cover* of G if every set in \mathcal{X} contains at least two vertices, and every edge of G is induced by at least one set in \mathcal{X} . A cover \mathcal{X} is *t-thin* if every pair of sets in \mathcal{X} intersect in at most t vertices. A *k-hinge* of \mathcal{X} is set of k vertices which lie in the intersection of at least two sets in \mathcal{X} . A *k-hinge* U of \mathcal{X} is *closed in* G if $G[U]$ is a complete graph. We use $\Theta_k(\mathcal{X})$, respectively $\hat{\Theta}_k(\mathcal{X})$, to denote the set of all k -hinges, respectively closed k -hinges, of \mathcal{X} . For $U \in \Theta_k(\mathcal{X})$, let $d_{\mathcal{X}}(U)$ denote the number of sets in \mathcal{X} which contain U . Note that if G is t -thin then $\Theta_k(\mathcal{X}) = \emptyset$ for all $k \geq t + 1$. Note also that $\Theta_0(\mathcal{X}) = \{\emptyset\}$ and $d_{\mathcal{X}}(\emptyset) = |\mathcal{X}|$.

Lemma 3.1 *Let $G = (V, E)$ be a graph, $H = (V, F)$ be a maximal d -sparse subgraph of G , and H_1, H_2, \dots, H_m be the d -critical components of H . Let X_i be the vertex set of H_i for $1 \leq i \leq m$. Then $\mathcal{X} = \{X_1, X_2, \dots, X_m\}$ is a $(d - 1)$ -thin cover of G and each $(d - 1)$ -hinge of \mathcal{X} is closed.*

Proof: The definition of a d -critical subgraph implies that each H_i has at least two vertices and that every edge of H belongs to at least one H_i . Thus \mathcal{X} is a cover of H . To see that \mathcal{X} also covers G we choose $e = uv \in E \setminus F$. The maximality of H implies that $H + e$ is not d -sparse. Hence $\{u, v\}$ is contained in some d -critical subgraph of H . Thus \mathcal{X} also covers G . The facts that \mathcal{X} is $(d - 1)$ -thin and that each $(d - 1)$ -hinge of \mathcal{X} is closed follow from Lemma 2.1 •

We refer to the closed $(d - 1)$ -thin cover of G described in Lemma 3.1 as the *H-critical cover* of G . When G is d -sparse (and so $H = G$), we refer to this cover as the *d-critical cover* of G . Note that the definition of a d -critical set implies that each set in a d -critical cover has size two or has size at least $d + 2$.

Lemma 3.2 *Let $H = (V, E)$ be a d -sparse graph and \mathcal{X} be its d -critical cover. Suppose that each critical component of H has at least $d + 2$ vertices and choose $W \in \Theta_k(\mathcal{X})$ for some $0 \leq k \leq d - 1$. Then:*

(a)

$$(d-k) \sum_{\substack{U \in \Theta_{k+1}(\mathcal{X}) \\ W \subset U}} (d_{\mathcal{X}}(U) - 1) - \sum_{\substack{U \in \Theta_{k+2}(\mathcal{X}) \\ W \subset U}} (d_{\mathcal{X}}(U) - 1) < \binom{d+1-k}{2} (d_{\mathcal{X}}(W) - 1);$$

(b)

$$\begin{aligned} (d-k)(k+1) \sum_{U \in \Theta_{k+1}(\mathcal{X})} (d_{\mathcal{X}}(U) - 1) & - \binom{k+2}{2} \sum_{U \in \Theta_{k+2}(\mathcal{X})} (d_{\mathcal{X}}(U) - 1) \\ & < \binom{d+1-k}{2} \sum_{U \in \Theta_k(\mathcal{X})} (d_{\mathcal{X}}(U) - 1); \end{aligned}$$

(c)

$$\begin{aligned} (d-k) \sum_{U \in \Theta_{k+1}(\mathcal{X})} (d_{\mathcal{X}}(U) - 1) & - (k+2) \sum_{U \in \Theta_{k+2}(\mathcal{X})} (d_{\mathcal{X}}(U) - 1) \\ & < \binom{d}{k+1} \left[(d+1)(|\mathcal{X}| - 1) - \sum_{U \in \Theta_1(\mathcal{X})} (d_{\mathcal{X}}(U) - 1) \right]; \end{aligned}$$

