The number of equivalent realisations of a rigid graph

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Abstract

Given a generic rigid realisation of a graph in $\mathbb{R}^2$, it is an open problem to determine the maximum number of pairwise non-congruent realisations which have the same edge lengths as the given realisation. This problem can be restated as finding the number of solutions of a related system of quadratic equations and in this context it is natural to consider the number of solutions in $\mathbb{C}^2$. We show that the number of complex solutions, $c(G)$, is the same for all generic realisations of a rigid graph G, characterise the graphs G for which $c(G) = 1$, and show that the problem of determining $c(G)$ can be reduced to the case when G is 3-connected and has no non-trival 3-edge-cuts. We also consider the effect of the the so called Henneberg moves on $c(G)$ and determine $c(G)$ exactly for two important families of graphs.

1 Introduction

Graphs with geometrical constraints provide natural models for a variety of applications, including Computer-Aided Design, sensor networks and flexibility in molecules. Given a graph G and prescribed lengths for its edges, a basic problem is to determine whether G has a straight line realisation

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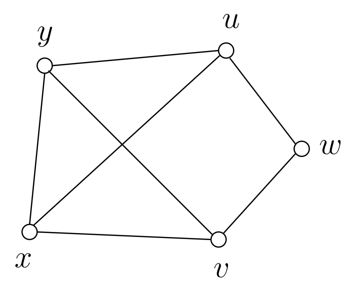

Figure 1: A realisation of a graph G in $\mathbb{R}^2$. The only other equivalent realisation is obtained by reflecting the vertex w in the line through $\{u, v\}$.

in Euclidean d-dimensional space with these given lengths. Given a realisation, one may also ask whether it is unique, and if not, determine how many distinct realisations exist with the same edge lengths. Saxe [21] has shown that both the existence and uniqueness problems are NP-hard. However, this hardness relies on algebraic relations between coordinates of vertices, and for practical purposes it is natural to study generic realisations.

A recent result of Gortler, Healy and Thurston [8] implies that the uniqueness of a generic realisation depends only on the structure of the underlying graph. It can be seen that a graph G has a unique generic realisation on the real line if and only if G is equal to K_2 or is 2-connected. Graphs with unique generic realisations in $\mathbb{R}^2$ are characterised by a combination of results due to Hendrickson [9], Connelly [5], and Jackson and Jordán [11]. No characterisations are known in $\mathbb{R}^d$ when $d \geq 3$.

In contrast, the number of realisations which are equivalent to, i.e. have the same edge lengths as, a given generic realisation of a graph in $\mathbb{R}^d$ may depend on both the graph and the realisation when $d \geq 2$, see Figures 1 and 2. Bounds on the maximum number of equivalent realisations, where the maximum is taken over all possible generic realisations of a given graph, are obtained by Borcea and Streinu in [4], and this number is determined exactly for an important family of graphs by Jackson, Jordán, and Szabadka in [12].

The set of all realisations which are equivalent to a given realisation can be represented as the set of solutions to a system of quadratic equations. In this setting it is natural to consider the number of complex solutions. This number gives an upper bound on the number of real solutions and, as we will see, is much better behaved than the number of real solutions.

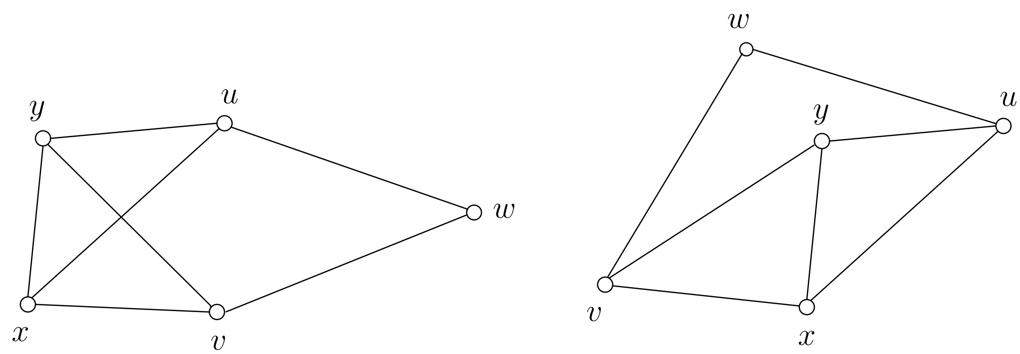

Figure 2: Two equivalent realisations of the graph G of Figure 1 in $\mathbb{R}^2$. Two other equivalent realisation can be obtained from these by reflecting the vertex w in the line through $\{u, v\}$, giving four different equivalent realisations in $\mathbb{R}^2$.

We will concentrate on the 2-dimensional case and consider realisations of rigid graphs i.e. graphs with the property that some/every generic realisation in $\mathbb{R}^2$ is locally unique. We show that the number, $c(G)$, of complex realisations of a rigid graph G which are equivalent to a given generic realisation is finite and is the same for all generic realisations. We then consider the affect of the so-called Henneberg moves on $c(G)$. We show that a type 1 move doubles $c(G)$ and that a type 2 move on a redundant edge does not increase $c(G)$. We use the latter result to characterise graphs G with $c(G) = 1$. Our characterisation is the same as the above mentioned characterisation of graphs with a unique generic realisation in $\mathbb{R}^2$, and explains the apparent inconsistency that having a unique real realisation is a generic property whereas the number of different real realisations is not. We next consider graphs G which can be separated into two pieces G_1, G_2 by deleting small sets of vertices or edges and show how $c(G)$ can be computed from $c(G_1)$ and $c(G_2)$. We then use these results to determine $c(G)$ for two important families of graphs. We close with a short section of open problems.

2 Definitions and notation

A *complex (real) realisation* of a graph $G = (V, E)$ is a map p from V to $\mathbb{C}^2$ ($\mathbb{R}^2$). We also refer to the ordered pair (G, p) as a *framework*. A framework

(G,p) is *generic* if the set of all coordinates of the points $p(v)$, $v \in V$, is algebraically independent over $\mathbb{Q}$.

For $P = (x,y) \in \mathbb{C}^2$ let $d(P) = x^2 + y^2$. Two frameworks (G,p) and (G,q) are *equivalent* if $d(p(u) - p(v)) = d(q(u) - q(v))$ for all $uv \in E$, and are *congruent* if $d(p(u) - p(v)) = d(q(u) - q(v))$ for all $u, v \in V$.

A real framework (G,p) is *rigid* if there exists an $\epsilon > 0$ such that every real framework (G,q) which is equivalent to (G,p) and satisfies $d(p(v) - q(v)) = \|p(v) - q(v)\|^2 < \epsilon$ for all $v \in V$, is congruent to (G,p). Equivalently, every continuous motion of the points $p(v)$, $v \in V$, in $\mathbb{R}^2$ which respects the length constraints results in a framework which is congruent to (G,p).

The *rigidity matrix* of a framework (G,p) is the matrix $R(G,p)$ of size $|E| \times 2|V|$, where, for each edge $v_iv_j \in E$, in the row corresponding to v_iv_j, the entries in the two columns corresponding to vertices v_i and v_j contain the two coordinates of $(p(v_i) - p(v_j))$ and $(p(v_j) - p(v_i))$, respectively, and the remaining entries are zeros. The framework is *infinitessimally rigid* if rank $R(G,p) = 2|V| - 3$.[1] Asimow and Roth [1] showed that infinitessimal rigidity is a sufficient condition for the rigidity of (G,p), and that the two properties are equivalent when (G,p) is generic. This implies that rigidity is a generic property of real realisations and we say that G is *rigid* if some/every generic real realisation of G is rigid. Rigid graphs are characterised by results of Laman [13] and Lovász and Yemini [14]. We refer the reader to [23] for more information on the rigidity of graphs.

Given a complex, respectively real, realisation (G,p) of a rigid graph G, let $c(G,p)$, respectively $r(G,p)$, denote the number of congruence classes in the set of all complex, respectively real, realisations of G which are equivalent to (G,p).

3 Congruent realisations

Given a complex realisation of a rigid graph it will be useful to have a 'canonical representative' for each congruence class in the set of all equivalent realisations. The following lemmas will enable us to do this.

[1]We always have rank $R(G,p) \leq 2|V| - 3$ since its null space always contains three linearly independent vectors coresponding to two translations and a rotation of the framework.

Lemma 3.1 *(a) Let* $P_0 \in \mathbb{C}^2$ *and* $\tau : \mathbb{C}^2 \to \mathbb{C}^2$ *by* $\tau(P) = P + P_0$*. Then* $d(P - Q) = d(\tau(P) - \tau(Q))$ *for all* $P, Q \in \mathbb{C}^2$*.*
(b) Let $z_1, z_2 \in \mathbb{C}$ *such that* $z_1^2 + z_2^2 = 1$ *and put*

$$M = \begin{pmatrix} z_1 & z_2 \\ -z_2 & z_1 \end{pmatrix}.$$

Let $\rho : \mathbb{C}^2 \to \mathbb{C}^2$ *by* $\rho(P) = MP$*. Then* $d(P - Q) = d(\rho(P) - \rho(Q))$ *for all* $P, Q \in \mathbb{C}^2$*.*
(c) Put

$$N = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let $\theta : \mathbb{C}^2 \to \mathbb{C}^2$ *by* $\theta(P) = NP$*. Then* $d(P - Q) = d(\theta(P) - \theta(Q))$ *for all* $P, Q \in \mathbb{C}^2$*.*

Proof. (a) is immediate since $\tau(P) - \tau(Q) = P - Q$. To prove (b) and (c), let $P - Q = (a, b)$. Then

$$\begin{aligned} d(\rho(P) - \rho(Q)) &= d(MP - MQ) = d(M(P - Q)) = d(z_1 a + z_2 b, -z_2 a + z_1 b) \\ &= (z_1 a + z_2 b)^2 + (-z_2 a + z_1 b)^2 = a^2 + b^2 = d(P - Q). \end{aligned}$$

since $z_1^2 + z_2^2 = 1$. Similarly

$$d(\theta(P) - \theta(Q)) = d(NP - NQ) = d(N(P - Q)) = d(a, -b) = d(P - Q).$$

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Three distinct points $P_1, P_2, P_3 \in \mathbb{C}^2$ are *collinear* if $P_2 - P_1 = z(P_3 - P_1)$ for some $z \in \mathbb{C}$.

Lemma 3.2 *Let* P_1, P_2, P_3 *be three distinct points in* $\mathbb{C}^2$ *which are not collinear. Suppose that* M, M' *are* 2×2 *complex matrices,* $t, t' \in \mathbb{C}^2$*, and that* $MP_i + t = M'P_i + t'$ *for all* $1 \le i \le 3$*. Then* $M = M'$ *and* $t = t'$*.*

Proof. Since P_1, P_2, P_3 are not collinear, $P_2 - P_1$ and $P_3 - P_1$ are linearly independent. Furthermore $(M - M')(P_2 - P_1) = 0 = (M - M')(P_3 - P_1)$. Hence $M - M' = 0$. Thus $M = M'$ and $t = t'$. •

Lemma 3.3 *Let (G,p) be a complex realisation of a graph $G=(V,E)$ with $V=\{v_1,v_2,\ldots,v_n\}$ and $n\geq 3$. Then (G,p) is congruent to a realisation (G,q) with $q(v_1)=(0,0)$. Furthermore, if $d(p(v_1)-p(v_2))=d_0^2$ for some $d_0\in\mathbb{C}$ with $d_0\neq 0$ and* Arg *$d_0\in(0,\pi]$, then there exists a realisation (G,q^*) which is congruent to (G,p) and satisfies $q^*(v_1)=(0,0)$, $q^*(v_2)=(0,d_0)$ and $q^*(v_3)=(a_3,b_3)$ for some $a_3,b_3\in\mathbb{C}$ with either $a_3=0$ or* Arg *$a_3\in(0,\pi]$.*

Proof. Define (G,q) by putting $q(v_i)=p(v_i)-p(v_1)$ for all $v_i\in V$. Then $q(v_1)=(0,0)$ and (G,q) is congruent to (G,p) by Lemma 3.1(a).

