A necessary condition for generic rigidity of bar-and-joint frameworks in d-space

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Abstract

A graph $G = (V, E)$ is d-sparse if each subset $X \subseteq V$ with $|X| \geq d$ induces at most $d|X| - \binom{d+1}{2}$ edges in G. Laman showed in 1970 that a necessary and sufficient condition for a realisation of G as a generic bar-and-joint framework in $\mathbb{R}^2$ to be rigid is that G should have a 2-sparse subgraph with $2|V| - 3$ edges. Although Laman's theorem does not hold when $d \geq 3$, Cheng and Sitharam recently showed that if G is generically rigid in $\mathbb{R}^3$ then every maximal 3-sparse subgraph of G must have $3|V| - 6$ edges. We extend their result to all $d \leq 5$ by showing that if G is generically rigid in $\mathbb{R}^d$ then every maximal d-sparse subgraph of G must have $d|V| - \binom{d+1}{2}$ edges.

1 Introduction

A *d-dimensional (bar-and-joint) framework* is a pair (G, p) where $G = (V, E)$ is a graph and $p : V \to \mathbb{R}^d$. It is a long standing open problem to determine when a given bar-and-joint framework is *rigid* i.e. every continuous motion of the points $p(v)$ which preserves the distances $\|p(u) - p(v)\|$ for all $uv \in E$ must also preserve the distances $\|p(u) - p(v)\|$ for all $u, v \in V$. It is not difficult to see that a 1-dimensional framework (G, p) is rigid if and only if the graph G is connected. Abbot [1] has recently shown that the problem of determining rigidity is NP-hard for all $d \geq 2$.

This problem becomes more tractable, however, if we assume that the framework is *generic* i.e. there are no algebraic dependencies between the coordinates of the points $p(v)$, $v \in V$. It is known that the rigidity of a d-dimensional generic framework (G, p) depends only on the graph G. Indeed we can define an $|E| \times d|V|$ matrix, the *d-dimensional rigidity matrix* $R_d(G)$, whose entries are linear combinations of indeterminates representing the coordinates of the points $p(v)$, in such a way that (G, p) is rigid if and only if the rank of $R_d(G)$, $r_d(G)$, is equal to $d|V| - \binom{d+1}{2}$. This naturally gives rise to a matroid on E, the *d-dimensional rigidity matroid* $\mathcal{R}_d(G)$ in which a set of edges $F \subseteq E$ is *independent/dependent* if and only if the corresponding rows of $R_d(G)$ are linearly

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independent/dependent. We refer the reader to [7] for a precise definition of the rigidity matrix, the rigidity matroid, and other information on the topic of rigidity.

Laman [4] characterized when a 2-dimensional generic framework is rigid (see also Lovász and Yemini [5]). His characterization is based on the following concept. We say that a subgraph H of G is *d-sparse* if each subset X of at least d vertices of Hinduces at most $d|W| - \binom{d+1}{2}$ edges of H. Maxwell [6] showed that being d-sparse is a necessary condition for the rows of $R_d(G)$ labeled by the edges of H to be linearly independent. Laman showed that that this condition is also sufficient when $d = 2$ and deduced that a 2-dimensional generic framework (G, p) is rigid if and only if it has a 2-sparse subgraph with $2|V|-3$ edges. Since every linearly independent set of rows of $R_2(G)$ can be extended to a basis for the row space of $R_2(G)$, Laman's theorem implies that every maximal 2-sparse subgraph of G has the same number of edges.

It is known that the condition that H is d-sparse is not sufficient for the rows of $R_d(G)$ labeled by the edges of H to be linearly independent when $d \geq 3$. Indeed it is not even true that all maximal d-sparse subgraphs of G have the same number of edges when $d \geq 3$. On the other hand Cheng and Sitharam [3] have recently shown that the number of edges in any maximal 3-sparse subgraph of G does at least give an upper bound on $r_3(G)$.

The purpose of this paper is to prove a result, Theorem 4.1 below, which extends the theorem of Cheng and Sitharam to all values of $d \leq 5$.

