MTH4108 Probability 1 – 2009/10– Exercise Sheet 9

These questions are on discrete random variables. You should write up your solution to the starred question, Q3,* **clearly**, *and hand it in during your week 11 exercise class for feedback. Put your* **full name and student number** *on the top of your solution. It is important that you make a serious attempt to do* **all** *of questions Q1-Q4 before week 10 lectures begin. Questions AQ1-AQ3 are for additional practice. You should attempt them when you have time.*

Q1. Let X be a discrete random variable with $\mathbb{E}(X) = 5$, $\mathrm{Var}(X) = 2/3$. Find the following:

(i) $\mathbb{E}(3X)$; (ii) $\mathrm{Var}(3X)$; (iii) $\mathbb{E}(4-3X)$; (iv) $\mathrm{Var}(4-3X)$; (v) $\mathbb{E}(4-3X^2)$.

Q2. A standard fair die is rolled repeatedly. For each of the following distributions write down in words a random variable related to this situation which has this distribution. In each case give the expectation and variance:

(i) $\mathrm{Bin}(6, 1/6)$; (ii) $\mathrm{Bin}(8, 1/2)$; (iii) $\mathrm{Geom}(1/6)$; (iv) $\mathrm{Geom}(2/3)$.

Q3*. Each match played by a football team is won by that team with probability 1/2, is a draw with probability 1/6, and is lost with probability 1/3, with the result of each match being independent of all other results. For each of the following random variables related to this say how they are distributed. (Some but not all of these take one of the special distributions studied in lectures; in these cases you should use the name of the distribution otherwise give the probability mass function).

(i) The number of drawn matches in a season lasting 38 matches.

(ii) The number of matches up to and including their first win.

(iii) The number of matches following their first defeat up to and including their next win.

(iv) The number of matches up to and including their second loss.

(v) The number of matches up to and including their mth loss where m is a fixed positive integer.

Note that in parts (ii)-(v) the numbers can accumulate over several seasons.

Q4. Suppose that $A \sim \mathrm{Geom}(1/3)$ and that $B \sim \mathrm{Poisson}(3)$. Find the following probabilities. (You can leave your answers involving powers of e but you should simplify all factorials and other powers.)

(i) $\mathbb{P}(A = 3)$; (ii) $\mathbb{P}(A \leq 3)$; (iii) $\mathbb{P}(B = 2)$; (iv) $\mathbb{P}(B > 2)$.

AQ1. A fair coin is tossed four times. Let N be the number of instances of a head followed by another head in the sequence of tosses.
A student argues as follows: There are three possible ways in which we could have a head followed by another head (at the first and second, the second and third, or third and fourth toss). We have a probability $1/2 \times 1/2 = 1/4$ of getting a head followed by another head at each of these positions. Hence N is the number of successes in three Bernoulli(1/4) trials and so $N \sim \text{Bin}(3, 1/4)$.

Explain what is wrong with this argument and determine the distribution of N.

How do the expectation and variance of N differ from the expectation and variance of a $\text{Bin}(3, 1/4)$ random variable? Comment on what this means.

AQ2. Let G be a $\text{Geom}(p)$ random variable.

(i) Show that for any $k, l \geq 1$

$$\mathbb{P}(G > k + l | G > k) = \mathbb{P}(G > l).$$

(ii) Why do you think this is sometimes called the "memoryless property" of the geometric distribution?

AQ3. Let X be the number of fish caught by a fisherman in one afternoon. Suppose that X is distributed $\text{Poisson}(\lambda)$. Each fish has probability p of being a salmon independently of all other fish caught. Let Y be the number of salmon caught. Show that $Y \sim \text{Poisson}(p\lambda)$.

Hint: Use the Theorem of total probability to first show that

$$\mathbb{P}(Y = m) = \sum_{i \geq m} \mathbb{P}(Y = m | X = i) \mathbb{P}(X = i).$$