Probability I – 2009/10

Solutions to Mid-Semester Test

Q1.

(i) We have $|A| = 3 \times 6 = 18$ and $|S| = 6 \times 6 = 36$. So $\mathbb{P}(A) = 18/36 = 1/2$. [5]

(ii) We have $|B| = 5 + 4 + 3 + 2 + 1 = 15$. So $\mathbb{P}(B) = 15/36 = 5/12$. [5]

(iii) We have $|A \cap B| = 5 + 3 + 1 = 9$. So $\mathbb{P}(A \cap B) = 9/36 = 1/4$. [5]

(iv) We have $\mathbb{P}(A \setminus B) = \mathbb{P}(A) - \mathbb{P}(A \cap B) = 1/2 - 1/4 = 1/4$. [5]

(v) We have $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) = 1/2 + 5/12 - 1/4 = 2/3$. [5]

Q2.

(i) Let A be the event that my selection begins with a vowel. We have $|A| = 2\times4\times3$ and $|S| = 5 \times 4 \times 3$. So $\mathbb{P}(A) = 2/5$. [10]

(ii) Let B be the event that my selection contains no vowels. We have $|B| = 3\times2\times1$. So $\mathbb{P}(B) = 6/60 \ = \ 1/10$. [5]

(iii) Let C be the event that my selection contains two vowels. We have $|C| = 3\times2\times3$. (There are three choices for the positions of the two vowels, two choices for the order in which these two vowels occur, and three choices for the remaining letter.) So $\mathbb{P}(C) = 18/60 \ = \ 3/10$. [10]

Note that since the events in (ii) and (iii) do not depend on the order the letters are chosen, we could also calculate their probabilities using unordered selection. Let S^* be the sample space of all unordered selections of three distinct letters from $\{a, b, c, d, e\}$, B^* be the unordered selections containing no vowels and C^* be the unordered selections containing two vowels. Then $|S| = \binom{5}{3} = 10$, $|B^*| = \binom{3}{3} = 1$, and $|C^*| = \binom{2}{2}\binom{3}{1} = 3$. So $\mathbb{P}(B^*) = |B^*|/|S^*| = 1/10$ and $\mathbb{P}(C^*) = |C^*|/|S^*| = 3/10$.

Q3. Let D, B, T, N, S be the events that the DLR, bus, train, Northern Line and District Line, respectively, are working. Let Q be the event that I can get to Queen Mary. Then

(i)

$$\begin{aligned}
\mathbb{P}(Q) &= \mathbb{P}([D \cap B] \cup [T \cap N \cap S]) \\
&= \mathbb{P}(D \cap B) + \mathbb{P}(T \cap N \cap S) - \mathbb{P}(D \cap B \cap T \cap N \cap S) \\
&= \mathbb{P}(D)\mathbb{P}(B) + \mathbb{P}(T)\mathbb{P}(N)\mathbb{P}(S) - \mathbb{P}(D)\mathbb{P}(B)\mathbb{P}(T)\mathbb{P}(N)\mathbb{P}(S) \\
&= \frac{4}{5} \times \frac{1}{2} + \frac{9}{10} \times \frac{2}{3} \times \frac{4}{5} - \frac{4}{5} \times \frac{1}{2} \times \frac{9}{10} \times \frac{2}{3} \times \frac{4}{5} \\
&= \frac{2}{5} + \frac{12}{25} - \frac{24}{125} = \frac{86}{125}
\end{aligned}$$

[10]

(ii) $\mathbb{P}(Q|D^c) = \mathbb{P}(T \cap N \cap S|D^c) = \mathbb{P}(T)\mathbb{P}(N)\mathbb{P}(S) = \frac{9}{10} \times \frac{2}{3} \times \frac{4}{5} = \frac{12}{25}$
since T, N, S, D^c are mutually independent. [5]

(iii)

$$\begin{aligned}\mathbb{P}(D^c|Q) &= \mathbb{P}(D^c \cap Q)/\mathbb{P}(Q) = \mathbb{P}(T \cap N \cap S \cap D^c)/\mathbb{P}(Q) \\ &= \mathbb{P}(T)\mathbb{P}(N)\mathbb{P}(S)\mathbb{P}(D^c)/\mathbb{P}(Q) \\ &= \frac{9}{10} \times \frac{2}{3} \times \frac{4}{5} \times \frac{1}{5} \times \frac{125}{86} = \frac{6}{43}\end{aligned}$$

since T, N, S, D^c are mutually independent. [10]

Q4.

(i)

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

[5]

(ii) Suppose that $\mathbb{P}(A|B) > \mathbb{P}(A)$. Then $\frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} > \mathbb{P}(A)$. Since $\mathbb{P}(B) > 0$ this implies that $\mathbb{P}(A \cap B) > \mathbb{P}(B)\mathbb{P}(A)$. Since $\mathbb{P}(A) > 0$ this implies that $\frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} > \mathbb{P}(B)$. Hence $\mathbb{P}(B|A) > \mathbb{P}(B)$. [10]

(iii) Suppose that A_1, A_2 are disjoint. Then

$$\begin{aligned}\mathbb{P}(A_1 \cup A_2|B) &= \frac{\mathbb{P}([A_1 \cup A_2] \cap B)}{\mathbb{P}(B)} \\ &= \frac{\mathbb{P}([A_1 \cap B] \cup [A_2 \cap B])}{\mathbb{P}(B)} \\ &= \frac{\mathbb{P}(A_1 \cap B) + \mathbb{P}(A_2 \cap B)}{\mathbb{P}(B)} \quad \text{[since } A_1 \cap B \text{ and } A_2 \cap B \text{ are disjoint]} \\ &= \frac{\mathbb{P}(A_1 \cap B)}{\mathbb{P}(B)} + \frac{\mathbb{P}(A_2 \cap B)}{\mathbb{P}(B)} \\ &= \mathbb{P}(A_1|B) + \mathbb{P}(A_2|B)\end{aligned}$$

[10]