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# MAS108 Probability I

2:30 pm, Thursday 3 May, 2007

Duration: 2 hours

Do not start reading the question paper until you are instructed to by the invigilators.

The paper has two Sections and you should attempt both Sections.

Please read carefully the instructions given at the beginning of each Section.

Calculators are NOT permitted in this examination. The unauthorized use of a calculator constitutes an examination offence.

This question paper may not be removed from the examination room.

**Section A: You should attempt all questions. Marks awarded are shown next to the question. This part of the examination carries 60% of the marks.**

**1.** [9 marks]

When I travel into work each morning I notice whether my train is late and by how much and also whether I am able to get a seat on it. Let $A$ be the event "the train is on time", $B$ be the event "the train is late but by not more than 15 minutes", and $C$ be the event "I am able to get a seat". Suppose that $\mathbb{P}(A) = 1/2$, $\mathbb{P}(B) = 1/4$, $\mathbb{P}(C) = 1/3$ and $\mathbb{P}(A \cap C) = 1/4$.

a) Express the event $A \cup B$ in words and calculate its probability.

b) Express the event "the train is late" in symbols and calculate its probability.

c) Find the conditional probability that I get a seat given that the train is late.

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**2.** [8 marks]

Let $f : X \to Y$ be a function where $X$ and $Y$ are finite sets.

a) Say what it means for $f$ to be injective.

Each member of a class of students goes to the library to reserve a book.

b) Explain how the information of which book each student reserves can be expressed as a function between two finite sets which should be specified.

c) What does it mean for the function of part b) to be injective?

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**3.** [8 marks]

A fair coin is tossed four times.

a) Are the events "the first toss is a head" and "exactly three heads are seen" independent?

b) Are the events "the first toss is a head" and "exactly two heads are seen" independent?

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**4.** [8 marks]

a) State (without proof) the theorem of total probability.

A coin which has probability 1/3 of coming up heads is tossed. If the coin comes up heads then an ordinary fair 6-sided die is rolled; if it comes up tails then a fair 4-sided die whose faces are labelled with the numbers 1,2,3 and 4 is rolled.

b) What is the probability that a 1 is rolled?

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**5.** [9 marks]

Let $X$ be the discrete random variable with distribution:

| $n$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $P(X=n)$ | $2/5$ | $3/10$ | $1/5$ | $1/10$ |

a) Find $\mathbb{E}(X)$.

b) Find $\mathrm{Var}(X)$.

Let $Y$ be a second random variable which has the same distribution as $X$ and is independent of $X$.

c) Find $\mathbb{P}(X=Y)$.

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**6.** [9 marks]

a) Write down the probability mass function of a $\mathrm{Bin}(n,p)$ random variable.

b) Show that the expectation of a $\mathrm{Bin}(n,p)$ random variable is $np$.

c) Give an example of a practical situation which is modelled by a $\mathrm{Bin}(5,1/4)$ random variable.

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**7.** [9 marks]

A continuous random variable $T$ has cumulative distribution function

$$F_T(t) = \begin{cases} 0 & \text{if } t < 1 \\ \frac{t-1}{2} & \text{if } 1 \leq t < 2 \\ \frac{t}{4} & \text{if } 2 \leq t < 4 \\ 1 & \text{if } 4 \leq t. \end{cases}$$

a) Find the probability density function of $T$.

b) Calculate the expectation of $T$.

c) Calculate the median of $T$.

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**Section B: You may attempt as many questions as you wish and all questions carry equal marks. Except for the award of a bare pass, only the best TWO questions answered will be counted. This part of the examination carries 40% of the marks.**

**8.**

a) An eccentric postman has three letters to deliver, one to each of house numbers 1, 2 and 3. He does this by ignoring the addresses and delivering one letter to each house with all possibilities equally likely.

   i) What is the cardinality of the sample space for this experiment?

   ii) What is the probability that no house receives the correct letter?

   iii) What is the probability that exactly two of the houses receive the correct letter?

b) The following day the same postman again has three letters to deliver, one to each of house numbers 1, 2 and 3. This time he chooses a house at random for each letter independently of where all other letters are delivered with all possibilities equally likely.

   i) What is the cardinality of the sample space for this experiment?

   ii) What is the probability that every house receives a letter?

   iii) What is the probability that no house receives the correct letter?

   iv) What is the probability that exactly two of the houses receive the correct letter?

   v) What is the conditional probability that every house receives a letter given that no house receives the correct letter.

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## 9.

A certain building is heated by an unreliable system. Each month the number of times the boiler breaks down has a Poisson distribution with parameter 1. Each time the boiler breaks an engineer must be called to repair it and this costs £100.

a) What is the probability that the boiler breaks exactly twice this month?

b) What is the probability that the boiler breaks at least once this month?

c) Find the expectation and variance of the random variable “the number of pounds which must be spent on repairing the boiler this month”?

d) Write down in words a random variable related to this situation which you would expect to have an exponential distribution.

The radiators are also unreliable and the number of pounds spent on repairs to them each month is a random variable with expectation 50 and variance 10. Let $C$ be the total amount of money spent on repairs to the boiler and the radiators this month.

e) Find the expectation of $C$.

f) Explain why this is not enough information to calculate the variance of $C$. What extra piece of information would be needed?

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## 10.

a) Say what it means for two discrete random variables to be independent of each other.

In my pocket are four ordinary fair coins and three coins which have tails on both sides. I take two coins from my pocket at random and toss them. Let $X$ be the random variable “the number of ordinary coins which I take out”. Let $Y$ be the random variable “the number of heads seen”.

b) Find the joint distribution of $X$ and $Y$ expressing your answer in a table.

c) Decide whether or not $X$ and $Y$ are independent of each other.

d) Find the marginal distribution of $Y$ and hence calculate the expectation of $Y$.

e) Find the conditional expectation of $X$ given that $Y = 1$.

f) Find the distribution of $X - Y$.

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## 11.

a) Define the cumulative distribution function (cdf) of a random variable.

b) Explain how to find the cdf of a continuous random variable whose probability density function (pdf) is given.

A continuous random variable $X$ has pdf

$$f_X(x) = \begin{cases} \frac{1}{2} & \text{if } 1 \leq x \leq 3 \\ 0 & \text{otherwise.} \end{cases}$$

c) What is the name of the distribution of $X$?

d) Find the cdf of $X$.

e) Calculate $\mathbb{P}(X^2 \leq 2)$.

f) Find the cdf of $X^2$.

g) Calculate $\mathbb{P}(X^{-1} \leq \frac{2}{3})$.

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END OF EXAM