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MAS108 Probability I

10:00 am, Monday 13 August, 2007

Duration: 2 hours

Do not start reading the question paper until you are instructed to by the invigilators.

The paper has two Sections and you should attempt both Sections.

Please read carefully the instructions given at the beginning of each Section.

Calculators are NOT permitted in this examination. The unauthorized use of a calculator constitutes an examination offence.

This question paper may not be removed from the examination room.

Section A: You should attempt all questions. Marks awarded are shown next to the question. This part of the examination carries 60% of the marks.

1. [8 marks]

A class of students sits two exams. A student is picked at random from the class. Let A be the event "the student passes the first exam", B be the event "the student passes the second exam", and and H be the event "the student is hard-working". Suppose that $\mathbb{P}(A) = 2/3$, $\mathbb{P}(B) = 3/4$, $\mathbb{P}(A \cap B) = 7/12$, and $\mathbb{P}(H) = 1/2$.

a) Express the following events in words and find their probabilities:

   i) $A \cup B$,

   ii) H^c.

b) Express the event "the student fails both exams" in symbols and find its probability.

c) You are told that all hard-working students pass the second exam. Express this statement in symbols.

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2. [8 marks]

Two standard fair 6 sided dice are rolled. Let A be the event "the number showing on the first die is even", B be the event "the number showing on the second die is even", and C be the event "the sum of the two numbers is even".

a) Show that A and C are independent.

b) Determine whether or not A, B and C are mutually independent.

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3. [9 marks]

A bag contains 4 red marbles and 3 blue marbles. I select a marble from the bag at random and record its colour. Then, without replacing the first marble, I take a second and record its colour.

a) Write down the sample space for this experiment explaining your notation carefully.

b) What is the probability that I end up with one marble of each colour?

c) Suppose now that I do replace the first marble before picking the second. How does the probability that I end up with one marble of each colour change?

4. [8 marks]

Let A and B be events with $P(A), P(B) > 0$.

a) Define the conditional probability $\mathbb{P}(A|B)$.

b) Determine $\mathbb{P}(A|B)$ in the case that A and B are independent.

c) Determine $\mathbb{P}(A|B)$ in the case that A and B are disjoint (mutually exclusive).

d) Determine $\mathbb{P}(A|B)$ in the case that B is a subset of A.

5. [9 marks]

Let X be the discrete random variable with distribution:

n	-2	-1	0	1	2
$P(X = n)$	$1/10$	$1/5$	$2/5$	$1/5$	$1/10$

a) Find $\mathbb{E}(X)$.

b) Find $\text{Var}(X)$.

c) Find $\mathbb{P}(X^2 > 1)$.

6. [9 marks]

Let X be a Poisson(λ) random variable.

a) Write down the probability mass function of X.

b) Show that $\mathbb{E}(X) = \lambda$ and $\mathbb{E}(X(X-1)) = \lambda^2$.

c) Use part b) to determine the variance of X.

7. [9 marks]

A continuous random variable T has probability density function

$$f_T(t) = \begin{cases} 0 & \text{if } t < 0 \\ 3t^2 & \text{if } 0 \leq t < 1 \\ 0 & \text{if } 1 < t. \end{cases}$$

a) Find $\mathbb{P}(T \leq 1/2)$.

b) Calculate the expectation of T.

c) Calculate the median of T.

Section B: You may attempt as many questions as you wish and all questions carry equal marks. Except for the award of a bare pass, only the best TWO questions answered will be counted. This part of the examination carries 40% of the marks.

8.

a) Define the cumulative distribution function (cdf) of a random variable X.

A continuous random variable T has cdf:

$$F_T(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 - e^{-3t} & \text{if } t \geq 0. \end{cases}$$

b) What is the name of this distribution?

c) Determine the probability density function (pdf) of T.

d) Calculate $\mathbb{P}(1/3 < T < 2/3)$.

e) Prove that for all $x, y > 0$ we have

$$\mathbb{P}(T > x + y | T > x) = \mathbb{P}(T > y).$$

f) Suppose now that the random variable T measures the lifetime in years of a lightbulb. Explain in words what the property you proved in part f) means in this context.

9.

a) Say what is meant by the joint distribution of two discrete random variables X and Y.

b) Explain how to find the marginal distribution of X given the joint distribution of X and Y.

Two numbers are chosen at random *with* replacement from the set $\{1,2,3\}$ with all choices equally likely. Let A denote the first number chosen and M denote the maximum of the two numbers chosen.

c) Determine the joint distribution of A and M.

d) Determine the marginal distribution of M.

e) Determine whether of not A and M are independent of each other.

f) Find the covariance of A and M.

10.

a) State and prove Baye's theorem.

I have two coins in my pocket. One of them is a fair coin and the other has probability $1/4$ of coming up heads when tossed.

b) I take a coin out of my pocket at random and toss it. What is the probability that the coin comes up heads?

c) I take a coin out of my pocket at random, toss it, and notice that it comes up heads. What is the conditional probability that I have the fair coin given this information?

d) I take a coin out of my pocket at random, toss it n times, and notice that it comes up heads every time. Show that the conditional probability that I have the fair coin given this information is:

$$\frac{2^n}{2^n+1}.$$

11.

a) Explain what it means for a function $f : X \to Y$ to be:

i) injective,

ii) surjective,

iii) bijective.

A manager has a set of tasks T and a set of employees E. She allocates the tasks to the employees by specifying a function from T to E which maps each task to the person who is responsible for it.

b) Explain in words what it means for this particular function to be:

i) injective,

ii) surjective,

iii) bijective.

c) Let f be a function from $\{1,2,3\}$ to $\{1,2,3\}$ chosen at random with all choices equally likely. What is the probability that f is bijective?

d) Let f be a function from $\{1,2,3,\ldots,n\}$ to $\{1,2,3,\ldots,n\}$ chosen at random with all choices equally likely. What is the probability that f is bijective?

END OF EXAM