2 Sparse subgraphs

Let $G = (V, E)$ be a graph and $d \geq 1$ be an integer. For $X \subseteq V$ we use $E_G(X)$ to denote the set, and $i_G(X)$ the number, of edges of G joining pairs of vertices of X. We simplify these to $E(X)$ and $i(X)$ when it is obvious to which graph we are referring. We may rewrite the condition for G to be d-sparse as $i(X) \leq d|X| - \binom{d+1}{2}$ for all $X \subseteq V$ with $|X| \geq d$. (Note that if $|X| \in \{d, d+1\}$ then we have $i(X) \leq \binom{|X|}{2} = d|X| - \binom{d+1}{2}$ and the inequality holds trivially.) A subgraph $H = (U, F)$ of a d-sparse graph G is *d-critical* if either $|U| = 2$ and $|F| = 1$, or $|U| \geq d+2$ and $|F| = d|X| - \binom{d+1}{2}$. The assumption that G is d-sparse implies that every d-critical subgraph of G is an induced subgraph. A *d-critical component* of G is a d-critical subgraph which is not properly contained in any other d-critical subgraph of G.

Lemma 2.1 *Let $G = (V, E)$ be a d-sparse graph and $H_1 = (U_1, F_1), H_2 = (U_2, F_2)$ be distinct critical components of G. Then $|U_1 \cap U_2| \leq d-1$ and, if equality holds, then $i_G(U_1 \cap U_2) = \binom{d-1}{2}$.*

Proof: Suppose that $|U_1 \cap U_2| \geq d-1$. When $|U_1 \cap U_2| \geq d$ we have $i(U_1 \cap U_2) \leq d|U_1 \cap U_2| - \binom{d+1}{2}$ since G is sparse. When $|U_1 \cap U_2| = d-1$, we have $i(U_1 \cap U_2) \leq \binom{d-1}{2} = d|U_1 \cap U_2| - \binom{d+1}{2} + 1$ trivially. The maximality of H_1, H_2 and the definition of a critical component imply that $|U_1|, |U_2| \geq d+2$, and $d(|U_1| + |U_2|) - 2\binom{d+1}{2} = i_G(U_1) + i_G(U_2) \leq i_G(U_1 \cup U_2) + i_G(U_1 \cap U_2) \leq$

$d|U_1 \cup U_2| - \binom{d+1}{2} - 1 + d|U_1 \cap U_2| - \binom{d+1}{2} + 1 = d(|U_1| + |U_2|) - 2\binom{d+1}{2}$. Equality must hold throughout. In particular we have $i_G(U_1 \cap U_2) = \binom{d+1}{2} + 1$. This implies that $|U_1 \cap U_2| = d - 1$ and $i_G(U_1 \cap U_2) = \binom{d-1}{2}$. •

3 Covers

Let k, t be nonnegative integers, $G = (V, E)$ be a graph and $\mathcal{X}$ be a family of subsets of V. We say that $\mathcal{X}$ is a *cover* of G if every set in $\mathcal{X}$ contains at least two vertices, and every edge of G is induced by at least one set in $\mathcal{X}$. A cover $\mathcal{X}$ is *t-thin* if every pair of sets in $\mathcal{X}$ intersect in at most t vertices. A *k-hinge* of $\mathcal{X}$ is set of k vertices which lie in the intersection of at least two sets in $\mathcal{X}$. A k-hinge U of $\mathcal{X}$ is *closed in G* if $G[U]$ is a complete graph. We use $\Theta_k(\mathcal{X})$ to denote the set of all k-hinges, respectively closed k-hinges, of $\mathcal{X}$. For $U \in \Theta_k(\mathcal{X})$, let $d_{\mathcal{X}}(U)$ denote the number of sets in $\mathcal{X}$ which contain U. Note that if G is t-thin then $\Theta_k(\mathcal{X}) = \emptyset$ for all $k \geq t+1$. Note also that $\Theta_0(\mathcal{X}) = \{\emptyset\}$ and $d_{\mathcal{X}}(\emptyset) = |\mathcal{X}|$.

