Rigid Frameworks

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- We assume that the vertices are at **generic positions** in the plane. Intuitively this means that there are no 'special relationships' between the positions e.g. no three vertices lie on a line.

Rigidity

The framework (G, p) is **rigid** if every continuous motion of the vertices which preserves the lengths of the edges, must preserve the distance between ALL pairs of vertices.

FIGURE 1

FIGURE 2

FIGURE 3

FIGURE 4

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- We can restrict the motion of the framework further by fixing one or both of the coordinates of SOME of its vertices.
- The **degrees of freedom** of (G, p) is the minimum number of coordinates we have to fix so that the framework has no motion at all. We denote this number by $df(G, p)$.

Independent Edges

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- Let F be a set of some of the edges of (G, p). When we add any edge of F to the empty framework (G_0, p) we reduce the degrees of freedom from $2n$ to $2n - 1$. We say that F is **independent** if adding ALL edges of F to (G_0, p) reduces the degrees of freedom from $2n$ to $2n - |F|$, where $|F|$ denotes the number of edges in $|F|$.

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Observation (J.C. Maxwell 1864)

A framework with n vertices is rigid if and only if it contains $2n - 3$ independent edges.

Examples Revisited

FIGURE 1

FIGURE 2

FIGURE 3

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Laman's Theorem

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Theorem (Laman 1970)

A set of edges F is independent if and only if for all sets of vertices X, the number of edges of F which join the vertices in X is at most $2|X| - 3$.

FIGURE 5

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  If it is, put e in F;
  If it isn't, delete e and move on to another edge.

  Stop when all edges have been considered and output F.

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NOTE (G, p) is rigid if and only if $|F| = 2n - 3$. More generally

$$df(G, p) = 2n - |F|.$$

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- (G, p) is **redundantly rigid** i.e. $(G - e, p)$ is rigid for all edges e.

Hendrickson's conjecture was verified 13 years later by T. Jordán and myself.

Theorem (Jackson and Jordán, 2005)

A 2-dimensional framework with at least four vertices is globally rigid if and only if it is 3-connected and redundantly rigid.