Probability I – 2009/10

Solutions to Exercise Sheet 8

Q1.

(a) (i) $\mathbb{P}(X=2)=1/20$ (just read it off the table)

(ii) $\mathbb{P}(X=3)=0$ (since 3 is not a value that X takes)

(iii) $\mathbb{P}(X\le 1)=\mathbb{P}(X=-2)+\mathbb{P}(X=-1)+\mathbb{P}(X=0)+\mathbb{P}(X=1)=19/20$

(iv) $\mathbb{P}(X<1)=\mathbb{P}(X=-2)+\mathbb{P}(X=-1)+\mathbb{P}(X=0)=3/4$

(v) $\mathbb{P}(X^2<1)=\mathbb{P}(X=0)=1/4$

(b)

$$\mathbb{E}(X)=(-2)\times(1/10)+(-1)\times(2/5)+(0)\times(1/4)+(1)\times(1/5)+(2)\times(1/20)=-3/10.$$

We have $\mathrm{Var}(X)=\mathbb{E}(X^2)-\mathbb{E}(X)^2$. Since

$$\mathbb{E}(X^2)=(4)\times(1/10)+(1)\times(2/5)+(0)\times(1/4)+(1)\times(1/5)+(4)\times(1/20)=6/5$$

this gives $\mathrm{Var}(X)=6/5-(-3/10)^2=6/5-(9/100)=111/100$.

(c) We have $-2^2+4=8$, $-1^2+4=5$, $0^2+4=4$, $1^2+4=5$ and $2^2+4=8$. Hence $\mathrm{Range}(Y)=\{4,5,8\}$.
The probability mass function of Y is given by

$$\mathbb{P}(Y=4)=\mathbb{P}(X=0)=1/4,$$
$$\mathbb{P}(Y=5)=\mathbb{P}(X=-1)+\mathbb{P}(X=1)=3/5,$$
$$\mathbb{P}(Y=8)=\mathbb{P}(X=-2)+\mathbb{P}(X=2)=3/20.$$

You could equally well express this as a table with 3 columns.

Q2* (i) The number of blue marbles I select can be 0, 1 or 2. Thus

$$\mathrm{Range}(B)=\{0,1,2\}.$$

Let $X=\{b_1,b_2,r_1,r_2,r_3,r_4,r_5,r_6\}$ and let S be the set of all unordered selections of five distinct elements from X. Then $|S|=\binom{8}{5}$. The event that we choose no blues marbles has cardinality $\binom{6}{5}$. Hence $\mathbb{P}(B=0)=\binom{6}{5}/\binom{8}{5}$. Similarly $\mathbb{P}(B=1)=\binom{6}{4}\times\binom{2}{1}/\binom{8}{5}$, and $\mathbb{P}(B=2)=\binom{6}{3}/\binom{8}{5}$. This gives the following probability mass function for B.

n	0	1	2
$P(B=n)$	6/56	30/56	20/56

Hence

$$\mathbb{E}(B)=0\times(6/56)+1\times(30/56)+2\times(20/56)=5/4.$$

We also have

$$\mathbb{E}(B^2) = 0 \times (6/56) + 1 \times (30/56) + 4 \times (20/56) = 110/56,$$

so $\text{Var}(X) = \mathbb{E}(B^2) - \mathbb{E}(B)^2 = (110/56) - (25/16) = 45/112$.
(ii) We have $R = 5 - B$. Hence Propositions 11.3 and 11.4 give

$$\mathbb{E}(R) = \mathbb{E}(5 - B) = 5 - \mathbb{E}(B) = 5 - 5/4 = 15/4$$

and

$$\text{Var}(R) = \text{Var}(5 - B) = \text{Var}(B) = 45/112.$$

Q3. (i) Let $\mathbb{P}(Z = 0) = \mathbb{P}(Z = 3) = a$, $\mathbb{P}(Z = 1) = \mathbb{P}(Z = 2) = b$. Then $2a + 2b = 1$ so $a + b = 1/2$. We have

$$\mathbb{E}(Z) = 0a + b + 2b + 3a = 3a + 3b = 3(a + b) = 3/2.$$

Alternatively, Proposition 11.3(v) tells us that if a random variable is symmetrically distributed about a real number b then its expectation is b. In this case Z is symmetrically distributed about $3/2$ so $\mathbb{E}(Z) = 3/2$.
(ii)

$$\text{Var}(Z) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = (0 \times a + 1 \times b + 4 \times b + 9 \times a) - 9/4 = 9a + 5b - 9/4.$$