Lemma 3.1 *Let $G = (V, E)$ be a graph, $H = (V, F)$ be a maximal d-sparse subgraph of G, and $H_1, H_2, \ldots, H_m$ be the d-critical components of H. Let X_i be the vertex set of H_i for $1 \leq i \leq m$. Then $\mathcal{X} = \{X_1, X_2, \ldots, X_m\}$ is a $(d-1)$-thin cover of G and each $(d-1)$-hinge of $\mathcal{X}$ is closed.*

Proof: The definition of a d-critical subgraph implies that each H_i has at least two vertices and that every edge of H belongs to at least one H_i. Thus $\mathcal{X}$ is a cover of H. To see that $\mathcal{X}$ also covers G we choose $e = uv \in E \setminus F$. The maximality of H implies that $H + e$ is not d-sparse. Hence $\{u, v\}$ is contained in some d-critical subgraph of H. Thus $\mathcal{X}$ also covers G. The facts that $\mathcal{X}$ is $(d-1)$-thin and that each $(d-1)$-hinge of $\mathcal{X}$ is closed follow from Lemma 2.1 •

We refer to the closed $(d-1)$-thin cover of G described in Lemma 3.1 as the *H-critical cover* of G. When G is d-sparse (and so $H = G$), we refer to this cover as the *d-critical cover* of G. Note that the definition of a d-critical set implies that each set in a d-critical cover has size two or has size at least $d + 2$.

Lemma 3.2 *Let $H = (V, E)$ be a d-sparse graph, $\mathcal{X}$ be its d-critical cover and $W \in \Theta_k(\mathcal{X})$ for some $0 \leq k \leq d-1$. Suppose that each critical component of H which contains W has at least $d + 2$ vertices. Then*

$$(d-k) \sum_{\substack{U \in \Theta_{k+1}(\mathcal{X}) \\ W \subset U}} (d_{\mathcal{X}}(U) - 1) - \sum_{\substack{U \in \Theta_{k+2}(\mathcal{X}) \\ W \subset U}} (d_{\mathcal{X}}(U) - 1) < \binom{d+1-k}{2} (d_{\mathcal{X}}(W) - 1).$$

Proof: Let $d_{\mathcal{X}}(W) = t$ and let $H_1, H_2, \ldots, H_t$ be the critical components of H which contain W. Put $H_i = (V_i, E_i)$ for $1 \leq i \leq t$. Let $H' = \bigcup_{i=1}^{t} H_i$ and put

$H' = (V', E')$. Then

$$|V'| = \sum_{i=1}^{t} |V_i| - k(t-1) - \sum_{\substack{U \in \Theta_{k+1}(\mathcal{X}) \\ W \subset U}} (d_{\mathcal{X}}(U) - 1) \tag{1}$$

since, for $v \in V'$, if $v \in W$ then v is counted t times in the sum $\sum_{i=1}^{t} |V_i|$, if $v \in U \setminus W$ for some $U \in \Theta_{k+1}$ with $W \subset U$ then v is counted $d_{\mathcal{X}}(U)$ times in this sum, and all other vertices of V' are counted exactly once in this sum. Similarly,

$$|E'| \geq \sum_{i=1}^{t} |E_i| - \binom{k}{2}(t-1) - k \sum_{\substack{U \in \Theta_{k+1}(\mathcal{X}) \\ W \subset U}} (d_{\mathcal{X}}(U) - 1) - \sum_{\substack{U \in \Theta_{k+2}(\mathcal{X}) \\ W \subset U}} (d_{\mathcal{X}}(U) - 1) \tag{2}$$

since, for $e = uv \in E'$, if $u, v \in W$ then e is counted t times in the sum $\sum_{i=1}^{t} |E_i|$ and there are at most $\binom{k}{2}$ such edges, if $u \in W$ and $v \in U \setminus W$ for some $U \in \Theta_{k+1}$ with $W \subset U$ then e is counted $d_{\mathcal{X}}(U)$ times in this sum and for each such v there are at most k choices for u, if $u, v \in U \setminus W$ for some $U \in \Theta_{k+2}$ with $W \subset U$ then e is counted $d_{\mathcal{X}}(U)$ times in this sum, and all other edges of E' are counted exactly once in this sum.

