Probability I 2009 Solutions

1. a) i) $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$

$= 5/6 + 7/12 - 1/2$

$= 11/12$

ii) $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$

$= 1 - 5/6$

$= 1/6$

iii) $\mathbb{P}(A \mid B) = \dfrac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \dfrac{1/2}{7/12} = 6/7$

b) i) B^c

ii) $B \setminus A$ (or $B \cap A^c$)

iii) $A \triangle B$

2. a) E, F are independent means that $\mathbb{P}(E \cap F) = \mathbb{P}(E)\,\mathbb{P}(F)$

b) Let E = even number rolled

F = multiple of 3 rolled

$\mathbb{P}(E) = \dfrac{|\{2,4,6\}|}{|\{1,\dots,6\}|} = \dfrac{1}{2}$

$\mathbb{P}(F) = \dfrac{|\{3,6\}|}{|\{1,\dots,6\}|} = \dfrac{1}{3}$

$\mathbb{P}(E \cap F) = \dfrac{|\{6\}|}{|\{1,\dots,6\}|} = \dfrac{1}{6}$

So $\mathbb{P}(E \cap F) = \frac{1}{2} \times \frac{1}{3} = \mathbb{P}(E) \times \mathbb{P}(F)$ So E, F are independent

c) $\mathbb{P}(E^c \cap F) = \mathbb{P}(F) - \mathbb{P}(E \cap F)$

$= \mathbb{P}(F) - \mathbb{P}(E)\mathbb{P}(F)$ (as E, F independent)

$= \mathbb{P}(F)(1 - \mathbb{P}(E)) = \mathbb{P}(F)\mathbb{P}(E^c)$

So E^c, F are independent. ~~The events "odd number rolled~~

d) In b) E^c is the event "odd number rolled".
So the events "odd number rolled" and "multiple of 3 rolled" are independent also.

3. a) If $E_1, \dots, E_n$ partition S and $\mathbb{P}(E_i) > 0$
then $\mathbb{P}(A) = \sum_{i=1}^{n} \mathbb{P}(A \mid E_i)\,\mathbb{P}(E_i)$
for any event A.

b) Let E_i be the event "pick bag i" $i = 1, 2, 3$.
R be the event "red ball picked"

$$\mathbb{P}(R) = \mathbb{P}(R \mid E_1)\mathbb{P}(E_1) + \mathbb{P}(R \mid E_2)\mathbb{P}(E_2) + \mathbb{P}(R \mid E_3)\mathbb{P}(E_3)$$
$$= 1 \times \frac{1}{3} + \frac{1}{2} \times \frac{1}{3} + \frac{1}{3} \times \frac{1}{3}$$
$$= \frac{11}{6} \times \frac{1}{3} = \frac{11}{18}$$

c) If k red balls are added to bag 3 we get
$$\mathbb{P}(R) = \frac{1}{3}\left(1 + \frac{1}{2} + \frac{2+k}{6+k}\right)$$
We need $\frac{1}{3}\left(1 + \frac{1}{2} + \frac{2+k}{6+k}\right) \geq \frac{2}{3}$
so $\frac{2+k}{6+k} \geq \frac{1}{2}$ $\quad k \geq 2$ $\quad$ So 2 red balls must be added

4. a) The cdf $F_X(x)$ is determined from the pdf $f_X(x)$ by
$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$$

b) If $x \leq 1$ then $F_C(x) = 0$
If $1 < x < 2$ then $F_C(x) = \int_1^x 1\,dt = x - 1$
If $2 \leq x$ then $F_C(x) = \int_1^2 1\,dt = 1$

c) Uniform $[1, 2]$

5.

a) If X is a discrete r.v. then

$$\mathbb{E}(X) = \sum_x x\,\mathbb{P}(X=x)$$ (where the sum is over all values X takes)

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2$$

b) $\mathbb{E}(Y) = \mathbb{E}(aX+b)$
$= \sum_x (ax+b)\,\mathbb{P}(X=x)$
$= a \sum x\,\mathbb{P}(X=x) + b \sum \mathbb{P}(X=x)$
$= a\,\mathbb{E}(X) + b$ $\square$

c) $\mathrm{Var}(Y) = a^2 \mathrm{Var}(X)$

proof: $\mathrm{Var}(Y) = \mathbb{E}((aX+b)^2) - (\mathbb{E}(aX+b))^2$
$= \mathbb{E}(a^2X^2 + 2abX + b^2) - (a\mathbb{E}(X)+b)^2$ (by b))
$= a^2\mathbb{E}(X^2) + 2ab\mathbb{E}(X) + b^2$ (by b))
$\quad - a^2(\mathbb{E}(X))^2 - 2ab\mathbb{E}(X) - b^2$
$= a^2 [\mathbb{E}(X^2) - (\mathbb{E}(X))^2]$
$= a^2 \mathrm{Var}(X)$ $\square$

d) Set $a = \frac{1}{\sqrt{\mathrm{Var}(X)}}$ [Can do this since $\mathrm{Var}(X) \neq 0$]

and $b = -\frac{\mathbb{E}(X)}{\sqrt{\mathrm{Var}(X)}}$

This clearly does the job.