(d)

$$(d-k) \sum_{U \in \Theta_{k+1}(\mathcal{X})} (d_{\mathcal{X}}(U) - 1) - (k+1) \sum_{U \in \Theta_{k+2}(\mathcal{X})} (d_{\mathcal{X}}(U) - 1) < \binom{d+1}{k+2} (|\mathcal{X}| - 1).$$

Proof: We first prove (a). Let $d_{\mathcal{X}}(W) = t$ and let H_1, H_2, \dots, H_t be the critical components of H which contain W . Put $H_i = (V_i, E_i)$ for $1 \leq i \leq t$. Let $H' = \bigcup_{i=1}^t H_i$ and put $H' = (V', E')$. Then

$$|V'| = \sum_{i=1}^t |V_i| - k(t-1) - \sum_{\substack{U \in \Theta_{k+1}(\mathcal{X}) \\ W \subset U}} (d_{\mathcal{X}}(U) - 1) \quad (1)$$

since, for $v \in V'$, if $v \in W$ then v is counted t times in the sum $\sum_{i=1}^t |V_i|$, if $v \in U \setminus W$ for some $U \in \Theta_{k+1}(\mathcal{X})$ with $W \subset U$ then v is counted $d_{\mathcal{X}}(U)$ times in this sum, and all other vertices of V' are counted exactly once in this sum. Similarly,

$$|E'| \geq \sum_{i=1}^t |E_i| - \binom{k}{2} (t-1) - k \sum_{\substack{U \in \Theta_{k+1}(\mathcal{X}) \\ W \subset U}} (d_{\mathcal{X}}(U) - 1) - \sum_{\substack{U \in \Theta_{k+2}(\mathcal{X}) \\ W \subset U}} (d_{\mathcal{X}}(U) - 1) \quad (2)$$

since, for $e = uv \in E'$, if $u, v \in W$ then e is counted t times in the sum $\sum_{i=1}^t |E_i|$ and there are at most $\binom{k}{2}$ such edges, if $u \in W$ and $v \in U \setminus W$ for some $U \in \Theta_{k+1}(\mathcal{X})$ with $W \subset U$ then e is counted $d_{\mathcal{X}}(U)$ times in this sum and for each such v there are at most k choices for u , if $u, v \in U \setminus W$ for some $U \in \Theta_{k+2}(\mathcal{X})$ with $W \subset U$ then e is counted $d_{\mathcal{X}}(U)$ times in this sum, and all other edges of E' are counted exactly once in this sum.

Since $H' \subseteq H$, H' is sparse and since $W \in \Theta_k(\mathcal{X})$ we have $t \geq 2$ so H' is not critical. Hence $|E'| < d|V'| - \binom{d+1}{2}$. We may substitute equations (1) and (2)

into this inequality and use the fact that $|E_i| = d|V_i| - \binom{d+1}{2}$ for all $1 \leq i \leq t$ to obtain

$$\begin{aligned}
(d-k) \sum_{\substack{U \in \Theta_{k+1}(\mathcal{X}) \\ W \subset U}} (d_{\mathcal{X}}(U) - 1) &= \sum_{\substack{U \in \Theta_{k+2}(\mathcal{X}) \\ W \subset U}} (d_{\mathcal{X}}(U) - 1) \\
&< \left[\binom{d+1}{2} + \binom{k}{2} - dk \right] (t-1) \\
&= \binom{d+1-k}{2} (t-1).
\end{aligned}$$

Hence (a) holds. Part (b) follows by summing the identity in (a) over all $W \in \Theta_k(\mathcal{X})$ using the facts that

$$\sum_{W \in \Theta_k(\mathcal{X})} \sum_{\substack{U \in \Theta_{k+1}(\mathcal{X}) \\ W \subset U}} (d_{\mathcal{X}}(U) - 1) = (k+1) \sum_{U \in \Theta_{k+1}(\mathcal{X})} (d_{\mathcal{X}}(U) - 1)$$

and

$$\sum_{W \in \Theta_k(\mathcal{X})} \sum_{\substack{U \in \Theta_{k+2}(\mathcal{X}) \\ W \subset U}} (d_{\mathcal{X}}(U) - 1) = \binom{k+2}{2} \sum_{U \in \Theta_{k+2}(\mathcal{X})} (d_{\mathcal{X}}(U) - 1).$$