Since $H' \subseteq H$, H' is sparse and since $W \in \Theta_k$ we have $t \geq 2$ so H' is not critical. Hence $|E'| < d|V'| - \binom{d+1}{2}$. We may substitute equations (1) and (2) into this inequality and use the fact that $|E_i| = d|V_i| - \binom{d+1}{2}$ for all $1 \leq i \leq t$ to obtain

$$\begin{aligned} (d-k) \sum_{\substack{U \in \Theta_{k+1}(\mathcal{X}) \\ W \subset U}} (d_{\mathcal{X}}(U) - 1) \quad &- \sum_{\substack{U \in \Theta_{k+2}(\mathcal{X}) \\ W \subset U}} (d_{\mathcal{X}}(U) - 1) \\ &< \left[\binom{d+1}{2} + \binom{k}{2} - dk \right] (t-1) \\ &= \binom{d+1-k}{2} (t-1). \end{aligned}$$

•

Lemma 3.3 *Let $H = (V, E)$ be a d-sparse graph and $\mathcal{X}$ be its d-critical cover. Suppose that each critical component of H has at least $d+2$ vertices. Put $a_k = \sum_{U \in \Theta_k(\mathcal{X})} (d_{\mathcal{X}}(U) - 1)$ for $0 \leq k \leq d$. Then for all $0 \leq k \leq d-2$ we have:*

(a) $(d-k)(k+1)a_{k+1} - \binom{k+2}{2} a_{k+2} < \binom{d+1-k}{2} a_k$;

(b) $(d-k)a_{k+1} - (k+1)a_{k+2} < \binom{d+1}{k+2}(|\mathcal{X}| - 1)$;

(c) $d(d-k)a_{k+1} < (k+2)(d-k-1)\binom{d+1}{k+2}(|\mathcal{X}| - 1)$.

Proof: Part (a) follows by summing the inequality in Lemma 3.2 over all $W \in \Theta_k$, and using the facts that

$$\sum_{W \in \Theta_k(\mathcal{X})} \sum_{\substack{U \in \Theta_{k+1}(\mathcal{X}) \\ W \subset U}} (d_{\mathcal{X}}(U) - 1) = (k+1) \sum_{U \in \Theta_{k+1}(\mathcal{X})} (d_{\mathcal{X}}(U) - 1) = (k+1)a_{k+1}$$

and

$$\sum_{W\in\Theta_k(\mathcal{X})}\ \sum_{\substack{U\in\Theta_{k+2}(\mathcal{X})\\ W\subset U}}(d_{\mathcal{X}}(U)-1)=\binom{k+2}{2}\sum_{U\in\Theta_{k+2}(\mathcal{X})}(d_{\mathcal{X}}(U)-1)=\binom{k+2}{2}a_{k+2}\,.$$

We prove (b) by induction on k. When $k=0$, (b) follows by putting $k=0$ in (a). Hence suppose that $k\geq 1$. Then (a) gives

$$2(d-k)a_{k+1}-2(k+1)a_{k+2}<\frac{(d-k+1)(d-k)}{k+1}a_k-ka_{k+2}\,. \tag{3}$$

We may also use (a) to obtain

$$ka_{k+2}>\frac{k(d-k)}{k+2}\left(2a_{k+1}-\frac{d-k+1}{k+1}a_k\right)\,. \tag{4}$$

Substituting (4) into (3) and using induction we obtain

$$\begin{aligned}(d-k)a_{k+1}-(k+1)a_{k+2} &< \tfrac{d-k}{k+2}\left[(d-k+1)a_k-ka_{k+1}\right]\\ &< \tfrac{d-k}{k+2}\tbinom{d+1}{k+1}(|\mathcal{X}|-1)\\ &= \tbinom{d+1}{k+2}(|\mathcal{X}|-1)\,.\end{aligned}$$

We prove (c) by induction on $d-k$. When $d-k=2$, (c) follows by putting $k=d-2$ in (b) and using the fact that $a_d=0$ by Lemma 3.1. Hence suppose that $d-k\geq 3$. Then (b) gives

$$d(d-k)a_{k+1}<d\tbinom{d+1}{k+2}(|\mathcal{X}|-1)+d(k+1)a_{k+2}\,.$$

We may now apply induction to a_{k+2} to obtain

$$\begin{aligned}d(d-k)a_{k+1} &< \left[d\tbinom{d+1}{k+2}+\tfrac{(k+1)(k+3)(d-k-2)}{d-k-1}\tbinom{d+1}{k+3}\right](|\mathcal{X}|-1)\\ &= (k+2)(d-k-1)\tbinom{d+1}{k+2}(|\mathcal{X}|-1)\,.\end{aligned}$$