Now suppose that $d(p(v_2)-p(v_1))=d_0^2\neq 0$ and let $q(v_2)=(a,b)$. Then

$$a^2+b^2=d(q(v_2)-q(v_1))=d(p(v_2)-p(v_1))=d_0^2\neq 0.$$

Put $z_1=b/d_0$ and $z_2=-a/d_0$. Then $z_1^2+z_2^2=1$. We may now define the matrix M as in Lemma 3.1(b) and define a realisation $(G,\tilde{q})$ by putting $\tilde{q}(v_i)=Mq(v_i)$ for all $v_i\in V$. We then have $\tilde{q}(v_1)=q(v_1)=(0,0)$ and $\tilde{q}(v_2)=(0,d_0)$. Then $(G,\tilde{q})$ is congruent to (G,p) by Lemma 3.1(b). Let $\tilde{q}(v_3)=(a_3,b_3)$. If $a_3=0$ or Arg $a_3\in(0,\pi]$ we put $q^*=\tilde{q}$; if Arg $a_3\in(-\pi,0]$ we put $q^*(v_i)=Nq(v_i)$ for all $v_i\in V$, where N is the matrix defined in Lemma 3.1(c). By Lemma 3.1(c), (G,q^*) is congruent to (G,p) and satisfies the conditions on q^* given in the statement of the lemma. •

4 Field extensions

In this section we obtain some preliminary results on field extensions of $\mathbb{Q}$. We will use these results in the next section to prove a key lemma: if (G,p) is a generic realisation of a rigid graph G, and (G,q) is an equivalent realisation in 'canonical position' i.e. satisfying the conclusions of Lemma 3.3, then the two field extensions we obtain by adding the coordinates of the points $q(v)$, $v\in V$, or the 'edge lengths' $d(p(u)-p(v))$, $uv\in E$, to $\mathbb{Q}$ have the same algebraic closure.

A point $x\in\mathbb{C}^n$ is *generic* if its components form an algebraically independent set over $\mathbb{Q}$. Given a field K we use $K[X_1,X_2,\ldots,X_n]$ to denote the ring of polynomials in the indeterminates $X_1,X_2,\ldots,X_n$ with coefficients in K and $K(X_1,X_2,\ldots,X_n)$ to denote its field of fractions. Given a multivariate polynomial function $f:\mathbb{C}^n\to\mathbb{C}^m$ we use $df|_x$ to denote the Jacobean matrix of f evaluated at a point $x\in\mathbb{C}^n$. We will obtain several results concerning

$\mathbb{Q}(p)$ and $\mathbb{Q}(f(p))$ when p is a generic point in $\mathbb{C}^n$. These will be applied to a generic realisation (G, p) by taking $f(p)$ to be the vector of 'squared edge lengths' in (G, p).

Lemma 4.1 *Let* $f : \mathbb{C}^n \to \mathbb{C}^m$ *by* $f(x) = (f_1(x), f_2(x), \ldots, f_m(x))$*, where* $f_i \in \mathbb{Q}[X_1, X_2, \ldots, X_n]$ *for* $1 \leq i \leq m$*. Suppose that* p *is a generic point in* $\mathbb{C}^n$ *and* $\operatorname{rank} df|_p = m$*. Then* $f(p)$ *is a generic point in* $\mathbb{C}^m$*.*

Proof. Relabelling if necessary, we may suppose that the first m columns of $df|_p$ are linearly independent. Let $p = (p_1, p_2, \ldots, p_n)$. Define $h : \mathbb{C}^m \to \mathbb{C}^m$ by $h(x_1, x_2, \ldots, x_m) = f(x_1, x_2, \ldots, x_m, p_{m+1}, \ldots, p_n)$. and let $p' = (p_1, p_2, \ldots, p_m)$. Then $h(p') = f(p)$ and $\operatorname{rank} dh|_{p'} = m$.

Let $h(p') = (\beta_1, \beta_2, \ldots, \beta_m)$. Suppose that $g(\beta_1, \beta_2, \ldots, \beta_m) = 0$ for some polynomial g with integer coefficients. Then $g(f_1(p), f_2(p), \ldots, f_m(p)) = 0$. Since p is generic, we have $g(h(x)) = 0$ for all $x \in \mathbb{C}^m$. By the inverse function theorem h maps a sufficiently small open neighbourhood U of p' bijectively onto $h(U)$. Thus, for each $y \in h(U)$, there exists $x \in U$ such that $h(x) = y$. This implies that $g(y) = g(h(x)) = 0$ for each $y \in h(U)$. Since g is a polynomial map and $h(U)$ is an open subset of $\mathbb{C}^m$, we have $g \equiv 0$. Hence $h(p') = f(p)$ is generic. •

Given a point $p \in \mathbb{C}^n$ we use $\mathbb{Q}(p)$ to denote the field extension of $\mathbb{Q}$ by the coordinates of p. Given fields $K \subseteq L$ the *transcendence degree* of L over K, $\operatorname{td}[L : K]$, is the cardinality of a largest subset of L which is algebraically independent over K, see [20, Section 18.1]. (It follows from the Steinitz exchange axiom, see [20, Lemma 18.4], that every set of elements of L which is algebraically independent over K can be extended to a set of $\operatorname{td}[L : K]$ elements which is algebraically independent over K.) We use $\overline{K}$ to denote the algebraic closure of K. Note that $\operatorname{td}[\overline{K} : K] = 0$.

Lemma 4.2 *Let* $f : \mathbb{C}^n \to \mathbb{C}^n$ *by* $f(x) = (f_1(x), f_2(x), \ldots, f_n(x))$*, where* $f_i \in \mathbb{Q}[X_1, X_2, \ldots, X_n]$ *for* $1 \leq i \leq n$*. Suppose that* $f(p)$ *is a generic point in* $\mathbb{C}^n$ *for some point* $p \in \mathbb{C}^n$*. Then* $\overline{\mathbb{Q}(f(p))} = \overline{\mathbb{Q}(p)}$*.*

Proof. Since f_i is a polynomial with rational coefficients, we have $f_i(p) \in \mathbb{Q}(p)$ for all $1 \leq i \leq n$. Thus $\mathbb{Q}(f(p)) \subseteq \mathbb{Q}(p)$. Since $f(p)$ is generic, $\operatorname{td}[\mathbb{Q}(f(p)) : \mathbb{Q}] = n$. Since $\mathbb{Q}(f(p)) \subseteq \mathbb{Q}(p)$ and $p \in \mathbb{C}^n$ we have $\operatorname{td}[\mathbb{Q}(p) : \mathbb{Q}] = n$. Thus $\overline{\mathbb{Q}(f(p))} \subseteq \overline{\mathbb{Q}(p)}$ and $\operatorname{td}[\overline{\mathbb{Q}(f(p))} : \mathbb{Q}] = n = \operatorname{td}[\overline{\mathbb{Q}(p)} : \mathbb{Q}]$. Suppose $\overline{\mathbb{Q}(f(p))} \neq \overline{\mathbb{Q}(p)}$, and choose $\gamma \in \overline{\mathbb{Q}(p)} - \overline{\mathbb{Q}(f(p))}$. Then γ is not algebraic over $\mathbb{Q}(f(p))$ so $S = \{\gamma, f_1(p), f_2(p), \ldots, f_n(p)\}$ is algebraically independent over $\mathbb{Q}$. This contradicts the facts that $S \subseteq \overline{\mathbb{Q}(p)}$ and $\mathrm{td}[\overline{\mathbb{Q}(p)} : \mathbb{Q}] = n$. •

Lemma 4.3 *Let $f : \mathbb{C}^n \to \mathbb{C}^m$ by $f(x) = (f_1(x), f_2(x), \ldots, f_m(x))$, where $f_i \in \mathbb{Q}[X_1, X_2, \ldots, X_n]$ for $1 \leq i \leq m$. Let p be a generic point in $\mathbb{C}^n$ and suppose that* rank $df|_p = n$. *Let $W = \{q \in \mathbb{C}^n : f(q) = f(p)\}$. Then W is finite and $\overline{\mathbb{Q}(p)} = \overline{\mathbb{Q}(q)}$ for all $q \in W$.*

Proof. Reordering the components of f if necessary, we may suppose that the first n rows of $df|_p$ are linearly independent. Let $g : \mathbb{C}^n \to \mathbb{C}^n$ by $g(x) = (f_1(x), f_2(x), \ldots, f_n(x))$. Then rank $dg|_p = n$ and the set $W' = \{q \in \mathbb{C}^n : g(q) = q(p)\}$ is finite by [17, Theorem 2.3]. Since $W \subseteq W'$, W is also finite. Furthermore, Lemma 4.1 implies that $g(p)$ is a generic point in $\mathbb{C}^n$. Lemma 4.2 and the fact that $g(p) = g(q)$ now give $\overline{\mathbb{Q}(p)} = \overline{\mathbb{Q}(g(p))} = \overline{\mathbb{Q}(q)}$. •

Lemma 4.4 *Let $f : \mathbb{C}^n \to \mathbb{C}^m$ by $f(x) = (f_1(x), f_2(x), \ldots, f_m(x))$, where $f_i \in \mathbb{Q}[X_1, X_2, \ldots, X_n]$ for $1 \leq i \leq m$. For each $y \in \mathbb{C}^n$ let*

$$W(y) = \{z \in \mathbb{C}^n : f(z) = f(y)\}.$$

Suppose that p and q are generic points in $\mathbb{C}^n$ and that rank $df|_p = n$. *Then $W(p)$ and $W(q)$ are both finite and $|W(p)| = |W(q)|$.*

Proof. The fact that $W(p)$ and $W(q)$ are finite follows from Lemma 4.3. Since $p = (p_1, p_2, \ldots, p_n)$ and $q = (q_1, q_2, \ldots, q_n)$ are both generic, $\mathbb{Q}(p)$ and $\mathbb{Q}(q)$ are both isomorphic to $\mathbb{Q}(X_1, X_2, \ldots, X_n)$ and we may define an isomorphism $\theta : \mathbb{Q}(p) \to \mathbb{Q}(q)$ by putting $\theta(c) = c$ for all $c \in \mathbb{Q}$ and $\theta(p_i) = q_i$ for all $1 \leq i \leq n$. We may extend θ to an isomorphism $\tilde{\theta} : \overline{\mathbb{Q}(p)} \to \overline{\mathbb{Q}(q)}$.[2] We may then apply $\tilde{\theta}$ to each component of $\overline{\mathbb{Q}(p)}^{\,n}$ to obtain an isomorphism $\Theta : \overline{\mathbb{Q}(p)}^{\,n} \to \overline{\mathbb{Q}(q)}^{\,n}$.

Suppose $z \in W(p)$. Then $f(z) = f(p)$ and Lemma 4.3 gives $\overline{\mathbb{Q}(p)} = \overline{\mathbb{Q}(z)}$. It follows that each component of z belongs to $\overline{\mathbb{Q}(p)}$ and hence $z \in \overline{\mathbb{Q}(p)}^{\,n}$.

[2]This follows from the fact that there is an isomorphism between any two algebraically closed fields of the same transcendence degree over $\mathbb{Q}$, which takes a given transcendence basis for the first to one for the second, see for example the proof of [15, Proposition 8.16].

Thus $W(p) \subseteq \overline{\mathbb{Q}(p)}^{\,n}$. In addition we have

$$\begin{aligned} f(\Theta(z)) &= [f_1(\Theta(z)), \ldots, f_m(\Theta(z))] = [\theta(f_1(z)), \ldots, \theta(f_m(z))] \\ &= [\theta(f_1(p)), \ldots, \theta(f_m(p))] = [f_1(\Theta(p)), \ldots, f_m(\Theta(p))] \\ &= [f_1(q), \ldots, f_m(q)] = f(q) \end{aligned}$$

so $\Theta(z) \in W(q)$. Since Θ is a bijection, this implies that $|W(p)| \leq |W(q)|$. By symmetry we also have $|W(q)| \leq |W(p)|$ and hence $|W(p)| = |W(q)|$. •

Lemma 4.5 *Let $X_1, X_2, \ldots, X_n$ and $D_1, D_2, \ldots, D_t$ be indeterminates and let $f_i \in K[X_1, X_2, \ldots, X_n, D_1, D_2, \ldots, D_t]$ for all $1 \leq i \leq m$, for some field K with $\mathbb{Q} \subseteq K \subseteq \mathbb{C}$. For each $d \in \mathbb{C}^t$ let $V_d = \{x \in \mathbb{C}^n : f_i(x, d) = 0$ for all $1 \leq i \leq m\}$. Then $V_d \neq \emptyset$ for some $d \in \mathbb{C}^t$ with $\mathrm{td}[K(d) : K] = t$ if and only if $V_d \neq \emptyset$ for all $d \in \mathbb{C}^t$ with $\mathrm{td}[K(d) : K] = t$.*

Proof. Let I be the ideal of $K(D)[X]$ generated by $\{f_i(X, D) : 1 \leq i \leq m\}$. For each $d \in \mathbb{C}^t$ with $\mathrm{td}[K(d) : K] = t$ let I_d be the ideal of $K(d)[X]$ generated by $\{f_i(X, d) : 1 \leq i \leq m\}$. There is an isomorphism from $K(D)(X)$ to $K(d)(X)$ which maps I onto I_d. Furthermore, Hilbert's Weak Nullstellensatz, see [6], tells us that $V_d \neq \emptyset$ if and only if I_d contains a non-zero element of $K(d)$. We may use the above isomorphism to deduce that $V_d \neq \emptyset$ if and only if I contains a non-zero element of $K(D)$. The lemma now follows since the latter condition is independent of the choice of d. •

5 Generic frameworks

Let $G = (V, E)$ be a graph and (G, p) be a complex realisation of G. Let $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{e_1, e_2, \ldots, e_m\}$. We view p as a point $p = (p(v_1), p(v_2), \ldots, p(v_n))$ in $\mathbb{C}^{2n}$. The *rigidity map* $d_G : \mathbb{C}^{2n} \to \mathbb{C}^m$ is given by $d_G(p) = (\ell(e_1), \ell(e_2), \ldots, \ell(e_m))$, where $\ell(e_i) = d(p(u) - p(v))$ when $e_i = uv$. Note that the evaluation of the Jacobian of the rigidity map at the point $p \in \mathbb{C}^{2n}$ is twice the rigidity matrix of the framework (G, p). We say that the framework (G, p) is *quasi-generic* if (G, p) is congruent to a generic framework.