To prove (c) we put $a_i = \sum_{U \in \Theta_i(\mathcal{X})} (d_{\mathcal{X}}(U) - 1)$ for all $0 \leq i \leq d+1$, and $b_i = (d+1-i)a_i - (i+1)a_{i+1}$ for $0 \leq i \leq d$. Then the identity in (b) can be rewritten as $b_{k+1} < \frac{d-k}{k+1} b_k$ for all $0 \leq k \leq d-1$. A simple induction on k gives $b_{k+1} < \binom{d}{k+1} b_0$. We may now replace b_{k+1} and b_0 by $(d-k)a_{k+1} - (k+2)a_{k+2}$ and $(d+1)a_0 - a_0$, respectively, and use the fact that $a_0 = |\mathcal{X}|$ to obtain (c).

We prove (d) by induction on k . When $k=0$, (d) follows by putting $k=0$ in (b). Hence suppose that $k \geq 1$. Then (b) gives

$$2(d-k)a_{k+1} - 2(k+1)a_{k+2} < \frac{(d-k+1)(d-k)}{k+1} a_k - ka_{k+2}. \quad (3)$$

We may also use (b) to obtain

$$ka_{k+2} > \frac{k(d-k)}{k+2} \left(2a_{k+1} - \frac{d-k+1}{k+1} a_k \right). \quad (4)$$

Substituting (4) into (3) and using induction we obtain

$$\begin{aligned}
(d-k)a_{k+1} - (k+1)a_{k+2} &< \frac{d-k}{k+2} [(d-k+1)a_k - ka_{k+1}] \\
&< \frac{d-k}{k+2} \binom{d+1}{k+1} (|\mathcal{X}| - 1) \\
&= \binom{d+1}{k+2} (|\mathcal{X}| - 1).
\end{aligned}$$

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Theorem 3.3 *Let $H = (V, E)$ be a d -sparse graph and \mathcal{X} be its d -critical cover. For each critical component H_i of H let $\theta_k(H_i)$ be the number of k -hinges of \mathcal{X} contained in H_i . Then:*

- (a) $\theta_1(H_1) \leq 2d + 1$ for some critical component H_1 of H ;
- (b) $d\theta_1(H_2) + (d - 2)\theta_2(H_2) \leq 2d^2 + 2d - 1$ for some critical component H_2 of H ;
- (c) $\theta_{d-1}(H_3) \leq d$ for some critical component H_3 of H .

Proof: The theorem is trivially true if some critical component of H has only two vertices. Hence we may suppose that every critical component of H has at least $d + 2$ vertices.

We first prove (a). Putting $k = d - 1$ in Lemma 3.2(c) and using the fact that $\Theta_d(\mathcal{X}) = \emptyset = \Theta_{d+1}(\mathcal{X})$ we obtain

$$\sum_{U \in \Theta_{d-1}(\mathcal{X})} (d_{\mathcal{X}}(U) - 1) < (d + 1)(|\mathcal{X}| - 1). \quad (5)$$

Since $d_{\mathcal{X}}(U) \geq 2$ for all $U \in \Theta_{d-1}(\mathcal{X})$ we have $d_{\mathcal{X}}(U) - 1 \geq d_{\mathcal{X}}(U)/2$ and hence (5) gives

$$\sum_{U \in \Theta_{d-1}(\mathcal{X})} d_{\mathcal{X}}(U) < 2(d + 1)|\mathcal{X}|.$$

This tells us that the average number of 1-hinges in a critical component is strictly less than $2(d + 1)$.

