Queen Mary
University of London

B. Sc. Examination by course unit 2010

MTH4108 Probability 1

Duration: 2 hours

Date and time: 10 May, 14:30

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt all questions. Marks awarded are shown next to the questions.

Calculators are NOT permitted in this examination. The unauthorized use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work which is not to be assessed.

Candidates should note that the Examination and Assessment Regulations state that possession of unauthorized materials by any candidate who is under examination conditions is an assessment offence. Please check your pockets now for any notes that you may have forgotten that are in your possession. If you have any, then please raise your hand and give them to an invigilator now.

Exam papers must not be removed from the examination room.

Examiner(s): Bill Jackson

Question 1 [10]

(a) Explain briefly what is meant by the terms *sample space* and *event*.

(b) I have a deck of five cards labelled 1, 2, 3, 4 and 5. I select three at random, one after the other, and without replacement.

(i) Define the sample space as a set.

(ii) Describe the event that the first card I select is labelled 2 and the last card I select is labelled 4 using set notation.

(iii) Let T be the event that one of the chosen cards is labelled 2 and F be the event that one of the chosen cards is labelled 4. Express the event $T^c \cap F$ in words and calculate the probability that it occurs.

Question 2 [10]

(a) State Kolmogorov's axioms for a probability function.

(b) Use Kolmogorov's axioms to prove that if A and B are events then

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B).$$

(c) I throw a fair 6-sided die twice and write down the two scores. Let A be the event that the first score is even and B be the event that the sum of the scores is even. Calculate $\mathbb{P}(A \cup B)$.

Question 3 [10]

(a) Let A, B, C be events in a sample space. Explain what it means to say that *A and B are independent* and that *A, B and C are mutually independent.*

(b) A lecture course contains 36 students. Of these, 18 are female students, 24 are home students and 12 are taking joint degrees. There are 12 female home students, 6 female students taking joint degrees, 8 home students taking joint degrees, and 3 female home students taking joint degrees. A student is selected at random. Let F, H, and J be the respective events that the student is female, home, or taking a joint degree.

(i) Determine whether F and H are independent.

(ii) Determine whether $F \cap H$ and J are independent.

(iii) Are F, H and J mutually independent? Justify your answer.

Question 4 [10]

(a) Let X be a discrete random variable.

(i) Define the *expectation* and *variance* of X.

(ii) Prove that $\mathrm{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$.

(b) A football team has probability 1/2 of winning its next match, 1/4 of drawing its next match, and 1/4 of losing its next match. Let X be the number of points the team receives after the next match (so $X = 3$ if the team wins, $X = 1$ if the team draws, and $X = 0$ if the team loses). Find $\mathbb{E}(X)$ and $\mathrm{Var}(X)$.

Question 5 [10]

(a) Explain how the Binomial(n, p) probability distribution arises from a sequence of Bernoulli trials.

(b) Use the description in (a) to derive the probability mass function for a Binomial(n, p) random variable.

(c) Suppose that $X \sim \mathrm{Binomial}(3, 1/4)$.

(i) Determine $\mathbb{P}(X \geq 2)$.

(ii) Determine $\mathbb{P}(X \geq 2 \,|\, X \geq 1)$.

Question 6 [10]

(a) Define the *cumulative distribution function* of a continuous random variable X.

(b) Suppose that the cumulative distribution function of X is

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 - e^{-\lambda x} & \text{if } x \geq 0, \end{cases}$$

where $\lambda > 0$ is a constant.

(i) Calculate the probability density function of X.

(ii) Calculate the expectation and variance of X.

(iii) What is the name of this probability distribution?

Question 7 [20]

(a) Let A and B be events in a sample space S with $\mathbb{P}(B) > 0$.

(i) Define the *conditional probability* of A given B.

(ii) Determine the conditional probability of A given B in the following cases:

- A and B are independent;
- A and B are disjoint;
- $B \subseteq A$.

(b) State and prove the Theorem of Total Probability.

(c) Two football teams M and C each have one game left to play in the season, and these last games are against two other teams. If M wins and C does not win, or if M draws and C loses, then M wins the championship. Otherwise C wins the championship. The probability that M wins, draws, or loses the last game is 1/2, 1/6, and 1/3, respectively. The probability that C wins, draws, or loses the last game is 2/3, 1/6, and 1/6, respectively.

(i) What is the probability that M wins the championship?

(ii) What is the probability that C draws the last game given that M wins the championship?

Question 8 [20]

(a) Let X and Y be two discrete random variables defined on the same sample space.

(i) Define the *joint probability mass function* of X and Y.

(ii) Explain what it means to say that X and Y are *independent*.

(iii) Define the *covariance* of X and Y and prove that it is equal to zero when X and Y are independent.

(b) A bag contains four balls labelled 0, 1, 2 and 3. Two balls are chosen at random, one after the other and without replacement. Let X be the label on the first ball and Y be the label on the second ball.

(i) Find the joint probability mass function of X and Y.

(ii) Calculate $\mathbb{E}(X)$, $\mathbb{E}(Y)$, $\mathbb{E}(XY)$ and $\mathrm{Cov}(X, Y)$.

(iii) Are X and Y independent? Justify your answer.

End of Paper