Lemma 5.1 *Suppose that (G, p) is a quasi-generic complex realisation of a graph G. If the rows of the rigidity matrix of G are linearly independent then $d_G(p)$ is generic.*

Proof. Choose a generic framework (G, q) congruent to (G, p). Since the rows of the rigidity matrix of G are linearly independent, rank $d(d_G)|_q = |E|$. Hence Lemma 4.1 implies that $d_G(q)$ is generic. The lemma now follows since $d_G(p) = d_G(q)$. •

A graph $G = (V, E)$ is *isostatic* if it is rigid and has $|E| = 2|V| - 3$. Note that if (G, p) is a generic realisation of an isostatic graph then its rigidty matrix has linearly independent rows so $d_G(p)$ is generic by Lemma 5.1.

Our next result allows us to choose a canonical representative for each congruence class in the set of all realisations which are equivalent to a given generic realisation of a rigid graph.

Lemma 5.2 *Suppose that (G, p) is a generic complex realisation of a rigid graph $G = (V, E)$ where $V = \{v_1, v_2, \ldots, v_n\}$ and $n \geq 3$. Let (G, q) be a realisation of G which is equivalent to (G, p). Then $d(q(v_1) - q(v_2)) \neq 0$. Furthermore, if we choose $d_0 \in \mathbb{C}$ with $d(q(v_1) - q(v_2)) = d_0^2$ and* Arg $d_0 \in (0, \pi]$*, then there exists a unique realisation (G, q^*) of G which is congruent to (G, q), and satisfies $q^*(v_1) = (0, 0)$, $q^*(v_2) = (0, d_0)$, and $q^*(v_3) = (a_3, b_3)$ for some $a_3, b_3 \in \mathbb{C}$ with* Arg $a_3 \in (0, \pi]$.

Proof. Suppose $d(q(v_1) - q(v_2)) = 0$. By Lemma 3.3, there exists a realisation $(G, \tilde{q})$ of G which is congruent to (G, q) and has $\tilde{q}(v_1) = (0, 0)$. Since $d(\tilde{q}(v_1) - \tilde{q}(v_2)) = d(q(v_1) - q(v_2)) = 0$ we have $\tilde{q}(v_2) = (b, \pm ib)$ for some $b \in \mathbb{C}$. Let H be a spanning isostatic subgraph of G. Lemma 5.1 implies that $d_H(p)$ is generic. Since $d_H(p) = d_H(q) = d_H(\tilde{q})$, $\text{td}[\mathbb{Q}(d_H(\tilde{q})) : \mathbb{Q}] = 2n - 3$. Since $\mathbb{Q}(d_H(\tilde{q})) \subseteq \mathbb{Q}(\tilde{q})$ we have $\text{td}[\mathbb{Q}(\tilde{q}) : \mathbb{Q}] \geq 2n - 3$. Since $i = \sqrt{-1}$ is algebraic over $\mathbb{Q}$, this implies that the set of coordinates of the points $\tilde{q}(v_i)$, $3 \leq i \leq n$, is algebraically independent over $\mathbb{Q}$. In particular $\tilde{q}(v_3) \neq (x, \pm ix)$ for all $x \in \mathbb{C}$. Lemma 3.3, now gives us a realisation (G, q') of G which is congruent to $(G, \tilde{q})$, has $q'(v_1) = (0, 0)$ and $q'(v_3) = (0, b_3)$ for some $b_3 \in \mathbb{C}$. Furthermore

$$d(q'(v_2)) = d(q'(v_2) - q'(v_1)) = d(\tilde{q}(v_2) - \tilde{q}(v_1)) = d(\tilde{q}(v_2)) = 0$$

so $q'(v_2) = (c, \pm ic)$ for some $c \in \mathbb{C}$. This implies that $\mathrm{td}[\mathbb{Q}(q') : \mathbb{Q}] \leq 2n-4$, and contradicts the facts that $d_H(q) = d_H(q')$ and $\mathbb{Q}(d_H(q')) \subset \mathbb{Q}(q')$ so $\mathrm{td}[\mathbb{Q}(q') : \mathbb{Q}] \geq \mathrm{td}[\mathbb{Q}(d_H(q)) : \mathbb{Q}] = 2n-3$. Hence $d(q(v_1) - q(v_2)) \neq 0$.

Lemma 3.3 now implies that there exists a realisation (G, q^*) which satisfies the conditions in the lemma with the possible exception that $a_3 = 0$. This latter alternative cannot occur since

$$\mathrm{td}[\mathbb{Q}(q^*) : \mathbb{Q}] \geq \mathrm{td}[\mathbb{Q}(d_H(q^*) : \mathbb{Q}] = \mathrm{td}[\mathbb{Q}(d_H(q)) : \mathbb{Q}] = 2n-3.$$

It remains to show that (G, q^*) is unique. Choose $d_1, d_2 \in \mathbb{C}$ such that $d(q(v_1) - q(v_3)) = d_1$ and $d(q(v_2) - q(v_3)) = d_2$. Since (G, q) and (G, q^*) are congruent, we have $a_3^2 + b_3^2 = d_1$ and $a_3^2 + (b_3 - d_0)^2 = d_2$. These equations imply that b_3 and a_3^2 are uniquely determined by q. Since we also have $\mathrm{Arg}\ a_3 \in (0, \pi]$, $q^*(v_3) = (a_3, b_3)$ is uniquely determined by q.

By applying a similar argument as in the proceeding paragraph to v_i for all $4 \leq i \leq n$, we have $q^*(v_i) = (\pm a_i, b_i)$ for some fixed $a_i, b_i \in \mathbb{C}$ which are uniquely determined by q. Furthermore, the facts that (G, q^*) is congruent to (G, q) and $d(a_3 - a_i, b_3 - b_i) \neq d(a_3 + a_i, b_3 - b_i)$ imply that $q^*(v_i)$ is also uniquely determined by q. Hence (G, q^*) is unique. •

We say that a realisation (G, q^*) of a rigid graph G is in *canonical position* if it satisfies the conditions of Lemma 5.2. This lemma implies that each congruence class in the set of all realisations of G which are equivalent to a given generic realisation (G, p) has a unique representative in canonical position. By symmetry, each congruence class also contains unique representatives (G, q^*) with $q^*(v_1) = (0, 0)$, $q^*(v_2) = (0, d_0)$, $q^*(v_3) = (a_3, b_3)$ and either: $\mathrm{Arg}\ d_0 \in (0, \pi]$ and $\mathrm{Arg}\ a_3 \in (-\pi, 0]$; $\mathrm{Arg}\ d_0 \in (-\pi, 0]$ and $\mathrm{Arg}\ a_3 \in (0, \pi]$; $\mathrm{Arg}\ d_0 \in (-\pi, 0]$ and $\mathrm{Arg}\ a_3 \in (-\pi, 0]$. This gives:

Corollary 5.3 *Suppose that (G, p) is a generic complex realisation of a rigid graph $G = (V, E)$ where $V = \{v_1, v_2, \ldots, v_n\}$ and $n \geq 3$. Let S be the set of all equivalent realisations of G. Then each congruence class in S has exactly four realisations (G, q) with $q(v_1) = (0, 0)$ and $q(v_2) = (0, z)$ for some $z \in \mathbb{C}$. Moreover, exactly two of these realisations have* $\mathrm{Arg}\ z \in (0, \pi]$.

•

Our next two results show that if (G, q) is equivalent to a generic realisation (G, p) of a rigid graph and is in canonical position then the algebraic closures of $\mathbb{Q}(q)$ and $\mathbb{Q}(d_G(p))$ are the same.

Lemma 5.4 *Let (G,p) be a complex realisation of an isostatic graph $G = (V,E)$ with $V = \{v_1, v_2, \ldots, v_n\}$. Suppose that $p(v_1) = (0,0)$, $p(v_2) = (0,y_2)$, $p(v_i) = (x_i, y_i)$ for $3 \le i \le n$, and $d_G(p)$ is generic. Then $\overline{\mathbb{Q}(p)} = \overline{\mathbb{Q}(d_G(p))}$.*

Proof. Let $f : \mathbb{C}^{2n-3} \to \mathbb{C}^{2n-3}$ be defined by putting $f(z_1, z_2, \ldots, z_{2n-3})$ equal to $d_G(0,0,0,z_1,z_2,\ldots,z_{2n-3})$. Let $p' = (y_2, x_3, y_3, \ldots, x_n, y_n)$. Then $f(p') = d_G(p)$ is generic, $\mathbb{Q}(p) = \mathbb{Q}(p')$ and $\mathbb{Q}(d_G(p)) = \mathbb{Q}(f(p'))$. Lemma 4.2 now implies that $\overline{\mathbb{Q}(p)} = \overline{\mathbb{Q}(p')} = \overline{\mathbb{Q}(f(p'))} = \overline{\mathbb{Q}(d_G(p))}$. •

Lemma 5.5 *Let (G,p) be a quasi-generic complex realisation of a rigid graph $G = (V,E)$ with $V = \{v_1, v_2, \ldots, v_n\}$ and $n \ge 3$. Suppose that $p(v_1) = (0,0)$ and $p(v_2) = (0,y)$ for some $y \in \mathbb{C}$. Let (G,q) be another realisation of G which is equivalent to (G,p) and has $q(v_1) = (0,0)$ and $q(v_2) = (0,z)$ for some $z \in \mathbb{C}$. Then $\overline{\mathbb{Q}(p)} = \overline{\mathbb{Q}(q)} = \overline{\mathbb{Q}(d_G(p))}$ and $\mathrm{td}[\mathbb{Q}(q) : \mathbb{Q}] = 2n-3$.*

Proof. Choose a spanning isostatic subgraph H of G. Lemma 5.1 implies that $d_H(p) = d_H(q)$ is generic. Lemma 5.4 now gives $\overline{\mathbb{Q}(p)} = \overline{\mathbb{Q}(d_H(p))} = \overline{\mathbb{Q}(q)}$ and $\mathrm{td}[\mathbb{Q}(q) : \mathbb{Q}] = \mathrm{td}[\mathbb{Q}(d_H(p) : \mathbb{Q}] = 2n-3$. •

Lemma 5.5 implies that $\mathrm{td}[\mathbb{Q}(d_G(p) : \mathbb{Q}] = 2|V| - 3$ for any generic realisation (G,p) of a rigid graph G. Our next result extends this to all graphs. Given a graph G we use $\mathrm{rank}(G)$ to denote the rank of the rigidity matrix of a generic realisation of G. A *rigid component* of G is a maximal rigid subgraph of G. It is known that the edge-sets of the rigid components $H_1, H_2, \ldots, H_t$ of G partition $E(G)$ and that $\mathrm{rank}(G) = \sum_{i=1}^{t}(2|V(H_i)|-3)$, see for example [11].

Lemma 5.6 *Let (G,p) be a quasi-generic complex realisation of a graph G. Then $\mathrm{td}[\mathbb{Q}(d_G(p)) : \mathbb{Q}] = \mathrm{rank}(G)$.*

Proof. Let $H_1, H_2, \ldots, H_t$ be the rigid components of G. By Lemma 5.5, $\mathrm{td}[\mathbb{Q}(d_{H_i}(p|_{V(H_i)})) : \mathbb{Q}] = 2|V(H_i)| - 3$ for all $1 \le i \le t$. Thus

$$\mathrm{td}[\mathbb{Q}(d_G(p)) : \mathbb{Q}] \le \sum_{i=1}^{t} \mathrm{td}[\mathbb{Q}(d_{H_i}(p|_{V(H_i)})) : \mathbb{Q}] = \sum_{i=1}^{t}(2|V(H_i)|-3) = \mathrm{rank}(G).$$

On the other hand, we may apply Lemma 5.1 to a spanning subgraph F of G whose edge set corresponds to a maximal set of linearly independent rows of the rigidity matrix of (G,p) to deduce that $\mathrm{td}[\mathbb{Q}(d_G(p)) : \mathbb{Q}] \geq \mathrm{td}[\mathbb{Q}(d_F(p)) : \mathbb{Q}] = \mathrm{rank}(G)$. Thus $\mathrm{td}[\mathbb{Q}(d_G(p)) : \mathbb{Q}] = \mathrm{rank}(G)$. •

We close this section by showing that the number of pairwise non-congruent realisations of a rigid graph G which are equivalent to a given generic realisation is the same for all generic realisations.