We next prove (b). Putting $k = 0$ in Lemma 3.2(d) we obtain

$$d \sum_{U \in \Theta_1(\mathcal{X})} (d_{\mathcal{X}}(U) - 1) + (d - 2) \sum_{U \in \Theta_2(\mathcal{X})} (d_{\mathcal{X}}(U) - 1) < d(d + 1)(|\mathcal{X}| - 1). \quad (6)$$

Since $d_{\mathcal{X}}(U) \geq 2$ we have $d_{\mathcal{X}}(U) - 1 \geq d_{\mathcal{X}}(U)/2$ for all $U \in \Theta_1(\mathcal{X}) \cup \Theta_2(\mathcal{X})$ and hence (6) gives

$$d \sum_{U \in \Theta_1(\mathcal{X})} d_{\mathcal{X}}(U) + (d - 2) \sum_{U \in \Theta_2(\mathcal{X})} d_{\mathcal{X}}(U) < 2d(d + 1)|\mathcal{X}|.$$

This tells us that the average value of $d\theta_1(H_i) + (d - 2)\theta_2(H_i)$ over all critical components H_i of H is strictly less than $2d(d + 1)$.

Finally we prove (c). Putting $k = d - 2$ in Lemma 3.2(d) and using the fact that $\Theta_d(\mathcal{X}) = \emptyset$ we obtain

$$2 \sum_{U \in \Theta_{d-1}(\mathcal{X})} (d_{\mathcal{X}}(U) - 1) < (d + 1)(|\mathcal{X}| - 1). \quad (7)$$

We can now proceed as in (a) and (b). •

4 An upper bound on the rank

Theorem 4.1 *Let $G = (V, E)$ be a graph, and $H = (V, F)$ be a maximal d -sparse subgraph of G . Then $r_d(G) \leq |F|$.*

Proof: We proceed by contradiction. Suppose the theorem is false and choose a counterexample (G, H) such that $|E|$ is as small as possible. Let H_1, H_2, \dots, H_m be the d -critical components of H where $H_i = (V_i, E_i)$ for $1 \leq i \leq m$. Then $\mathcal{X} = \{V_1, V_2, \dots, V_m\}$ is the H -critical cover of G . For all $1 \leq i \leq m$ let E_i^* be the set of all edges $uv \in E_i$ such that $\{u, v\}$ is a 2-hinge of \mathcal{X} .

Claim 4.2 *For all $1 \leq i \leq m$ either $E_i^* = E_i$ or E_i^* is a dependent set of edges in the d -dimensional rigidity matroid $\mathcal{R}_d(G)$.*

Proof: We proceed by contradiction. Suppose that $E_i^* \neq E_i$ and E_i^* is an independent set of edges in $\mathcal{R}_d(G)$ for some $1 \leq i \leq m$. Let $G' = G - (E_G(V_i) \setminus E_i^*)$, $H' = H - (E_i \setminus E_i^*)$ and $F' = F \setminus (E_i \setminus E_i^*)$. Then H' is a maximal sparse subgraph of G' (H' is sparse since $H' \subseteq H$, and H' is maximal since for each edge $e = uv$ of $G' - F'$ we have $\{u, v\} \subseteq V_j$ for some $1 \leq j \leq m$ with $j \neq i$ so $H_j + e \subseteq G' + e$ is not sparse). By the minimality of the counterexample (G, H) ,

$$r_d(G') \leq |F'| = |F| - |E_i| + |E_i^*|. \quad (8)$$

Choose a base B' for $\mathcal{R}_d(G')$ which contains E_i^* . We may extend B' to a base B for $\mathcal{R}_d(G)$. Then $E_i^* \subseteq B$. Since B' spans $E(G')$ and since B can contain at most $|E_i|$ edges between the vertices of V_i we have

$$r_d(G) = |B| \leq |B'| + |E_i| - |E_i^*| = r(G') + |E_i| - |E_i^*|. \quad (9)$$

We may now combine (8) and (9) to obtain

$$r_d(G) \leq |F'| + |E_i| - |E_i^*| = |F|.$$

This contradicts the choice of (G, H) as a counterexample to the theorem. •

Claim 4.3 *$|V_i| \geq d + 2$ and $|E_i^*| \geq \binom{d+2}{2} - 1$ for all $1 \leq i \leq m$.*

Proof: If $|V_i| = 2$ then $E_i^* = \emptyset$. This would contradict Claim 4.2 and hence $|V_i| \geq d + 2$. Suppose $E_i^* = E_i$. Then

$$|E_i^*| = |E_i| = d|V_i| - \binom{d+1}{2} \geq d(d+2) - \binom{d+1}{2} = \binom{d+2}{2} - 1.$$