Since a and b are both probabilities we have $a, b \geq 0$ and $2a+2b = 1$. It is not difficult to see that the smallest $9a + 5b - 9/4$ can be is when $a = 0$ and $b = 1/2$. Then Z has probability mass function $\mathbb{P}(Z = 0) = \mathbb{P}(Z = 3) = 0$, $\mathbb{P}(Z = 1) = \mathbb{P}(Z = 2) = 1/2$.
Similarly, for the largest variance, we take $a = 1$ and $b = 0$. Then Z has probability mass function $\mathbb{P}(Z = 0) = \mathbb{P}(Z = 3) = 1/2$, $\mathbb{P}(Z = 1) = \mathbb{P}(Z = 2) = 0$.
Note that these agree with the idea that smallest variance means most sharply concentrated while largest variance means most spread out.
(iii) We have from part (ii) that $\text{Var}(Z) = 9a + 5b - 9/4$. Since $\text{Var}(Z) = 1$, this gives $9a + 5b = 13/4$. We also have $a + b = 1/2$. We may solve these simultaneous equations for a and b to deduce that $a = 3/16$ and $b = 5/16$ giving probability mass function

n	0	1	2	3
$P(Z = n)$	3/16	5/16	5/16	3/16

Q4.

(i)

$$\mathbb{P}(A \geq i) = \mathbb{P}(A = i) + \mathbb{P}(A = i+1) + \cdots + \mathbb{P}(A = n).$$

So,

$$\begin{aligned}
\sum_{i \geq 1}^{n} \mathbb{P}(A \geq i) &= \mathbb{P}(A \geq 1) + \mathbb{P}(A \geq 2) + \cdots + \mathbb{P}(A \geq n) \\
&= \mathbb{P}(A = 1) + \mathbb{P}(A = 2) + \mathbb{P}(A = 3) + \cdots + \mathbb{P}(A = n) \\
&+ \mathbb{P}(A = 2) + \mathbb{P}(A = 3) + \cdots + \mathbb{P}(A = n) \\
&+ \mathbb{P}(A = 3) + \cdots + \mathbb{P}(A = n) \\
&\vdots \\
&+ \mathbb{P}(A = n-1) + \mathbb{P}(A = n) \\
&+ \mathbb{P}(A = n).
\end{aligned}$$

For all $1 \leq i \leq n$, the term $\mathbb{P}(A = k)$ occurs precisely k times in the above expression and so

$$\sum_{i=1}^{n} \mathbb{P}(A \geq i) = \sum_{i=1}^{n} i\,\mathbb{P}(A = i) \ = \ \mathbb{E}(A)$$

as required.

(ii) If $\mathbb{E}(A) < 1$ then by part (i)

$$1 \ > \ \sum_{i=1}^{n} \mathbb{P}(A \geq i).$$

Since all the terms in this sum are non-negative we have that

$$\sum_{i=1}^{n} \mathbb{P}(A \geq i) \ \geq \ \mathbb{P}(A \geq 1).$$

Hence

$$1 \ > \ \mathbb{P}(A \geq 1) \ = \ 1 - \mathbb{P}(A = 0).$$

Rearranging gives $\mathbb{P}(A = 0) \ > 0$. In other words A takes the value 0 with positive probability.

AQ1

n	0	1	2
$P(X = n)$	1/8	1/4	5/8

Please let me know if you have any comments or corrections