Theorem 5.7 *Suppose (G,p) is a generic complex realisation of a rigid graph $G = (V,E)$. Let S be the set of all equivalent realisations of G. Then the number of congruence classes in S is finite. Furthermore, this number is the same for all generic realisations of G.*

Proof. Let $V = \{v_1, v_2, \ldots, v_n\}$ and let (G,q) be another generic realisation of G. Let (G,p^*) and (G,q^*) be realisations in canonical position which are congruent to (G,p) and (G,q) respectively. Let $p^*(v_i) = (x_i, y_i)$ and $q^*(v_i) = (x'_i, y'_i)$ for $1 \leq i \leq n$. Let $\tilde{p} = (y_2, x_3, y_3, \ldots, x_n, y_n)$ and $\tilde{q} = (y'_2, x'_3, y'_3, \ldots, x'_n, y'_n)$. Then $\tilde{p}$ and $\tilde{q}$ are generic by Lemma 5.5.

Let $f : \mathbb{C}^{2n-3} \to \mathbb{C}^{2n-3}$ be defined by putting $f(z_1, z_2, \ldots, z_{2n-3})$ equal to $d_G(0,0,0,z_1,z_2,\ldots,z_{2n-3})$. Then rank $df|_{\tilde{p}} =$ rank $d(d_G)|_p = 2n-3$ since (G,p) is infinitesimally rigid (and hence the only vector in the null space of $R(G,p^*)$ which has a zero in its first three components is the zero vector). For each $y \in \mathbb{C}^n$ let $W(y) = \{z \in \mathbb{C}^n : f(z) = f(y)\}$. By Lemma 4.3, $W(\tilde{p})$ is finite. Since $W(\tilde{p}) = 4c(G,p)$ by Corollary 5.3, $c(G,p)$ is finite. Since $|W(\tilde{q})| = |W(\tilde{q})|$ by Lemma 4.4, $c(G,q)$ is also finite and $c(G,p) = c(G,q)$. •

As mentioned in the Introduction, we denote the common value of $c(G,p)$ over all generic realisations of G by $c(G)$.

6 Henneberg moves

We consider the effect of Henneberg moves on the number of equivalent complex realisations of a rigid graph. The *type 1 Henneberg move* on a graph H adds a new vertex v and two new edges vx, vy from v to distinct vertices x, y of H. The *type 2 Henneberg move* deletes an edge xy from H and adds a new vertex v and three new edges vx, vy, vz from v to x, y and another vertex z of H distinct from x, y.

We first consider type 1 moves.

Lemma 6.1 *Let $G = (V, E)$ be a rigid graph with at least four vertices, $v_n \in V$ with $N(v_n) = \{v_1, v_2\}$, and $H = G - v_n$. Then $c(G) = 2c(H)$.*

Proof. Let (G, p) be a generic realisation of G and $v_3 \in V \setminus \{v_1, v_2, v_n\}$. Let S be the set of all realisations (G, q) which are equivalent to (G, p) and satisfy $q(v_1) = (0, 0)$, $q(v_2) = (0, b_2)$ and $q(v_3) = (a_3, b_3)$ for some $b_2, a_3, b_3 \in \mathbb{C}$ with $\text{Arg } b_2, \text{Arg } a_3 \in (0, \pi]$. Similarly, let S^* be the set of all realisations (H, q^*) which are equivalent to $(H, p|_{V-v_n})$ and satisfy $q^*(v_1) = (0, 0)$, $q^*(v_2) = (0, b_2)$ and $q^*(v_3) = (a_3, b_3)$ for some $b_2, a_3, b_3 \in \mathbb{C}$ with $\text{Arg } b_2, \text{Arg } a_3 \in (0, \pi]$. By Lemma 5.2, $|S| = c(G)$ and $|S^*| = c(H)$.

Let $\theta : S \to S^*$ defined by $\theta(G, q) = (H, q|_{V-v_n})$ for all $(G, q) \in S$. For each $(H, q^*) \in S^*$ we can construct a $(G, q) \in S$ with $\theta(G, q) = (G, q^*)$ by putting $q(v_i) = (a_i, b_i) = q^*(v)$ for all $v_i \in V - v_n$ and $q(v_n) = (a_n, b_n)$ where $a_n^2 + b_n^2 = d(p(v_n) - p(v_1))$ and $a_n^2 + (b_n - b_2)^2 = d(p(v_n) - p(v_2))$. Since this system of equations has exactly two solutions, each $(H, q^*) \in S^*$ has exactly two pre-images in S. Hence $c(G) = |S| = 2|S^*| = 2c(H)$. •

We next consider type 2 moves. We need the following result which is an extension of [12, Lemma 4.1] to complex frameworks. Its proof uses ideas from simplified versions of the proof of [12, Lemma 4.1] given in [16, 22].

Lemma 6.2 *Let (G, p) be a quasi-generic complex framework and $v_n \in V$ with $N(v_n) = \{v_1, v_2, v_3\}$. Suppose that (G, q) is a complex realisation of G which is equivalent to (G, p). If $G - v_n$ is rigid then $d(p(v_i) - p(v_j)) = d(q(v_i) - q(v_j))$ for all $1 \leq i < j \leq 3$.*

Proof. By symmetry we need only show that $d(p(v_1) - p(v_2)) = d(q(v_1) - q(v_2))$. Label the vertices of G as $v_1, \ldots, v_n$ and put $p(v_i) = p_i = (p_{i,1}, p_{i,2})$ and $q(v_i) = q_i = (q_{i,1}, q_{i,2})$ for all $1 \leq i \leq n$. Since $G - v_n$ is rigid and $d(v_n) = 3$, G is rigid. By applying Lemma 5.2 to both (G, p) and (G, q), we may suppose that $p_{1,1} = p_{1,2} = p_{2,2} = 0$ and $q_{1,1} = q_{1,2} = q_{2,2} = 0$. [3] Then

$$d(p_1 - p_2) - d(q_1 - q_2) = p_{2,1}^2 - q_{2,1}^2$$

so it will suffice to show that $p_{2,1}^2 - q_{2,1}^2 = 0$.

Let $p' = p|_{V-v_n}$, $q' = q|_{V-v_n}$, $K = \mathbb{Q}(p')$ and $L = \mathbb{Q}(q')$. Consider the equivalent frameworks $(G - v_n, p')$ and $(G - v_n, q')$. Applying Lemma 5.5 to

[3] We have switched the order of the coordinate axes from that given in Lemma 5.2 since it makes the remainder of the proof more straightforward.

$G - v_n$, we have $\overline{K} = \overline{L}$. Thus $q_{2,1}, q_{3,1}, q_{3,2} \in \overline{K}$. Since (G, q) is equivalent to (G, p), we have the following equations.

$$
\begin{aligned}
q_{n,1}^2 + q_{n,2}^2 &= p_{n,1}^2 + p_{n,2}^2 & (1)\\
(q_{n,1} - q_{2,1})^2 + q_{n,2}^2 &= (p_{n,1} - p_{2,1})^2 + p_{n,2}^2 & (2)\\
(q_{n,1} - q_{3,1})^2 + (q_{n,2} - q_{3,2})^2 &= (p_{n,1} - p_{3,1})^2 + (p_{n,2} - p_{3,2})^2 & (3)
\end{aligned}
$$

Subtracting (1) from (2) and (3) we obtain

$$
\begin{aligned}
q_{n,1} &= \frac{p_{2,1}}{q_{2,1}} p_{n,1} + \frac{q_{2,1}^2 - p_{2,1}^2}{2q_{2,1}} & (4)\\
q_{n,2} &= \frac{p_{3,1}}{q_{3,2}} p_{n,1} + \frac{p_{3,2}}{q_{3,2}} p_{n,2} - \frac{q_{3,1}}{q_{3,2}} q_{n,1} + \frac{q_{3,1}^2 - p_{3,1}^2 + q_{3,2}^2 - p_{3,2}^2}{2q_{3,2}} & (5)
\end{aligned}
$$

We may use (4) to eliminate $q_{n,1}$ from the right hand side of (5) to obtain a matrix equation for q_n of the form

$$q_n = Ap_n + b \qquad (6)$$

where A is a 2×2 lower triangular matrix with entries in $\overline{K}$ and $b \in \overline{K}^2$. Rewriting (1) as $q_n^T q_n = p_n^T p_n$ and then substituting for q_n using (6) we obtain

$$p_n^T (A^T A - I) p_n + 2b^T A p_n + b^T b = 0. \qquad (7)$$

This is a polynomial equation for the components of p_n with coefficients in $\overline{K}$. Since $\text{td}[\mathbb{Q}(p) : \mathbb{Q}] = 2n - 3$ by Lemmas 5.1 and 5.4, $\{p_{n,1}, p_{n,2}\}$ is algebraically independent over $\overline{K}$. Thus all coefficients of $p_{n,1}, p_{n,2}$ in (7) must be zero. In particular $A^T A = I$ and, since A is lower triangular, A must be a diagonal matrix with ± 1 entries on the diagonal. In particular $a_{1,1} = p_{2,1}/q_{2,1} = \pm 1$ and hence $p_{2,1}^2 - q_{2,1}^2 = 0$. •

Lemma 6.3 *Let $G = (V, E)$ be a rigid graph, $v_n \in V$ with $N(v_n) = \{v_1, v_2, v_3\}$, and $H = (G - v_n) \cup \{e_1, e_2, e_3\}$ where $e_1 = v_1 v_2$, $e_2 = v_2 v_3$ and $e_3 = v_1 v_3$. Suppose that $G - v_n$ is rigid. Then $c(G) = c(H)$.*

Proof. Let (G, p) be a generic realisation of G and (G, p') be a realisation which is congruent to (G, p) and in canonical position. Let S be the set of all realisations (G, q) which are equivalent to (G, p) and in canonical position.

Similarly, let S^* be the set of all realisations (H, q^*) which are equivalent to $(H, p|_{V-v_n})$ and in canonical position. By Lemma 5.2, $|S| = c(G)$ and $|S^*| = c(H)$.

Let F be a complete graph with vertex set $\{v_1, v_2, v_3, v_n\}$. Then Lemma 6.2 implies that $(F, q|_{V(F)})$ is congruent to $(F, p'|_{V(F)})$ for all $(G, q) \in S$. Lemma 5.2 now gives $q(v_i) = p'(v_i)$ for all $i \in \{1, 2, 3, n\}$ and all $(G, q) \in S$. We may use a similar argument to deduce that $q^*(v_i) = p'(v_i)$ for all $i \in \{1, 2, 3\}$ and all $(H, q^*) \in S^*$. This implies that the map $\theta : S \to S^*$ defined by $\theta(G, q) = (H, q|_{V-v_n})$ for all $(G, q) \in S$ is a bijection. Hence $c(G) = |S| = |S^*| = c(H)$. •

Corollary 6.4 *Let $G = (V, E)$ be a rigid graph, $v_n \in V$ with $N(v_n) = \{v_1, v_2, v_3\}$, and $H = (G - v_n) + e_1$ where $e_1 = v_1v_2$. Suppose that $G - v_n$ is rigid. Then $c(G) \leq c(H)$.*

Proof. By Lemma 6.3, $c(G) = c(H \cup \{e_2, e_3\}) \leq c(H)$, where $e_2 = v_2v_3$ and $e_3 = v_1v_3$. •

An edge e in a rigid graph G is *redundant* if $G - e$ is rigid. Corollary 6.4 tells us that if we extend a graph H by performing a Henneberg type 2 move on a redundant edge of H then we do not increase $c(H)$. It is an open problem to determine the effect that performing a Henneberg type 2 move on a non-redundant edge has on $c(H)$.

Problem 6.5 *Do there exist universal constants $k_1, k_2 > 0$ such that if $H = (V, E)$ is a rigid graph, $u, v, w \in V$, $e = uv \in E$ such that $H - e$ is not rigid, and G is obtained from $H - e$ by adding a new vertex x and new edges xu, xv, xw, then $k_1c(H) \leq c(G) \leq k_2c(H)$?*

7 Uniquely realisable graphs and globally linked pairs of vertices

We first use Corollary 6.4 to characterise graphs G with $c(G) = 1$. Our characterization is the same as that given in [11] for *globally rigid graphs* i.e. graphs G with the property that $r(G, p) = 1$ for all generic real realisations (G, p).

