Probability I – 2009/10

Solutions to Exercise Sheet 9

Q1. This question uses the basic properties of expectation and variance that we proved in Propositions 11.3 and 11.4. Make sure you can see which parts of those results are being used at each step. (I've told you for the first two).

(i) $\mathbb{E}(3X) = 3\mathbb{E}(X) = 15$ (using part (i) of Proposition 11.3)

(ii) $\text{Var}(3X) = 3^2\text{Var}(X) = 9 \times 2/3 = 6$ (using part (ii) of Proposition 11.4)

(iii) $\mathbb{E}(4 - 3X) = \mathbb{E}(4) + \mathbb{E}(-3X) = 4 - 3\mathbb{E}(X) = 4 - 15 = -11$

(iv) $\text{Var}(4 - 3X) = \text{Var}(-3X) = (-3)^2\text{Var}(X) = 9 \times 2/3 = 6$

(v) $\mathbb{E}(4 - 3X^2) = 4 - 3\mathbb{E}(X^2)$ but what is $\mathbb{E}(X^2)$?

By the definition of variance $\text{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2$ so $\mathbb{E}(X^2) = 2/3 + 5^2 = 77/3$. Hence,

$$\mathbb{E}(4 - 3X^2) = 4 - 3 \times 77/3 = -73.$$

(Hopefully nobody was tempted to write $\mathbb{E}(X^2) = 5^2$.)

Q2. There are several equally correct answers to this question. The key point is that the binomial distribution is used for the number of successes in a fixed number of trials while the geometric distribution is used for the number of trials up to and including the first success.

(i) The number of 1s seen in the first 6 rolls has the $\text{Bin}(6, 1/6)$ distribution. Its expectation is $6 \times 1/6 = 1$ and its variance is $6 \times 1/6 \times 5/6 = 5/6$

(ii) The number of odd numbers seen in the first 8 rolls has the $\text{Bin}(8, 1/2)$ distribution. Its expectation is $8 \times 1/2 = 4$ and its variance is $8 \times 1/2 \times 1/2 = 2$

(iii) The number of rolls needed to get a 6 has the $\text{Geom}(1/6)$ distribution. Its expectation is $\frac{1}{1/6} = 6$ and its variance is $\frac{5/6}{(1/6)^2} = 30$.

(iv) The number of rolls needed to get a number bigger than 2 has the $\text{Geom}(2/3)$ distribution. Its expectation is $\frac{1}{2/3} = 3/2$ and its variance is $\frac{1/3}{(2/3)^2} = 3/4$

Q3*.

(i) If we regard a draw as "suuccess" then we have 38 independent Bernoulli trials with the probability of success being 1/6 in each. The distribution is therefore $\text{Bin}(38, 1/6)$.

(ii) If we regard a win as "success" then this is the number of $\text{Bernoulli}(1/2)$ trials up to and including the first success. The distribution is therefore $\text{Geom}(1/2)$.

(iii) Starting from the first defeat we have a sequence of Bernoulli(1/2) trials. So the number of games up to and including the next win has the Geom(1/2) distribution. The fact that we start from the first defeat is irrelevant to the distribution of the number of games until the next win. See question 7 for another expression of this.

(iv) This is not a distribution we have studied so we will have to work out the probability mass function directly. Let N be the number of matches up to and including the second loss. Then N takes values $2, 3, 4, \ldots$ and if $k \geq 2$ we have

$$\begin{aligned}\mathbb{P}(N = k) &= \mathbb{P}(1 \text{ loss and } k-2 \text{ others in the first } k-1 \text{ matches followed by a loss in the } k\text{th}) \\ &= (k-1)\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^{k-2}\left(\frac{1}{3}\right) \\ &= (k-1)\left(\frac{1}{3}\right)^{2}\left(\frac{2}{3}\right)^{k-2}.\end{aligned}$$

(v) Similarly if M is the number of matches up to and including the mth loss. Then M takes values $m, m+1, m+2, \ldots$ and if $k \geq m$ we have

$$\begin{aligned}\mathbb{P}(M = k) &= \mathbb{P}(m-1 \text{ losses and } k-m \text{ others in the first } k-1 \text{ matches then a loss in the } k\text{th}) \\ &= \binom{k-1}{m-1}\left(\frac{1}{3}\right)^{m-1}\left(\frac{2}{3}\right)^{k-m}\left(\frac{1}{3}\right) \\ &= \binom{k-1}{m-1}\left(\frac{1}{3}\right)^{m}\left(\frac{2}{3}\right)^{k-m}.\end{aligned}$$

Q4.

(i) $\mathbb{P}(A = 3) = \frac{2}{3} \times \frac{2}{3} \times \frac{1}{3} = \frac{4}{27}$.

(ii) $\mathbb{P}(A \leq 3) = 1 - \mathbb{P}(A > 3) = 1 - \left(\frac{2}{3}\right)^3 = \frac{19}{27}$.
You could equally well say
$\mathbb{P}(A \leq 3) = \mathbb{P}(A = 1) + \mathbb{P}(A = 2) + \mathbb{P}(A = 3) = \frac{1}{3} + \frac{2}{3} \times \frac{1}{3} + \frac{2}{3} \times \frac{2}{3} \times \frac{1}{3} = \frac{19}{27}$.

(iii) $\mathbb{P}(B = 2) = e^{-3}\frac{3^2}{2!} = e^{-3}\frac{9}{2} = \frac{9}{2e^3}$

(iv)

$$\begin{aligned}\mathbb{P}(B > 2) &= 1 - \mathbb{P}(B = 0) - \mathbb{P}(B = 1) - \mathbb{P}(B = 2) \\ &= 1 - e^{-3}\frac{3^0}{0!} - e^{-3}\frac{3^1}{1!} - e^{-3}\frac{3^2}{2!} \\ &= 1 - e^{-3} - 3e^{-3} - e^{-3}\frac{9}{2} \\ &= 1 - \frac{17}{2e^3}.\end{aligned}$$

AQ1 The Range of N is $\{0, 1, 2, 3\}$ and the probability mass function is

n	0	1	2	3
$P(N = n)$	1/2	5/16	1/8	1/16

Please let me know if you have any comments or corrections