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Theorem 3.4 *Let $H=(V,E)$ be a d-sparse graph and $\mathcal{X}$ be its d-critical cover. For each critical component H_i of H let $\theta_k(H_i)$ be the number of k-hinges of $\mathcal{X}$ contained in H_i. Then:*
(a) $\theta_1(H_1)\leq 2d-1$ for some critical component H_1 of H;
(b) $\theta_2(H_2)\leq (d-2)(d+1)-1$ for some critical component H_2 of H;
(c) $\theta_{d-1}(H_3)\leq d$ for some critical component H_3 of H.

Proof: The theorem is trivially true if some critical component of H has only two vertices. Hence we may suppose that every critical component of H has at least $d+2$ vertices.

We first prove (a). Putting $k = 0$ in Lemma 3.3(c) we obtain

$$d \sum_{U \in \Theta_1(\mathcal{X})} (d_{\mathcal{X}}(U) - 1) < (d-1)(d+1)(|\mathcal{X}| - 1)\,. \tag{5}$$

Since $d_{\mathcal{X}}(U) \geq 2$ for all $U \in \Theta_1(\mathcal{X})$ we have $d_{\mathcal{X}}(U) - 1 \geq d_{\mathcal{X}}(U)/2$ and hence (5) gives

$$\sum_{U \in \Theta_1(\mathcal{X})} d_{\mathcal{X}}(U) < 2d\,|\mathcal{X}|\,.$$

This tells us that the average number of 1-hinges in a critical component is strictly less that $2d$.

We next prove (b). Putting $k = 1$ in Lemma 3.3(c) we obtain

$$\sum_{U \in \Theta_2(\mathcal{X})} (d_{\mathcal{X}}(U) - 1) < (d-2)(d+1)(|\mathcal{X}| - 1)/2\,. \tag{6}$$

We can now proceed as in (a).

Finally we prove (c). Putting $k = d - 2$ in Lemma 3.3(c) gives

$$2 \sum_{U \in \Theta_{d-1}(\mathcal{X})} (d_{\mathcal{X}}(U) - 1) < (d+1)(|\mathcal{X}| - 1)\,. \tag{7}$$

We can now proceed as in (a).

4 An upper bound on the rank

Theorem 4.1 *Let $G = (V, E)$ be a graph, $d \leq 5$ be an integer and $H = (V, F)$ be a maximal d-sparse subgraph of G. Then $r_d(G) \leq |F|$.*

Proof: We proceed by contradiction. Suppose the theorem is false and choose a counterexample (G, H) such that $|E|$ is as small as possible. Let $H_1, H_2, \ldots, H_m$ be the d-critical components of H where $H_i = (V_i, E_i)$ for $1 \leq i \leq m$. Then $\mathcal{X} = \{V_1, V_2, \ldots, V_m\}$ is the H-critical cover of G. For all $1 \leq i \leq m$ let E_i^* be the set of all edges $uv \in E_i$ such that $\{u, v\}$ is a 2-hinge of $\mathcal{X}$.

Claim 4.2 *For all $1 \leq i \leq m$ either $E_i^* = E_i$ or E_i^* is a dependent set of edges in the d-dimensional rigidity matroid $\mathcal{R}_d(G)$.*

Proof: We proceed by contradiction. Suppose that $E_i^* \neq E_i$ and E_i^* is an independent set of edges in $\mathcal{R}(G)$ for some $1 \leq i \leq m$. Let $G' = G - (E_G(V_i) \setminus E_i^*)$, $H' = H - (E_i \setminus E_i^*)$ and $F' = F \setminus (E_i \setminus E_i^*)$. Then H' is a maximal sparse subgraph of G' (H' is sparse since $H' \subseteq H$, and H' is maximal since for each edge $e = uv$ of $G' - F'$ we have $\{u, v\} \subseteq V_j$ for some $1 \leq j \leq m$ with $j \neq i$ so $H_j + e \subseteq G' + e$ is not sparse). By the minimality of the counterexample (G, H),