Theorem 7.1 *Let $G = (V, E)$ be a graph with at least four vertices. Then $c(G) = 1$ if and only if G is 3-connected and redundantly rigid.*

Proof. Necessity was proved for real (and hence also for complex) generic realisations in [9]. We prove sufficiency by induction on $|V| + |E|$. If G has four vertices then $G = K_4$ and $c(G) = 1$ since G is complete. Hence suppose that $|V| \geq 5$. If $G - e$ is 3-connected and redundantly rigid for some $e \in E$, then $c(G - e) = 1$ by induction, and hence $c(G) = 1$. Thus we may suppose that $G - e$ is not both 3-connected and redundantly rigid. By [11, Theorem 6.1] there exists a vertex $v_n \in V$ with $N(v) = \{v_1, v_2, v_3\}$ such that $H = G - v_n + v_1v_2$ is 3-connected and redundantly rigid. This implies in particular that $G - v_n$ is rigid. Induction and Corollary 6.4 now give $c(G) \leq c(H) = 1$. •

Let (G, p) be a complex realisation of a rigid graph $G = (V, E)$ and $u, v \in V$. We say that $\{u, v\}$ is *globally linked* in (G, p) if every equivalent complex realisation (G, q) of G has $d(p(u) - p(v)) = d(q(u) - q(v))$. It can be seen that u, v is globally linked in (G, p) if and only if $c(G, p) = c(G + e, p)$, where $e = uv$. Theorem 5.7 now implies that the property of being globally linked is a *generic property* i.e. if $\{u, v\}$ is globally linked in some generic complex realisation of G then $\{u, v\}$ is globally linked in all such realisations. We say that $\{u, v\}$ *is globally linked in* G if $\{u, v\}$ is globally linked in some/all generic complex realisations of G.

The analogous concept for real realisations was introduced in [12]. (The situation for generic real realisations is more complicated as it is not necessarily true that if $\{u, v\}$ is globally linked in some generic real realisation of G then $\{u, v\}$ is globally linked in all generic real realisations. For example the pair u, v is globally linked in the real realisation in Figure 1, but not in Figure 2. The authors get round this problem by defining $\{u, v\}$ *to be globally linked in G in* $\mathbb{R}^2$ if $\{u, v\}$ is globally linked in *all* generic real realisations of G.)

Our next result is analogous to a result for real realisations given in [12, Theorem 4.2].

Theorem 7.2 *Let (G, p) be a generic complex realisation of a graph $G = (V, E)$ and $u, v, v_1, v_2, v_3, v_n \in V$ with $N(v_n) = \{v_1, v_2, v_3\}$ and $v_n \neq u, v$. Let $H = G - v_n + v_1v_2$. Suppose that $G - v_n$ is rigid and that $\{u, v\}$ is globally linked in $(H, p|_{V - v_n})$. Then $\{u, v\}$ is globally linked in (G, p).*

Proof. Suppose (G, q) is equivalent to (G, p). Let $p^* = p|_{V-v_n}$ and $q^* = q|_{V-v_n}$. Since $G - v_n = H - v_1v_2$ is rigid, Lemma 6.2 implies that $d(p(v_1) - p(v_2)) = d(q(v_1) - q(v_2))$. Hence (H, p^*) and (H, q^*) are equivalent. Since $\{u, v\}$ is globally linked in (H, p^*), we have

$$d(p(u) - p(v)) = d(p^*(u) - p^*(v)) = d(q^*(u) - q^*(v)) = d(q(u) - q(v)).$$

Thus $\{u, v\}$ is globally linked in (G, p). •

The real analogue of Theorem 7.2 was used in [12, Section 5] to characterize when two vertices in a generic real realisation of an '$\mathcal{RM}$-connected graph' are globally linked in $\mathbb{R}^2$. We can show that the same characterization holds for complex realisations. We first need to introduce some new terminology.

A *matroid* $\mathcal{M} = (E, \mathcal{I})$, consists of a set E together with a family $\mathcal{I}$ of subsets of E, called *independent sets*, which satisfy three simple axioms which capture the properties of linear independence in vector spaces, see [19]. Given a complex realisation (G, p) of a graph $G = (V, E)$, its *rigidity matroid* $\mathcal{R}(G, p) = (E, \mathcal{I})$ is defined by taking $\mathcal{I}$ to be the family of all subsets of E which correspond to linearly independent sets of rows in the rigidity matrix of (G, p). It is not difficult to see that the set of independent subsets of E is the same for all generic complex realisations of G. We refer to the resulting matroid as the *rigidity matroid of* G and denote it by $\mathcal{R}(G)$.

Given a *matroid* $\mathcal{M} = (E, \mathcal{I})$ we may define an equivalence relation on E by saying that $e, f \in E$ are related if $e = f$ or if there is a *circuit*, i.e. minimal dependent set, C of $\mathcal{M}$ with $e, f \in C$. The equivalence classes are called the *components* of $\mathcal{M}$. If $\mathcal{M}$ has at least two elements and only one component then $\mathcal{M}$ is said to be *connected*. We say that a graph $G = (V, E)$ is *$\mathcal{RM}$-connected* if its rigidity matroid $\mathcal{R}(G)$ is connected. The *$\mathcal{RM}$-components* of G are the subgraphs of G induced by the components of $\mathcal{R}(G)$. For more examples and basic properties of $\mathcal{RM}$-connected graphs see [11]. An efficient algorithm for constructing the $\mathcal{RM}$-components of a graph is given in [3].

Theorem 7.3 *Let $G = (V, E)$ be a an $\mathcal{RM}$-connected graph and $u, v \in V$. Then $\{u, v\}$ is globally linked in G if and only if u and v are joined by three internally disjoint paths in G.*

Proof. Necessity follows for real (and hence also complex) generic realisations by [12, Lemma 5.6]. Sufficiency follows by applying the same proof technique as for [12, Theorem 5.7] but using Theorem 7.2 in place of [12, Theorem 4.2] $\bullet$

The following conjecture is a complex version of [12, Conjecture 5.9]. It would characterise when two vertices in a rigid graph are globally linked.

Conjecture 7.4 *Let $G = (V, E)$ be a rigid graph and $u, v \in V$. Then $\{u, v\}$ is globally linked in G if and only if either $uv \in E$ or u and v are joined by three internally disjoint paths in some $\mathcal{RM}$-connected component of G.*

Note that the 'sufficiency part' of Conjecture 7.4 follows from Theorem 7.3.

8 Separable graphs

A *k-separation* of a graph $G = (V, E)$ is a pair (G_1, G_2) of edge-disjoint subgraphs of G each with at least $k + 1$ vertices such that $G = G_1 \cup G_2$ and $|V(G_1) \cap V(G_2)| = k$. If (G_1, G_2) is a k-separation of G, then we say that G *is k-separable* and that $V(G_1) \cap V(G_2)$ is a *k-separator* of G. We will obtain expressions for $c(G)$ when G is a rigid graph with a 2-separation, and also when G has a 3-separation induced by a 3-edge-cut.

Lemma 8.1 *Let (G_1, G_2) be a 2-separation of a rigid graph G with $V(G_1) \cap V(G_2) = \{v_1, v_2\}$ and let $H_i = G_i + e$ where $e = v_1v_2$ for $i = 1, 2$. Suppose that $\{v_1, v_2\}$ is globally linked in G. Then $c(G) = 2c(H_1)c(H_2)$.*

Proof. Let (G, p) be a generic realisation of G and choose $d_0 \in \mathbb{C}$ with $d(p(v_1) - p(v_2)) = d_0^2$ and $\mathrm{Arg}\, d_0 \in (0, \pi]$. Let S be the set of all realisations (G, q) which are equivalent to (G, p) and satisfy $q(v_1) = (0, 0)$ and $q(v_2) = (0, d_0)$. Corollary 5.3 and the hypothesis that $\{v_1, v_2\}$ is globally linked in G imply that $|S| = 2c(G)$.

For $i = 1, 2$, let S_i be the set of all realisations (H_i, q_i) which are equivalent to $(H_i, p|_{V(H_i)})$ and satisfy $q_i(v_1) = (0, 0)$ and $q_i(v_2) = (0, d_0)$. Corollary 5.3 and the fact that $v_1v_2 \in E(H_i)$ imply that $|S_i| = 2c(H_i)$. It is straightforward to check that the map $\theta : S \to S_1 \times S_2$ defined by $\theta(G, q) = [(H_1, q|_{V(H_1)}), (H_2, q|_{V(H_2)})]$ is a bijection. Hence $2c(G) = |S| = |S_1| \times |S_2| = 4c(H_1)c(H_2)$. $\bullet$

We next show that we can apply Lemma 8.1 when G has a 2-separation (G_1, G_2) in which G_1 and G_2 are both rigid. We need one more piece of matroid terminology. An $\mathcal{RM}$-*circuit* in a graph G is a subgraph H such that $E(H)$ is a circuit in the rigidity matroid of G.

Lemma 8.2 *Let (G_1, G_2) be a 2-separation of a rigid graph G with $V(G_1) \cap V(G_2) = \{v_1, v_2\}$ and let $H_i = G_i + e$ where $e = v_1v_2$ for $i = 1, 2$. Suppose that G_1 and G_2 are both rigid. Then $\{u, v\}$ is globally linked in G and $c(G) = 2c(H_1)c(H_2)$.*

Proof. We first show that $\{v_1, v_2\}$ is globally linked in G. This holds trivially if $e \in E(G)$ and hence we may suppose that $e \notin E(G)$. Since G_i is rigid, e_i is contained in an $\mathcal{RM}$-circuit C_i of H_i for each $i = 1, 2$. Then $C = (C_1 - e) \cup (C_2 - e)$ is an $\mathcal{RM}$-circuit of G by [2, Lemma 4.1]. We may now use Theorem 7.3 to deduce that $\{u, v\}$ is globally linked in C. Since $C \subseteq G$, $\{u, v\}$ is globally linked in G. The fact that $c(G) = 2c(H_1)c(H_2)$ now follows immediately from Lemma 8.1. •

We next consider 2-separations (G_1, G_2) in which G_1 and G_2 are not both rigid. We need some results concerning the number of realisations of a rigid graph satisfying given 'distance' constraints.

Lemma 8.3 *Let $G = (V, E)$ be a rigid graph with $V = \{v_1, v_2, \ldots, v_n\}$, $E = \{e_1, e_2, \ldots, e_m\}$ and $e_i = v_{i_1}v_{i_2}$ for all $1 \leq i \leq m$. Suppose that $T = \{e_1, e_2, \ldots, e_t\} \subseteq E$ is such that $\operatorname{rank}(G - T) = \operatorname{rank}(G) - t$. Let (G, p) be a generic realisation of G and $d^*_T = \{d^*_1, d^*_2, \ldots, d^*_t\} \subset \mathbb{C}$ be algebraically independent over $\mathbb{Q}(d_{G-T}(p))$. Then the number of pairwise non-congruent realisations (G, q) of G with $(G - T, q)$ equivalent to $(G - T, p)$ and $d(p(v_{i_1}) - p(v_{i_2})) = d^*_i$ for all $e_i \in T$ is $c(G)$.*

Proof. Let $K = \mathbb{Q}(d_{G-T}(p))$. We will define polynomials $f_i \in K[X, Y, D]$ for $1 \leq i \leq m$, where $X = (X_1, X_2, \ldots, X_n)$, $Y = (Y_1, Y_2, \ldots, Y_n)$, and $D = (D_1, D_2, \ldots, D_t)$ are indeterminates. We first associate two variables X_i, Y_i with each $v_i \in V$ and a variable D_i with each $e_i \in T$. We then put $f_i = (X_{i_1} - X_{i_2})^2 + (Y_{i_1} - Y_{i_2})^2 - D_i$ for each $e_i \in T$ and $f_i = (X_{i_1} - X_{i_2})^2 + (Y_{i_1} - Y_{i_2})^2 - d(p(v_{i_1}) - p(v_{i_2}))$ for each $e_i \in E \setminus T$.

We now apply Lemma 4.5. We need to find $x, y \in \mathbb{C}^n$ and $d \in \mathbb{C}^t$ such that $f_i(x, y, d) = 0$ for all $1 \leq i \leq m$, and $\operatorname{td}[K(d), K] = t$. This is easy since we can just put $(x_i, y_i) = p(v_i)$ for all $v_i \in V$ and $d_i = d(p(v_{i_1}) - p(v_{i_2}))$ for all $e_i \in T$, and use the definition of the polynomials f_i to deduce that $f_i(x, y, d) = 0$ for all $1 \le i \le m$. Since G is rigid $\mathrm{td}[\mathbb{Q}(d_G(p)), \mathbb{Q}] = 2n - 3$ and $\mathrm{td}[\mathbb{Q}(d_{G-T}(p)), \mathbb{Q}] = \mathrm{rank}(G - T) = 2n - 3 - t$ by Lemma 5.6. Since $\mathrm{td}[\mathbb{Q}(d_G(p)), \mathbb{Q}] = \mathrm{td}[K(d), K] + \mathrm{td}[K, \mathbb{Q}]$ we have $\mathrm{td}[K(d), K] = t$. Since we also have $\mathrm{td}[K(d_T^*), K] = t$, Lemma 4.5 implies that there exists a realisation (G, q) with $(G - T, q)$ equivalent to $(G - T, p)$ and $d(p(v_{i_1}) - p(v_{i_2})) = d_i^*$ for all $e_i \in T$.

