Probability I – 2009/10

Solutions to Exercise Sheet 7

Q1. (a)

(i) These events do not in general partition S since some pair of them may intersect (it may be for instance that $A \cap B \neq \emptyset$ or $A \cap B^c \neq \emptyset$).

(ii) These events do partition S. The four events are "A but not B","B but not A", "both A and B", "neither A nor B". Every element of S is in exactly one of these.

(iii) These events do partition S. The three events are "both A and B", "exactly one of A and B", "neither A nor B". Every element of S is in exactly one of these.

(b) There are many possibilities $E_1 = \{1, 2, 3\}, E_2 = \{4, 5, 6\}, E_3 = \{7, 8, 9, 10\}$ is one.

Q2. Consider picking a voter at random with all choices equally likely. Let O be the event "the chosen voter voted for Obama" and U be the event "the chosen voter was aged under 30".

(i) By the Theorem of Total Probability (Theorem 9.3) using the partition of S into the two events U and U^c

$$\mathbb{P}(O) = \mathbb{P}(O|U)\mathbb{P}(U) + \mathbb{P}(O|U^c)\mathbb{P}(U^c)$$

The question asks for $\mathbb{P}(O|U^c)$. Putting in the values from the question

$$\frac{52.6}{100} = \frac{66}{100} \times \frac{18}{100} + \mathbb{P}(O|U^c) \times \left(1 - \frac{18}{100}\right)$$

which gives $\mathbb{P}(O|U^c) = 0.497$ to 3 decimal places.

(ii) This asks for $\mathbb{P}(U|O)$. By Bayes' Theorem (Theorem 9.5)

$$\mathbb{P}(U|O) = \frac{\mathbb{P}(O|U)\mathbb{P}(U)}{\mathbb{P}(O)} = \frac{\frac{66}{100} \times \frac{18}{100}}{\frac{52.6}{100}} = 0.226$$

to 3 decimal places.

Q3. Let E_0, E_1, E_2 be the events "neither player recovers", "exactly one player recovers" and "both players recover" respectively. Let W be the event "the match is won". By the Theorem of Total Probability (Theorem 9.3)

$$\mathbb{P}(W) = \mathbb{P}(W|E_0)\mathbb{P}(E_0) + \mathbb{P}(W|E_1)\mathbb{P}(E_1) + \mathbb{P}(W|E_2)\mathbb{P}(E_2).$$

Since the recoveries of the two players are independent we have

$$\mathbb{P}(E_0) = \left(\frac{2}{3}\right)^2 = \frac{4}{9}$$
$$\mathbb{P}(E_1) = 2\left(\frac{1}{3}\right)\left(\frac{2}{3}\right) = \frac{4}{9}$$
$$\mathbb{P}(E_2) = \left(\frac{1}{3}\right)^2 = \frac{1}{9}.$$

We are told the relevant conditional probabilities in the question and so

$$\mathbb{P}(W) = \frac{1}{16} \times \frac{4}{9} + \frac{1}{2} \times \frac{4}{9} + \frac{3}{4} \times \frac{1}{9} = \frac{1}{3}.$$

Q4*.

(i) Let D be the event "the double headed coin is chosen" and N be the event "the normal coin is chosen". Clearly $\mathbb{P}(N) = 2/3$, $\mathbb{P}(D) = 1/3$. Let H_1 be the event "the chosen coin comes down heads". The question asks for $\mathbb{P}(H_1)$.

By the Theorem of Total Probability (Theorem 9.3)

$$\mathbb{P}(H_1) = \mathbb{P}(H_1|D)\mathbb{P}(D) + \mathbb{P}(H_1|N)\mathbb{P}(N) = 1 \times \frac{1}{3} + \frac{1}{2} \times \frac{2}{3} = \frac{2}{3}.$$

(ii) We want $\mathbb{P}(D|H_1)$. By Bayes' Theorem (Theorem 9.5)

$$\mathbb{P}(D|H_1) = \frac{\mathbb{P}(H_1|D)\mathbb{P}(D)}{\mathbb{P}(H_1)} = \frac{1 \times 1/3}{2/3} = \frac{1}{2}.$$