Thus we may suppose that $E_i^* \neq E_i$. By Claim 4.2, E_i^* is a dependent set of edges in the d -dimensional rigidity matroid. The claim now follows since the smallest dependent set of edges in this matroid has size $\binom{d+2}{2}$. •

Claim 4.3 implies that the number of 2-hinges of \mathcal{X} in each H_i is at least $\binom{d+2}{2} - 1$. It follows that the number 1-hinges in each H_i is at least $d + 2$. We may now apply Theorem 3.3(b) to obtain the required contradiction. •

5 Closing remarks

An improved upper bound on the rank

Given a graph G , let $s_d(G)$ be the minimum number of edges in a maximal d -sparse subgraph of G . Theorem 4.1 tells us that $r_d(G) \leq s_d(G)$. It is not difficult to construct graphs for which strict inequality holds. We use the following operation. Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with $V_1 \cap V_2 = \{u, v\}$ and $E_1 \cap E_2 = \{uv\}$, we refer to the graph $G = G_1 \cup G_2$ as the *parallel connection of G_1 and G_2 along the edge uv* .

The graph G obtained by taking the parallel connection of two copies of K_5 along an edge uv and then deleting uv , is 3-sparse and is not rigid in \mathbb{R}^3 . Hence $s_3(G) = |E(G)| = 18 > 17 = r_3(G)$. On the other hand we may improve the upper bound on $r_3(G)$ in this example by considering the graph $G^* = G + uv$. A maximal sparse subgraph of G^* which contains uv has 17 edges. Thus we have $17 = r_3(G) \leq r_3(G^*) \leq s_3(G^*) = 17$.

More generally, for any graph G we have the improved upper bound

$$r_d(G) \leq \min\{s_d(G^*) : G \subseteq G^*\} =: s_d^*(G). \quad (10)$$

The following example shows that strict inequality can also hold in (10). Let G be obtained from K_5 by taking parallel connections with 10 different K_5 's along each of the edges of the original K_5 . We have $r_3(G) = 89$. On the other hand, $s_3(G) = 90$ (obtained by taking a maximal sparse subgraph which contains 9 of the edges of the original K_5). Furthermore we have $s_3(G^*) \geq r_3(G^*) > r_3(G)$ for all graphs G^* which properly contain G . Thus $s_3^*(G) = 90 > r_3(G)$.

Algorithmic considerations

For fixed d , we can use network flow algorithms to test whether a graph is d -sparse in polynomial time, see for example [2]. This means we can greedily construct a maximal d -sparse subgraph H of a graph G in polynomial time and hence obtain an upper bound on $r_d(G)$ via Theorem 4.1. We do not know whether $s_d(G)$ or $s_d^*(G)$ can be determined in polynomial time.

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References

- [1] Abbot, T. G., *Generalizations of Kempe's Universality Theorem*. MSc thesis, MIT (2008). <http://web.mit.edu/tabbott/www/papers/mthesis.pdf>
- [2] Berg A. and Jordán T., *Algorithms for graph rigidity and scene analysis*, in: Proceedings of the 11th Annual European Symposium on Algorithms 2003, Springer Lecture Notes in Computer Science, vol. 2832, 2003, 78–89.

- [3] Cheng M. J. and Sitharam M., *Maxwell independence: a better rank estimate for 3D rigidity matroids*, submitted.
- [4] Laman, G., *On graphs and rigidity of plane skeletal structures*. J. Engineering Math., **4** (1970) 331-340
- [5] Lovász, L. and Yemini, Y., *On generic rigidity in the plane*. SIAM J. Algebraic Discrete Methods, **3** (1982) 91-98.
- [6] Maxwell, J. C., *On the calculation of the equilibrium and stiness of frames*. Philosophical Magazine **27**, (1864), 294 - 299
- [7] Whiteley, W., *Some matroids from discrete applied geometry*, in Matroid Theory, Bonin, J. E. et al., (eds.), Contemp. Math., vol 197, Amer. Math. Soc., Providence, RI (1996) 171–311.