We may assume that (G, q) is in canonical position. Then $\mathrm{td}[\mathbb{Q}(d_G(q)) : \mathbb{Q}] = \mathrm{td}[\mathbb{Q}(d_G(q)) : K] + \mathrm{td}[K : \mathbb{Q}] \ge \mathrm{td}[\mathbb{Q}(d_T^*) : K] + \mathrm{td}[K : \mathbb{Q}] \ge |T| + 2n - 3 - |T| = 2n - 3$. Since (G, q) is in canonical position and $\mathbb{Q}(d_G(q)) \subseteq \mathbb{Q}(q)$ we must have $\mathrm{td}[\mathbb{Q}(q) : \mathbb{Q}] = 2n - 3$. We may now use Lemma 3.1(a,b) to construct a generic framework which is congruent to (G, q). Hence (G, q) is quasi-generic and and the number of pairwise non-congruent realisations of G which are equivalent to (G, q) is $c(G)$. •

Our next result is needed to apply Lemma 8.3 to k-separations.

Lemma 8.4 *Let H_1, H_2 be rigid graphs. Put $H = H_1 \cup H_2$, $H_3 = H_1 \cap H_2$, and $T = E(H_3)$. Suppose that H_3 is isostatic and that* $\mathrm{rank}(H_2 - T) = \mathrm{rank}(H_2) - |T|$. *Let (H, p) be a quasi-generic realisation of H, G_1 be a spanning rigid subgraph of H_1, and (G_1, q_1) be a realisation of G_1 which is equivalent to $(G_1, p|_{V(G_1)})$. Then $d_T^* = \{d(q_1(u) - q_1(v)) : uv \in T\}$ is algebraically independent over $\mathbb{Q}(d_{H_2 - T}(p|_{V(H_2)}))$.*

Proof. If $T = \emptyset$ there is nothing to prove so we may suppose that $|T| \ge 1$ and hence $|V(H_3)| \ge 2$. We may also assume that (H, p) and (H_1, q_1) are both in canonical position with $p(u) = (0, 0) = q_1(u)$, $p(v) = (0, y)$ and $q_1(v) = (0, z)$ for some $y, z \in \mathbb{C}$ and some $u, v \in V(H_3)$.

Let F be a spanning isostatic subgraph of H which contains T and let $F_i = F \cap H_i$. Then

$$\begin{aligned} |E(F)| &= |E(F_1)| + |E(F_2)| - |T| \\ &\le (2|V(H_1)| - 3) + 2(|V(H_2)| - 3) - 2(|V(H_3)| - 3) \\ &= 2|V(H)| - 3. \end{aligned}$$

Equality must occur throughout and hence F_i is a spanning isostatic subgraph of H_i for $i = 1, 2$. Lemma 5.5 now implies that

$$\overline{\mathbb{Q}(d_{F_1}(q_1))} = \overline{\mathbb{Q}(q_1)} = \overline{\mathbb{Q}(d_{G_1}(q_1))} \tag{8}$$

and

$$\overline{\mathbb{Q}(d_{F_1}(p|_{V(F_1)}))} = \overline{\mathbb{Q}(p|_{V(F_1)})} = \overline{\mathbb{Q}(d_{H_1}(p|_{V(H_1)})))} = \overline{\mathbb{Q}(d_{G_1}(p|_{V(G_1)})))}. \quad (9)$$

Since (G_1, q_1) and $(G_1, p|_{G_1})$ are equivalent $d_{G_1}(q_1) = d_{G_1}(p|_{V(G_1)})$. Equations (8) and (9) now give $\overline{\mathbb{Q}(d_{F_1}(p|_{V(F_1)}))} = \overline{\mathbb{Q}(d_{F_1}(q_1))}$ and hence

$$\begin{aligned}
\overline{\mathbb{Q}(d_H(p))} &= \overline{\mathbb{Q}(d_{H_1}(p|_{V(H_1)}), d_{H_2-T}(p|_{V(H_2)}))} \\
&= \overline{\mathbb{Q}(d_{F_1}(p|_{V(F_1)}), d_{H_2-T}(p|_{V(H_2)}))} \\
&= \overline{\mathbb{Q}(d_{F_1}(q_1), d_{H_2-T}(p|_{V(H_2)}))}.
\end{aligned}$$

Thus

$$\text{td}[\mathbb{Q}(d_H(p)) : \mathbb{Q}] = \text{td}[\mathbb{Q}(d_{H_2-T}(p|_{V(H_2)}) : \mathbb{Q}] + \text{td}[\mathbb{Q}(d_{F_1}(q_1) : \mathbb{Q}(d_{H_2-T}(p|_{V(H_2)})].$$

By Lemma 5.6, $\text{td}[\mathbb{Q}(d_H(p)) : \mathbb{Q}] = \text{rank}(H) = 2|V(H)| - 3$ and

$$\text{td}[\mathbb{Q}(d_{H_2-T}(p|_{V(H_2)})) : \mathbb{Q}] = \text{rank}(H_2 - T) = 2|V(H_2)| - 3 - |T|.$$

Thus

$$\begin{aligned}
\text{td}[\mathbb{Q}(d_{F_1}(q_1) : \mathbb{Q}(d_{H_2-T}(p|_{V(H_2)})] &= 2|V(H)| - 3 - (2|V(H_2)| - 3 - |T|) \\
&= 2|V(F_1)| - 3 = |E(F_1)|.
\end{aligned}$$

Hence $d_{F_1}(q_1)$ is algebraically independent over $\mathbb{Q}(d_{H_2-T}(p|_{V(H_2)})$. Since $T \subseteq E(F_1)$, d_T^* is also algebraically independent over $\mathbb{Q}(d_{H_2-T}(p|_{V(H_2)})$. •

Lemma 8.5 *Let (G_1, G_2) be a 2-separation of a rigid graph G with $V(G_1) \cap V(G_2) = \{v_1, v_2\}$. Suppose that G_2 is not rigid and put $H_2 = G_2 + e$ where $e = v_1v_2$. Then G_1 and H_2 are both rigid and $c(G) = 2c(G_1)c(H_2)$.*

Proof. Let F be a spanning isostatic subgraph of G. We have $|E(F) \cap E(G_1)| \leq 2|V(G_1)| - 3$, and $|E(F) \cap E(G_2)| \leq 2|V(G_2)| - 4$ since G_2 is not rigid. Thus

$$\begin{aligned}
|E(F)| &= |E(F) \cap E(G_1)| + |E(F) \cap E(G_2)| \\
&\leq 2|V(G_1)| - 3 + 2|V(G_2)| - 4 = 2|V(F)| - 3.
\end{aligned}$$

Since F is rigid, we must have equality throughout. In particular $|E(F) \cap E(G_1)| = 2|V(G_1)| - 3$ so G_1 is rigid.

Consider the 2-separation (G_1, H_2) of $H = G+e$, and let F' be a spanning isostatic subgraph of H which contains e. Then $|E(F') \cap E(H_2)| \leq 2|V(H_2)| - 3$ and, since $e \in E(F')$, $|E(F') \cap E(G_1)| \leq 2|V(G_1)| - 4$. Thus

$$\begin{aligned} |E(F')| &= |E(F') \cap E(G_1)| + |E(F') \cap E(H_2)| \\ &\leq 2|V(G_1)| - 4 + 2|V(H_2)| - 3 = 2|V(F')| - 3. \end{aligned}$$

Since F' is rigid, we must have equality throughout. In particular $|E(F') \cap E(H_2)| = 2|V(H_2)| - 3$ so H_2 is rigid.

Let (G, p) be a generic realisation of G. For each $z \in \mathbb{C} \setminus \{0\}$ with Arg $z \in (0, \pi]$ let $S(z)$ be the set of all realisations (G, q) of G such that (G, q) is equivalent to (G, p), $q(v_1) = (0, 0)$ and $q(v_2) = (0, z)$. Define $S_1(z)$ and $S_2(z)$ similarly by replacing (G, p) by $(G_1, p|_{V(G_1)})$ and $(H_2, p|_{V(H_2)})$ respectively. Lemma 5.2 and Theorem 5.7 imply that $S(z)$, $S_1(z)$ and $S_2(z)$ are finite, and are non-empty for only finitely many values of z. In addition we have

$$2c(G) = \sum_{S(z) \neq \emptyset} |S(z)| \;\text{ and }\; 2c(G_1) = \sum_{S_1(z) \neq \emptyset} |S_1(z)|. \tag{10}$$

We will show that

$$|S(z)| = 2|S_1(z)|\, c(H_2) \tag{11}$$

for all $z \in \mathbb{C} \setminus \{0\}$ with Arg $z \in (0, \pi]$. If $S_1(z) = \emptyset$ then we must also have $S(z) = \emptyset$, since for any $(G, q) \in S(z)$ we would have $(G_1, q|_{V(G_1)}) \in S_1(z)$, so (11) holds trivially.

We next consider the case when $S_1(z) \neq \emptyset$. Choose $(G_1, q_1) \in S_1(z)$. We may apply Lemma 8.4 with $H = G + e$, $H_1 = G_1 + e$, $T = \{e\}$ and $d_T^* = \{d(q_1(v_1) - q_1(v_2))\}$ to deduce that d_T^* is algebraically independent over $\mathbb{Q}(d_{H_2}(p|_{V(H_2)}))$. We may then apply Lemma 8.3 (with $G = H_2$) and Corollary 5.3 to deduce that $|S_2(z)| = 2c(H_2)$. Since the map $\theta : S(z) \to S_1(z) \times S_2(z)$ by $\theta(G, q) = [(G_1, q|_{V(G_1)}), (H_2, q|_{V(G_2)})]$ is a bijection, we have

$$|S(z)| = |S_1(z)|\, |S_2(z)| = 2\, |S_1(z)|\, c(H_2).$$

Thus (11) also holds when $S_1(z) \neq \emptyset$.

Equation (11) and the fact that $c(H_2) \neq 0$ imply that $S_1(z) = \emptyset$ if and only if $S(z) = \emptyset$. We can now use equations (10) and (11) to deduce that

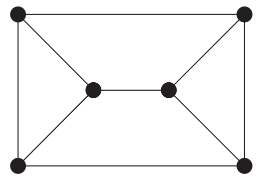

Figure 3: The triangular prism.

$$c(G) = \sum_{S(z)\neq\emptyset} |S(z)| = 2 \sum_{S_1(z)\neq\emptyset} |S_1(z)|\, c(H_2) = 2\, c(G_1)\, c(H_2).$$

•

Note that Lemma 6.1 is the special case of Lemma 8.5 when G_2 is a path of length two.

We next determine $c(G)$ when G has a 3-edge-cut. We first solve the case when G is the triangular prism[4] i.e. the graph on six vertices consisting of two disjoint triangles joined by a perfect matching, see Figure 3.

Lemma 8.6 *Let P be the triangular prism. Then $c(P) = 12$*

Proof. Label the vertices of P so that one triangle T_1 has vertices v_1, v_2, v_3, the second triangle T_2 has vertices v_4, v_5, v_6 and $v_1v_4, v_2v_5, v_3v_6 \in E(P)$. Let (P, p') be a generic realisation of P and (P, p) be the unique framework which is congruent to (P, p') and is in canonical position. Let (P, q) be a realisation which is equivalent to (P, p) and is also in canonical position. Since T_1 is globally rigid we have $q(v_i) = p(v_i)$ for $i = 1, 2, 3$. Let $p(v_i) = (x_i, y_i)$ for $1 \leq i \leq 6$, and $d_{i,j} = d(p(v_i) - p(v_j))$ for all $v_iv_j \in E(P)$. Since P is isostatic, the $d_{i,j}$ are algebraically independent over $\mathbb{Q}$.

Since T_2 is globally rigid, there is a unique realisation in canonical position $(T_2, \tilde{p})$ which is congruent to $(T_2, p|_{T_2})$. Let $\tilde{p}(v_i) = (\tilde{x}_i, \tilde{y}_i)$ for $4 \leq i \leq 6$. Note that $\tilde{x}_4 = \tilde{y}_4 = \tilde{x}_5 = 0$ and that the values of $\tilde{y}_5, \tilde{x}_6, \tilde{y}_6$ are uniquely determined by $d_{4,5}, d_{5,6}, d_{6,4}$. Since $(T, q|_T)$ is congruent to $(T, \tilde{p})$,we may use Lemmas 3.1 and 3.2 to deduce that there exist unique $a, b, u, v \in \mathbb{C}$ and $c \in \{1, -1\}$ such that

$$u^2 + v^2 = 1 \tag{12}$$

[4]This graph is referred to as the *doublet* in [17] and the *Desargues graph* in [4].

and

$$q(v_i) = (u\tilde{x}_i + v\tilde{y}_i + a, c(-v\tilde{x}_i + u\tilde{y}_i) + b) \tag{13}$$

for $i = 4, 5, 6$. Since (P, p) and (P, q) are congruent and $q(v_i) = p(v_i)$ for $1 \leq i \leq 3$, we also have

$$d(p(v_i) - q(v_{i+3})) = d_{i,i+3} \tag{14}$$

for $i = 1, 2, 3$. Equations (12), (13) and (14) determine all (P, q) in canonical position which are equivalent to (P, p).