$$r_d(G') \leq |F'| = |F| - |E_i| + |E_i^*|. \tag{8}$$

Choose a base B' for $\mathcal{R}_d(G')$ which contains E_i^*. We may extend B' to a base B for $\mathcal{R}_d(G)$. Then $E_i^* \subseteq B$. Since B' spans $E(G')$ and since B can contain at most $|E_i|$ edges between the vertices of V_i we have

$$r_d(G) = |B| \leq |B'| + |E_i| - |E_i^*| = r(G') + |E_i| - |E_i^*|. \tag{9}$$

We may now combine (8) and (9) to obtain

$$r_d(G) \leq |F'| + |E_i| - |E_i^*| = |F|.$$

This contradicts the choice of (G, H) as a counterexample to the theorem. •

Claim 4.3 *$|V_i| \geq d + 2$ and $|E_i^*| \geq \binom{d+2}{2} - 1$ for all $1 \leq i \leq m$.*

Proof: If $|V_i| = 2$ then $E_i^* = \emptyset$. This would contradict Claim 4.2 and hence $|V_i| \geq d + 2$. Suppose $E_i^* = E_i$. Then

$$|E_i^*| = |E_i| = d|V_i| - \binom{d+1}{2} \geq d(d+2) - \binom{d+1}{2} = \binom{d+2}{2} - 1.$$

Thus we may suppose that $E_i^* \neq E_i$. By Claim 4.2, E_i^* is a dependent set of edges in the d-dimensional rigidity matroid. The claim now follows since the smallest dependent set of edges in this matroid has size $\binom{d+2}{2}$. •

Claim 4.3 implies that the number of 2-hinges of $\mathcal{X}$ in each H_i is at least $\binom{d+2}{2} - 1$. We may now apply Theorem 3.4(b) to obtain the required contradiction. •

5 Closing remarks

An improved upper bound on the rank

Given a graph G, let $s_d(G)$ be the minimum number of edges in a maximal d-sparse subgraph of G. Theorem 4.1 tells us that $r_d(G) \leq s_d(G)$ when $d \leq 5$. It is not difficult to construct graphs for which strict inequality holds. We use the following operation. Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with $V_1 \cap V_2 = \{u, v\}$ and $E_1 \cap E_2 = \{uv\}$, we refer to the graph $G = G_1 \cup G_2$ as the *parallel connection of G_1 and G_2 along the edge uv*.

The graph G obtained by taking the parallel connection of two copies of K_5 along an edge uv and then deleting uv, is 3-sparse and is not rigid in $\mathbb{R}^3$. Hence $s_3(G) = |E(G)| = 18 > 17 = r_3(G)$. On the other hand we may improve the upper bound on $r_3(G)$ in this example by considering the graph $G^* = G + uv$. A maximal sparse subgraph of G^* which contains uv has 17 edges. Thus we have $17 = r_3(G) \leq r_3(G^*) \leq s_3(G^*) = 17$.

More generally, for any graph G we have the improved upper bound

$$r_d(G) \leq \min\{s_d(G^*) \ : \ G \subseteq G^*\} =: s_d^*(G) \tag{10}$$

for all $d \leq 5$. The following example shows that strict inequality can also hold in (10). Let G be obtained from K_5 by taking parallel connections with 10 different K_5's along each of the edges of the original K_5. We have $r_3(G) = 89$. On the other hand, $s_3(G) = 90$ (obtained by taking a maximal sparse subgraph which contains 9 of the edges of the original K_5). Furthermore we have $s_3(G^*) \geq r_3(G^*) > r_3(G)$ for all graphs G^* which properly contain G. Thus $s_3^*(G) = 90 > r_3(G)$.

Algorithmic considerations

For fixed d, we can use network flow algorithms to test whether a graph is d-sparse in polynomial time, see for example [2]. This means we can greedily construct a maximal d-sparse subgraph H of a graph G in polynomial time and hence obtain an upper bound on $r_d(G)$ via Theorem 4.1. We do not know whether $s_d(G)$ or $s_d^*(G)$ can be determined in polynomial time.

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