6. a) $\mathbb{P}(R=0) = e^{-\lambda}\frac{\lambda^0}{0!} = e^{-\lambda}$

b) $\mathbb{P}(R \le 1) = \mathbb{P}(R=0) + \mathbb{P}(R=1) = e^{-\lambda} + e^{-\lambda}\frac{\lambda^1}{1!} = (1+\lambda)e^{-\lambda}$

c) $\mathbb{P}(R=0 \mid R\le 1) = \frac{\mathbb{P}(R=0 \text{ and } R\le 1)}{\mathbb{P}(R\le 1)} = \frac{\mathbb{P}(R=0)}{\mathbb{P}(R\le 1)} = \frac{1}{1+\lambda}$

d) $\mathbb{P}(R=1 \mid R\le 1) = \frac{\mathbb{P}(R=1)}{\mathbb{P}(R\le 1)} = \frac{\lambda}{1+\lambda}$ so conditional pmf is

n	0	1
$\mathbb{P}(R=n \mid R\le 1)$	$\frac{1}{1+\lambda}$	$\frac{\lambda}{1+\lambda}$

i.e. Bernoulli $\left(\frac{\lambda}{1+\lambda}\right)$

7.

a) A Bernoulli (p) trial is an experiment with 2 outcomes usually called "success" and "failure" with

$\mathbb{P}(\text{success}) = p$

b) Let T be the number of trials up to and including the first success in a sequence of independent Bernoulli (p) trials. Then T has a Geometric (p) distribution

c) If T is as above then for $k = 1, 2, 3$

$$\mathbb{P}(T = k) = \mathbb{P}(k-1 \text{ failures followed by a success})$$
$$= (1-p)^{k-1} p \qquad \text{(as trials are independent)}$$

d) $$\mathbb{P}(X < 4) = 1 - \mathbb{P}(X \geq 4)$$
$$= 1 - \mathbb{P}(3 \text{ failures})$$
$$= 1 - (1-p)^3 = 1 - \left(\tfrac{2}{3}\right)^3 = \frac{19}{27}$$

e) $\mathbb{E}(X) = \frac{1}{p} = 3$

so $\mathbb{P}(X > \mathbb{E}(X)) = \mathbb{P}(X > 3) = \left(\tfrac{2}{3}\right)^3 = \frac{8}{27}$

f) $$\mathbb{P}(X \text{ odd}) = \mathbb{P}(X=1) + \mathbb{P}(X=3) + \mathbb{P}(X=5) + \cdots$$
$$= \tfrac{1}{3} + \left(\tfrac{2}{3}\right)^2 \tfrac{1}{3} + \left(\tfrac{2}{3}\right)^4 \tfrac{1}{3} + \left(\tfrac{2}{3}\right)^6 \tfrac{1}{3} + \cdots$$
$$= \frac{1/3}{1 - \left(\tfrac{2}{3}\right)^2} \qquad \text{(G.P.)}$$
$$= \frac{1/3}{5/9}$$
$$= 3/5$$

g) Since $\left(\tfrac{2}{3}\right)^k < 1$ for all $k \geq 1$, $\underset{\frac{1}{3}}{\mathbb{P}(X=1)} \geq \underset{\frac{1}{3}\left(\frac{2}{3}\right)^{n-1}}{\mathbb{P}(X=n)}$ for all n.

So $\mathbb{P}(X=1)$ is the largest

8.

a) The sample space for a experiment is the set of all possible outcomes.
An event is a subset of the sample space.

b) $S = \{12, 13, 14, 21, 23, 24, 31, 32, 34, 41, 42, 43\}$

c) The event $\{12, 13, 14\}$ is "the 1st number picked is 1"

d)

n	1	2	3	4
$\mathbb{P}(A = n)$	$1/4$	$1/4$	$1/4$	$1/4$

n	1	2	3	4
$\mathbb{P}(B = n)$	$1/4$	$1/4$	$1/4$	$1/4$

e) $\mathbb{E}(A) = \frac{1}{4}(1+2+3+4) = 5/2$

$A \sim B$ so $\mathbb{E}(B) = 5/2$ as well

f) $T = 1+2+3+4 - A - B = 10 - (A+B)$

$$\begin{aligned}\mathbb{E}(T) &= \mathbb{E}(10 - (A+B)) \\ &= 10 - \mathbb{E}(A+B) \\ &= 10 - \mathbb{E}(A) - \mathbb{E}(B) \quad \text{(by linearity of expectation} \\ &= 5\end{aligned}$$

g) We have $P = \dfrac{4 \times 3 \times 2 \times 1}{A \times B} = \dfrac{24}{AB}$

but $\mathbb{E}\left(\frac{24}{AB}\right)$ is not determined by $\mathbb{E}(A)$, $\mathbb{E}(B)$

[In particular it may not be $\dfrac{24}{\mathbb{E}(A)\mathbb{E}(B)}$]