Since we have $\tilde{x}_4 = \tilde{y}_4 = 0$ the equation from (13) with $i = 4$ implies that $q(v_4) = (a, b)$. The equation from (14) with $i = 1$ and the fact that $x_1 = x_2 = 0$ now give $a^2 + b^2 - d_{1,4} = 0$, which is independent of u, v and c. The equations from (14) with $i = 2, 3$ may be expanded (using $u^2 + v^2 = 1$) to give two linear equations of the form $\alpha_i u + \beta_i v + \gamma_i = 0$ where

$$\begin{aligned}
\alpha_i &= (x_i - a)\tilde{x}_{i+3} + c(y_i - b)\tilde{y}_{i+3}, \\
\beta_i &= (x_i - a)\tilde{y}_{i+3} - c(y_i - b)\tilde{x}_{i+3}, \\
2\gamma_i &= d_{i,i+3} - \tilde{x}_{i+3}^2 - \tilde{y}_{i+3}^2 - (x_i - a)^2 - (y_i - b)^2
\end{aligned}$$

for $i = 2, 3$. We may solve these two linear equations for u and v and then substitute into (14) to obtain

$$U(a, b, c)^2 + V(a, b, c)^2 = W(a, b, c)^2 \tag{15}$$

where $U = (\alpha_2\gamma_3 - \alpha_3\gamma_2)$, $V = (\beta_2\gamma_3 - \beta_3\gamma_2)$ and $W = (\alpha_2\beta_3 - \alpha_3\beta_2)$. Note that equation (15) is independent of u, v.

For each value of $c = \pm 1$, (15) describes a curve C in the (a, b)-plane. The intersections of these curves with the circle $a^2 + b^2 - d_{1,4} = 0$ determine the coordinates of all possible (P, q). Note that the polynomial (15) which defines the curve C is irreducible over $\mathbb{C}[a, b]$ since p is generic. (see for example [7]). This implies that the curve C and the circle do not have a common component.

The terms of (15) which have highest total degree in a and b may be collected into a single term $(a^2 + b^2)^3((\tilde{x}_5 - \tilde{x}_6)^2 + (\tilde{y}_5 - \tilde{y}_6)^2)$, which has a non-zero coefficient since $p|_T$ is generic and hence $\tilde{y}_5, \tilde{x}_6, \tilde{y}_6$ are algebraically independent over $\mathbb{Q}$. Thus (15) has degree 6. By Bezout's theorem, see for example [10, Section 7.3], there are 12 intersections of each degree 6 curve with the circle in $\mathbb{P}^2(\mathbb{C})$ provided that each intersection is counted with its correct multiplicity. The degree 6 curves each have two intersections $(1, i)$ and $(1, -i)$ at infinity with any circle and each of these intersections has multiplicity 3 because the term in C of highest total degree is a constant multiple of $(a^2 + b^2)^3$. Thus each of the two degree 6 curves has 6 affine intersections with the circle provided they are counted with the correct multiplicity. We will show that each of these affine intersections has multiplicity one.

Two curves in the (a, b)-plane intersect with multiplicity greater than one only if they have parallel tangents at an intersection. Let $C_a(a, b) = \partial C(a, b)/\partial a$ and $C_b(a, b) = \partial C(a, b)/\partial b$. Then C and the circle have an intersection with multiplicity greater than one only if there is an $(a, b) \in \mathbb{C}^2$ such that $C(a, b) = 0$, $a^2 + b^2 - d_{1,4} = 0$ and $aC_b(a, b) - bC_a(a, b) = 0$. The curve $C(a, b) = 0$ and the curve C' defined by $aC_b(a, b) - bC_a(a, b) = 0$ do not have a common factor over $\mathbb{C}$ because C is irreducible over $\mathbb{C}[a, b]$ and the total degree of C is greater than the total degree of C'. Thus C and C' intersect in a finite set of points S. Since the coefficients of C and C' belong to $\mathbb{Q}(y_2, x_3, y_3, \tilde{y}_5, \tilde{x}_6, \tilde{y}_6)$ they are contained in $K = \overline{\mathbb{Q}(d_{T_1}(p|_{T_1}), d_{T_2}(p|_{T_2}))}$. Hence the coordinates of the points in S also belong to K. Since $d_{1,4}$ is generic over K, the circle $a^2 + b^2 - d_{1,4} = 0$ does not pass through any of the points in S. We may conclude that there are six distinct affine intersection points on each of the two curves corresponding to $c = +1$ and $c = -1$. We denote these two curves by C^+ and C^-, respectively. It remains to show that none of the affine intersections of C^+ with the circle coincide with an intersecion of C^- with the circle.

Since both C^+ and C^- are irreducible they have no common component and they intersect in a finite set of points S'. We may now deduce that the circle $a^2 + b^2 - d_{1,4} = 0$ does not pass through any of the points S' by a similar argument to that given in the previous paragraph. •

Lemma 8.7 *Suppose that G is a rigid graph and $G = G_1 \cup G_2 \cup \{e_1, e_2, e_3\}$ where $V(G_1) \cap V(G_2) = \emptyset$, $e_i = u_i v_i$ for $1 \le i \le 3$, u_1, u_2, u_3 are distinct vertices of G_1, and v_1, v_2, v_3 are distinct vertices of G_2. Then G_1 and G_2 are rigid and $c(G) = 12\, c(G_1)\, c(G_2)$.*

Proof. Let F be a spanning isostatic subgraph of G. We have $|E(F) \cap E(G_1)| \le 2|V(G_1)| - 3$ and $|E(F) \cap E(G_2)| \le 2|V(G_2)| - 3$. Thus

$$\begin{aligned} |E(F)| &\le |E(F) \cap E(G_1)| + |E(F) \cap E(G_2)| + 3 \\ &\le 2|V(G_1)| - 3 + 2|V(G_2)| - 3 + 3 = 2|V(F)| - 3. \end{aligned}$$

Since F is rigid, we must have equality throughout. In particular $|E(F) \cap E(G_i)| = 2|V(G_i)| - 3$ so G_i is rigid for $i = 1, 2$.

Claim 1 *Let H_2 be obtained from G_2 by adding the vertices u_1, u_2, u_3 and edges $u_1u_2, u_2u_3, u_3u_1, u_1v_1, u_2v_2, u_3v_3$. Then $c(G) = c(G_1)\, c(H_2)$.*

Proof. Let (G, p) be a generic realisation of G. For each fixed $b_2, a_3, b_3 \in \mathbb{C}\setminus\{0\}$ with $\operatorname{Arg} b_2, \operatorname{Arg} a_3 \in (0, \pi]$ let $S(b_2, a_3, b_3)$ be the set of all realisations (G, q) of G such that (G, q) is equivalent to (G, p), $q(u_1) = (0, 0)$, $q(u_2) = (0, b_2)$ and $q(u_3) = (a_3, b_3)$. Define $S_1(b_2, a_3, b_3)$ and $S_2(b_2, a_3, b_3)$ similarly by replacing (G, p) by $(G_1, p|_{V(G_1)})$ and $(H_2, p|_{V(H_2)})$ respectively. Lemma 5.2 and Theorem 5.7 imply that $S(b_2, a_3, b_3)$, $S_1(b_2, a_3, b_3)$ and $S_2(b_2, a_3, b_3)$ are finite, and are non-empty for only finitely many values of b_2, a_3, b_3. In addition we have

$$c(G) = \sum_{S(b_2,a_3,b_3)\neq\emptyset} |S(b_2, a_3, b_3)| \ \text{ and } \ c(G_1) = \sum_{S_1(b_2,a_3,b_3)\neq\emptyset} |S_1(b_2, a_3, b_3)|. \tag{16}$$

We will show that

$$|S(b_2, a_3, b_3)| = |S_1(b_2, a_3, b_3)|\, c(H_2) \tag{17}$$

for all $b_2, a_3, b_3 \in \mathbb{C} \setminus \{0\}$ with $\operatorname{Arg} b_2, \operatorname{Arg} a_3 \in (0, \pi]$. If $S_1(b_2, a_3, b_3) = \emptyset$ then we must also have $S(b_2, a_3, b_3) = \emptyset$, since for any $(G, q) \in S(b_2, a_3, b_3)$ we would have $(G_1, q|_{V(G_1)}) \in S_1(b_2, a_3, b_3)$, so (17) holds trivially.

We next consider the case when $S_1(b_2, a_3, b_3) \neq \emptyset$. Choose $(G_1, q_1) \in S_1(b_2, a_3, b_3)$. Let $T = \{u_1u_2, u_2u_3, u_3u_1\}$ and $d^*_T = \{d(q_1(u_i) - q_1(u_j)) : u_iu_j \in T\}$. We may apply Lemma 8.4 with $(H, p) = (G \cup T, p)$ and $(H_1, q_1) = (G_1 \cup T, q_1)$ to deduce that d^*_T is algebraically independent over $\mathbb{Q}(d_{H_2-T}(p|_{V(H_2)}))$. We may then apply Lemma 8.3 (with $G = H_2$) to deduce that $|S_2(b_2, a_3, b_3)| = c(H_2)$. Since the map $\theta : S(b_2, a_3, b_3) \to S_1(b_2, a_3, b_3) \times S_2(b_2, a_3, b_3)$ by $\theta(G, q) = [(G_1, q|_{V(G_1)}), (G_2, q|_{V(G_2)})]$ is a bijection, we have

$$|S(b_2, a_3, b_3)| = |S_1(b_2, a_3, b_3)|\, |S_2(b_2, a_3, b_3)| = |S_1(b_2, a_3, b_3)|\, c(H_2).$$

Thus (17) also holds when $S_1(b_2, a_3, b_3) \neq \emptyset$.

Equation (17) and the fact that $c(H_2) \neq 0$ imply that $S_1(b_2, a_3, b_3) = \emptyset$ if and only if $S(b_2, a_3, b_3) = \emptyset$. We can now use equations (16) and (17) to deduce that

$$c(G) = \sum_{S(b_2,a_3,b_3)\neq\emptyset} |S(b_2,a_3,b_3)| = \sum_{S_1(b_2,a_3,b_3)\neq\emptyset} |S_1(b_2,a_3,b_3)|\, c(H_2) = c(G_1)c(H_2).$$

This completes the proof of Claim 1 •

We may apply the argument of Claim 1 to H_2 to deduce that $c(H_2) = c(G_2)c(P)$, where P is the triangular prism. Lemma 8.6 and Claim 1 now give $c(G) = 12\, c(G_1)\, c(G_2)$. •

9 Two families of graphs

We use the results from the previous section to determine $c(G)$ for two families of rigid graphs.

Quadratically solvable graphs

Let $G = (V, E)$ be an isostatic graph with $E = \{e_1, e_2, \ldots, e_m\}$ and $e_i = u_i v_i$ for $1 \leq i \leq m$. Then G is *quadratically solvable* if for all $d = (d_1, d_2, ..., d_m) \in \mathbb{C}^m$ such that $\{d_1, d_2, d_3, \ldots, d_m\}$ is algebraically independent over $\mathbb{Q}$, there exists a realisation (G, p) of G with $d(p(u_i) - p(v_i)) = d_i$ for all $1 \leq i \leq m$, in which $\mathbb{Q}(p)$ is contained in a quadratic extension of $\mathbb{Q}(d)$ i.e. there exists a sequence of field extensions $K_1 \subset K_2 \subset \ldots \subset K_m$ such that $K_1 = \mathbb{Q}(d)$, $K_m = \mathbb{Q}(p)$ and $K_{i+1} = K_i(x)$ for some $x^2 \in K_i$ for all $1 \leq i < m$. These graphs are important in the theory of equation solving in Computer Aided Design.

We may recursively construct an infinite family $\mathcal{QS}$ of quadratically solvable isostatic graphs as follows. We first put the complete graph on three vertices K_3 in $\mathcal{QS}$. Then, for any two graphs $G_1, G_2 \in \mathcal{QS}$, any two vertices u_1, v_1 in G_1, and any edge $e = u_2 v_2$ of G_2, we construct a new graph G by 'gluing' G_1 and $G_2 - e$ together along $u_1 = u_2$ and $v_1 = v_2$, and add G to $\mathcal{QS}$. The second author conjectured in [18] that an isostatic graph G is quadratically solvable if and only if it belongs to $\mathcal{QS}$. This conjecture was subsequently verified for isostatic planar graphs in [17].