(iii) Let H_2 be the event "a second toss of the same coin gives a head". We know that event H_1 occurred so we want $\mathbb{P}(H_2|H_1)$. By Theorem 9.4 (total probability for conditional probabilties),

$$\mathbb{P}(H_2|H_1) = \mathbb{P}(H_2|H_1 \cap D)\mathbb{P}(D|H_1) + \mathbb{P}(H_2|H_1 \cap N)\mathbb{P}(N|H_1)$$

The only term in this that we don't know is $\mathbb{P}(N|H_1)$ but $N = D^c$ and so $\mathbb{P}(N|H_1) = 1 - \mathbb{P}(D|H_1) = 1/2$. Hence,

$$\mathbb{P}(H_2|H_1) = 1 \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} = \frac{3}{4}.$$

AQ1(ii) 0.12

AQ2. Let X be the event "Andre wins the game", W_1 be the event "Andre wins the first point" and L_1 be the event "Andre loses the first point". The events W_1 and L_1 partition the sample space (since $L_1^c = W_1$) and so by the Theorem of Total Probability (Theorem 9.3):

$$\begin{aligned}\mathbb{P}(X) &= \mathbb{P}(X|W_1)\mathbb{P}(W_1) + \mathbb{P}(X|L_1)\mathbb{P}(L_1)\\ x &= u \times \frac{1}{4} + v \times \frac{3}{4}\\ 4x &= u + 3v.\end{aligned}$$

This is the first equation we wanted. To derive the second we need to consider u, that is $\mathbb{P}(X|W_1)$. Let W_2 be the event "Andre wins the second point" and L_2 be the event "Andre loses the second point". The events W_2 and L_2 partition the sample space and so by Theorem 9.4 (total probability applied to conditional probabilities)

$$\begin{aligned}u = \mathbb{P}(X|W_1) &= \mathbb{P}(X|W_1 \cap W_2)\mathbb{P}(W_2|W_1) + \mathbb{P}(X|W_1 \cap L_2)\mathbb{P}(L_2|W_1)\\ &= \mathbb{P}(X|W_1 \cap W_2)\mathbb{P}(W_2) + \mathbb{P}(X|W_1 \cap L_2)\mathbb{P}(L_2)\end{aligned}$$

since the outcome of the second point is independent of the outcome of the first. If Andre wins the first two points then he wins the game and so $\mathbb{P}(X|W_1 \cap W_2) = 1$. If he wins the first point and loses the second then the score is back to deuce (each player needs to establish a lead of 2 to win) and so $\mathbb{P}(X|W_1 \cap L_2) = x$. So

$$\begin{aligned}u &= 1 \times \frac{1}{4} + x \times \frac{3}{4}\\ 4u &= 1 + 3x\end{aligned}$$

Finally a similar argument applied to v gives

$$\begin{aligned}v = \mathbb{P}(X|L_1) &= \mathbb{P}(X|L_1 \cap W_2)\mathbb{P}(W_2|L_1) + \mathbb{P}(X|L_1 \cap L_2)\mathbb{P}(L_2|L_1)\\ &= \mathbb{P}(X|L_1 \cap W_2)\mathbb{P}(W_2) + \mathbb{P}(X|L_1 \cap L_2)\mathbb{P}(L_2)\\ &= x \times \frac{1}{4} + 0 \times \frac{3}{4}\\ 4v &= x.\end{aligned}$$

This gives the three equations. Solving them (say by using the second and third equation to substitute for u and v respectively in the first equation) we obtain $x = 1/10$.

This is considerably smaller that the probability that Andre wins each point. The conclusion is that the scoring system in tennis exaggerates the difference between players. In other words a player with a lower probability than their opponent of winning each point will have an even lower probability of winning the game.

Of course we have only checked this for probability 1/4 but the same is true in general. Try doing the same question but with Andre having probability p of winning each point.

The question considered a game starting from deuce but the same principle holds for a whole game. If you know the scoring system in tennis then you could try working out the probability that Andre wins a game.

Please let me know if you have any comments or corrections