Queen Mary
University of London

BSc Examination 2008 By Course Units

MAS108 Probability I

2:30 pm, Thursday 14 August, 2008

Duration: 2 hours

Do not start reading the question paper until you are instructed to by the invigilators.

The paper has two Sections and you should attempt both Sections.

Please read carefully the instructions given at the beginning of each Section.

Calculators are NOT permitted in this examination. The unauthorized use of a calculator constitutes an examination offence.

Section A: You should attempt all questions. Marks awarded are shown next to the question. This part of the examination carries 60% of the marks.

1. [9 marks]

A student is picked at random from a class and asked which sports they play. Let B be the event "the student plays badminton" and C be the event "the student plays cricket". Suppose that $\mathbb{P}(B) = 1/3$, $\mathbb{P}(C) = 1/2$ and $\mathbb{P}(B \cap C) = 1/12$.

a) Express the event B^c in words and calculate its probability.

b) Express the event "the student plays cricket but not badminton" in symbols and calculate its probability.

c) Calculate the conditional probability that the student plays badminton given that they play cricket.

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2. [8 marks]

A number is picked at random from the set $\{1,2,3,4,5,6,7,8\}$ with all choices equally likely.

a) Are the events "the chosen number is greater than 4" and "the chosen number is even" independent? Justify your answer carefully.

b) Are the events "the chosen number is greater than 3" and "the chosen number is even" independent? Justify your answer carefully.

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3. [8 marks]

A biased coin which has probability $1/3$ of coming up heads is tossed repeatedly.

a) Write down a random variable related to this which has a $\text{Bin}(5,1/3)$ distribution.

b) Write down a random variable related to this which has a $\text{Bin}(5,2/3)$ distribution.

c) Write down a random variable related to this which has a $\text{Geom}(1/3)$ distribution.

d) Write down a random variable related to this which has a $\text{Bin}(5,1/3)$ distribution and is independent of the random variable you described in part a).

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4. [9 marks]

Let X be a discrete random variable with probability mass function (pmf)

n	1	2	3	4
$P(X=n)$	$1/12$	$5/12$	$1/4$	$1/4$

a) Calculate the following:

i) $\mathbb{P}(X > 2)$,

ii) $\mathbb{E}(X)$,

iii) $\mathrm{Var}(X)$.

b) Find the pmf of $(X-3)^2$.

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5. [9 marks]

Let A and B be sets and $f : A \to B$ be a function.

a) Say what it means for f to be surjective.

b) Write down a function from $\{1,2,3,4\}$ to $\{1,2,3\}$ which is surjective.

c) Write down a function from $\{1,2,3,4\}$ to $\{1,2,3\}$ which is not surjective.

d) If S is the sample space of some experiment then what is the name given to a function $N : S \to \mathbb{R}$?

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6. [8 marks]

a) State (without proof) the inclusion-exclusion formula for two events.

b) Say what it means for two events to be independent.

c) Suppose that A and B are independent events with $\mathbb{P}(A) = \mathbb{P}(B)$ and $\mathbb{P}(A \cup B) = 1/2$. Find $\mathbb{P}(A)$.

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7. [9 marks]

a) Write down two properties which the cumulative distribution function (cdf) of a random variable must have.

Let X be a continuous random variable with cdf

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ x^2 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$$

b) Calculate the median of X.

c) Calculate the expectation of X.

Section B: You may attempt as many questions as you wish and all questions carry equal marks. Except for the award of a bare pass, only the best TWO questions answered will be counted. This part of the examination carries 40% of the marks.

8.

a) Write down the probability mass function of a Geom(p) random variable.

b) Suppose that $A \sim \text{Geom}(1/4)$. Calculate the following probabilities:

i) $\mathbb{P}(A = 2)$,

ii) $\mathbb{P}(A \geq 3)$.

c) Derive carefully the formula for the expectation of a Geom(p) random variable.

Amanda and Brian each have a standard fair six-sided die. They play the following game. Each of them rolls their die repeatedly until they get a 6. The player who takes the smallest number of rolls to do this is the winner. If they take the same number of rolls then the game is a draw.

d) Show that the probability that the game ends in a draw is

$$\frac{1}{36} \sum_{k=1}^{\infty} \left(\frac{5}{6}\right)^{2(k-1)}$$

and hence calculate this probability.

e) Calculate the probability that Amanda wins the game.

9.

a) What does it mean to say that the events $E_1, \ldots, E_n$ partition the sample space?

b) State and prove the theorem of total probability.

c) Deduce that if X is any event and A and B are independent events with $\mathbb{P}(A), \mathbb{P}(B) > 0$ then

$$\begin{aligned}\mathbb{P}(X) =& \mathbb{P}(X|A\cap B)\mathbb{P}(A)\mathbb{P}(B) + \mathbb{P}(X|A\cap B^c)\mathbb{P}(A)\mathbb{P}(B^c) + \\ & \mathbb{P}(X|A^c\cap B)\mathbb{P}(A^c)\mathbb{P}(B) + \mathbb{P}(X|A^c\cap B^c)\mathbb{P}(A^c)\mathbb{P}(B^c).\end{aligned}$$

My right-hand pocket contains one ordinary fair coin and one coin which has heads on both sides. My left-hand pocket contains one ordinary fair coin and two coins which have heads on both sides.

d) I take one coin out of each pocket and toss them both. What is the probability that they both come up heads?

e) Given that they do both come up heads what is the conditional probability that they are both ordinary coins?

10.

a) Say what it means for two discrete random variables to be independent of each other.

b) Suppose that $X \sim \text{Bernoulli}(1/4)$ and $Y \sim \text{Bernoulli}(1/3)$ are independent. Find the joint distribution of X and Y expressing your answer as a table.

Let A and B be random variables with joint distribution:

		A	
		0	1
B	0	$1/3$	$1/6$
	1	$1/6$	$1/3$

c) Find the marginal distributions of A and B.

d) Find the covariance of A and B.

e) Write down (as a table) the joint distribution of a pair of random variables C and D with $C \sim \text{Bernoulli}(1/2)$, $D \sim \text{Bernoulli}(1/2)$ and $\text{Cov}(C,D) = -1/8$.

11.

a) Define the expectation of a continuous random variable in terms of its probability density function (pdf).

Let $X \sim \text{Uniform}[0, 10]$.

b) Write down the pdf of X.

c) Compute the expectation of X.

d) Find the cumulative distribution function (cdf) of X.

Define new random variables by $Y = \frac{X}{10}$ and $Z = \frac{X^2}{100}$.

e) By computing the cdf of Y show that $Y \sim \text{Uniform}[0, 1]$.

f) Is $Z \sim \text{Uniform}[0, 1]$? Justify your answer.

END OF EXAM