Theorem 9.1 *Suppose $G \in \mathcal{QS}$. Then $c(G) = 2^{|V(G)|-3}$.*

Proof. We use induction on $|V(G)|$. If $|V(G)| = 3$ then $G = K_3$ and $c(G) = 1$. Hence we may assume that $|V(G)| > 3$. It follows from the recursive definition of $\mathcal{QS}$ that there exists a 2-separation (G_1, G_2) of G with $V(G_1) \cap V(G_2) = \{u, v\}$ and such that G_1 and $G_2 + uv$ both belong to $\mathcal{QS}$. The theorem now follows from Lemma 8.5 and induction. •

$\mathcal{RM}$-connected graphs

We will determine $c(G)$ when G is an $\mathcal{RM}$-connected graph. We need some new terminology. For each $\{u, v\} \subset V$, let $w_G(u, v)$ denote the number of connected components of $G - \{u, v\}$ and put $b(G) = \sum_{\{u,v\} \subset V} (w_G(u, v) - 1)$. Note that $w_G(u, v) - 1 = 0$ if $\{u, v\}$ is not a 2-separator of G, so we can assume that the summation in the definition of $b(G)$ is restricted to pairs $\{u, v\}$ which are 2-separators of G.

Theorem 9.2 *Let G be an $\mathcal{RM}$-connected graph. Then $c(G) = 2^{b(G)}$.*

Proof. We use induction on $b(G)$. Suppose $b(G) = 0$. Then G is 3-connected and, since G is $\mathcal{RM}$-connected, it is also redundantly rigid. Hence $c(G) = 1$ by Theorem 7.1. Thus we may assume that $b(G) \geq 1$.

Choose vertices u, v of G with $w_G(u, v) \geq 2$ and let (G_1, G_2) be a 2-separation in G with $V(G_1) \cap V(G_2) = \{u, v\}$. Let $H_i = G_i + uv$ for $i = 1, 2$. By [12, Lemma 5.3(b)], H_i is $\mathcal{RM}$-connected for $i = 1, 2$. In addition, [11, Lemma 3.6] implies that every 2-separator $\{u', v'\}$ of G which is distinct from $\{u, v\}$ is a 2-separator of H_i for exactly one value of $i \in \{1, 2\}$, and, for this value of i, satisfies $w_G(u', v') = w_{H_i}(u', v')$. Since we also have $w_G(u, v) = w_{H_1}(u, v) + w_{H_2}(u, v)$, we may deduce that $b(G) = b(H_1) + b(H_2) - 1$. Using induction and Lemma 8.1 we have

$$c(G) = 2\, c(H_1)\, c(H_2) = 2 \times 2^{b(H_1)} \times 2^{b(H_2)} = 2^{b(G)}.$$

•

Our expression for $c(G)$ in Theorem 9.2 is identical to that given for $r(G, p)$ in [12, Theorem 8.2] when (G, p) is a generic real realisation of G, and provides an explanation for the fact that $r(G, p)$ is the same for all generic real realisations (G, p) of an $\mathcal{RM}$-connected graph G.

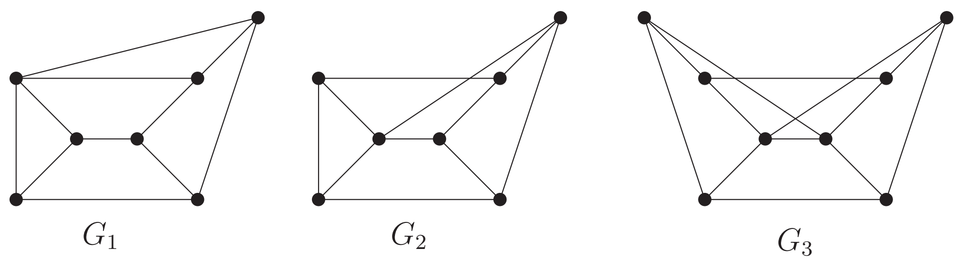

Figure 4: The graphs G_1, G_2 and G_3.

10 Open Problems

The obvious open problem is:

Problem 10.1 *Can $c(G)$ be determined efficiently for an arbitrary rigid graph G?*

Theorem 9.2 gives an affirmative answer to this problem when G is $\mathcal{RM}$-connected and the results of Section 8 allow us to reduce the problem to the case when G is 3-connected and all 3-edge-cuts of G are trivial i.e. consist of three edges incident to the same degree three vertex. On the other hand, the isostatic graphs G_1, G_2 and G_3 of Figure 4 indicate that it may be difficult to obtain an affirmative answer to Problem 10.1 for all graphs. Computer calculations indicate with high probability that $c(G_1) = 28$, $c(G_2) = 22$ and $c(G_3) = 45$, but we cannot see how these numbers could be deduced from the structures of G_1, G_2 and G_3.

If we cannot determine $c(G)$ precisely then we could ask for tight asymptotic upper bounds on $c(G)$.

Problem 10.2 *Determine the smallest $k \in \mathbb{R}$ such that $c(G) = O(k^n)$ for all rigid graphs G with n vertices.*

Clearly $c(G)$ will be maximised when G is isostatic, and hence it follows from [4, Theorem 1.1] that $c(G) \leq \frac{1}{2}\binom{2n-4}{n-2} \approx 4^n$ for all rigid graphs G with n vertices. Borcea and Streinu [4, Proposition 5.6] also construct an infinite family of isostatic graphs G with $c(G) = 12^{(n-3)/3} \approx 2.28^n$. (Such a family can be obtained recursively from a triangle by joining two disjoint graphs already in the family by three disjoint edges. The fact that $c(G) = 12^{(n-3)/3}$

for this family can be deduced from Lemma 8.7.) It follows that $12^{1/3} \leq k \leq 4$. It may be true that $12^{1/3}$ is the correct value of k, but the graph G_1 of Figure 4 shows that $12^{(n-3)/3}$ is (probably) not a universal upper bound on $c(G)$ since we (probably) have $c(G_1) = 28 > 12^{4/3}$.

It would also be of interest to determine a tight lower bound on $c(G)$ when G is isostatic.

Conjecture 10.3 *For all isostatic graphs G with n vertices, $c(G) \geq 2^{n-3}$.*

Lemma 6.1 shows that the equality $c(G) = 2^{n-3}$ holds when G is constructed from a triangle by type 1 Henneberg moves. Since every isostatic graph can be obtained from a triangle by type 1 or 2 Henneberg moves, it is tempting to try to prove Conjecture 10.3 by showing that if we perform a type 2 move on an isostatic graph G then we will increase $c(G)$ by at least a factor of two. Unfortunately this is (probably) not the case: the graph G_2 of Figure 4 can be obtained from the triangular prism P by a type 2 Henneberg move and we (probably) have $c(G_2) = 22 < 2c(P) = 24$.

We may also consider realisations (G, p) which are not generic.

Conjecture 10.4 *Suppose (G, p) is a realisation of an isostatic graph G and $c(G, p)$ is finite. Then $c(G, p) \leq c(G)$.*

This conjecture would be useful for obtaining lower bounds on $c(G)$.

It is not difficult to construct realisations which show that strict inequality can hold in Conjecture 10.4. For example label the vertices of K_4 as v_1, v_2, v_3, v_4, let $H = K_4 - v_3v_4$ and let G be obtained by adding a new vertex v_5 and two new edges v_5v_3, v_5v_4 to H. Then $G \in \mathcal{QS}$ so $c(G) = 4$ by Theorem 9.1. However the realisation (G, p) given by $p(v_1) = (0, 0)$, $p(v_2) = (0, 1)$, $p(v_3) = (1, 1)$, $p(v_4) = (-1, 1)$, and $p(v_5) = (2, 3)$ has $c(G, p) = 2$. This follows because every realisation (H, q) which is equivalent but not congruent to $(H, p|_H)$ has $q(v_3) = q(v_4)$ and hence cannot be extended to a realisation of G which is equivalent to (G, p) (because $d(p(v_5) - p(v_3)) \neq d(p(v_5) - p(v_4))$). Thus all realisations equivalent to (G, p) are extensions of $(H, p|_H)$ and there are exactly two ways to do this.

Note also that Conjecture 10.4 does not hold for all rigid graphs. For example, label the vertices of K_5 as v_1, v_2, v_3, v_4, v_5, and let $G = K_5 - v_4v_5$. Then $c(G) = 1$ because G is globally rigid. On the other hand, any realisation (G, p) with $p(v_1)$, $p(v_2)$ and $p(v_3)$ collinear has $c(G, p) \geq 2$ since we may obtain an equivalent but non-congruent realisation by reflecting $p(v_4)$ in the line joining $p(v_1)$, $p(v_2)$ and $p(v_3)$.

Our final problem was posed by Dylan Thurston at a workshop on global rigidity held at Cornell University in February 2011.

Problem 10.5 *Does every rigid graph G have a generic real realisation (G,p) such that $r(G,p) = c(G)$?*

The graph G_3 in Figure 4 suggests that the answer to this problem is most likely negative since the proof technique used by Hendrickson [9] to obtain necessary conditions for global rigidity can be adapted to show that $r(G,p)$ is even for all generic real realisations (G,p) of a graph G which is rigid but not globally rigid.[5] On the other hand, we (probably) have $c(G_3) = 45$ which is odd.

We may say a bit more about this parity argument. Let $G = (V,E)$ be a graph which is rigid but not globally rigid and S be the set of all realisations which are in canonical position and are equivalent to a given generic real realisation (G,p) of G. Since all edge lengths in (G,p) are real, the map $(G,q) \mapsto (G,q^*)$, where q^* is obtained by taking the complex conjugates of the coordinates of q and then, if necessary, reflecting the resulting framework in the axes to return to canonical position, is an involution on S.

Suppose (G,q^*) is equal to (G,q) and let $q(v_1) = (0,0)$, $q(v_2) = (0,y_2)$ and $q(v_3) = (x_3,y_3)$. Then $q^*(v_2) = (0,\pm\bar{y_2}) = (0,y_2)$. Hence y_2 is either real or imaginary. We first consider the case when y_2 is real. We have $q^*(v_3) = (\pm\bar{x_3},\bar{y_3}) = (x_3,y_3)$ so x_3 is either real or imaginary and y_3 is real. If x_3 is real then we have $q^*(v_j) = (\bar{x_j},\bar{y_j}) = (x_j,y_j)$ for all $v_j \in V$ so q is real. If x_3 is imaginary then $q^*(v_j) = (-\bar{x_j},\bar{y_j}) = (x_j,y_j)$ so $q(v_j) = (x_j,y_j)$ where x_j is imaginary and y_j is real for all $v_j \in V$. We next consider the the case when y_2 is imaginary. We have $q^*(v_3) = (\pm\bar{x_3},-\bar{y_3}) = (x_3,y_3)$ so x_3 is either real or imaginary and y_3 is imaginary. If x_3 is imaginary then we have $q^*(v_j) = (-\bar{x_j},-\bar{y_j}) = (x_j,y_j)$ for all $v_j \in V$ so q is imaginary. This is impossible since (G,q) is equivalent to (G,p) and so we must have

[5] Let S be the set of all real realisations which are in canonical position and are equivalent to (G,p). If G is not redundantly rigid then $G - e$ is not rigid for some edge e. In this case each component of the real configuration space of $(G - e, p)$ will contain an even number of elements of S. If G is redundantly rigid then, since G is not globally rigid, G has a 2-separation. In this case reflecting one of the sides of the 2-separation in the line through the two vertices of the corresponding 2-separator gives an involution on S with no fixed points.

$d(q(u) - q(v)) > 0$ for all $uv \in E$. If x_3 is real then $q^*(v_j) = (\bar{x_j}, -\bar{y_j}) = (x_j, y_j)$ so $q(v_j) = (x_j, y_j)$ where x_j is real and y_j is imaginary for all $v_j \in V$.

In summary (G, q^*) is equal to (G, q) if and only if q is real, or we have $q(v_j) = (x_j, iy_j)$ where $x_j, y_j \in \mathbb{R}$ for all $v_j \in V$, or we have $q(v_j) = (ix_j, y_j)$ where $x_j, y_j \in \mathbb{R}$ for all $v_j \in V$. We will refer to the latter two such realisations as *Minkowski realisations*.[6] It follows that the number of realisations in S which are neither real nor Minkowski must be even. As noted above, the number of real realisations is also even. Thus it is the number of Minkowski realisations which can be odd.

Although the answer to Problem 10.5 seems to be negative, it could still be of interest to find families of graphs for which the answer is positive. For example Theorem 9.2 and [12, Theorem 8.2] give a positive answer when G is $\mathcal{RM}$-connected, and indeed show that $r(G, p) = c(G)$ for *all* generic real realisations when G is $\mathcal{RM}$-connected. Lemma 6.1 and [4, Proposition 5.2] show that we also have a positive answer to Problem 10.5 when G can be obtained from a triangle by type 1 Henneberg moves.

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