Probability I – 2009/10

Solutions to Exercise Sheet 6

Q1. Let A be the event "they have two girls" and B be the event "their first child is a boy".

(i) You worked this out as part of question 1 on the previous sheet. Here is the calculation again:

$$\begin{aligned}\mathbb{P}(A) &= \mathbb{P}(\{gg, gbg, bgg, gbbg, bgbg, bbgg\}) \\ &= \left(\frac{1}{2}\right)^2 + 2\left(\frac{1}{2}\right)^3 + 3\left(\frac{1}{2}\right)^4 \\ &= \frac{11}{16}.\end{aligned}$$

(ii) We want $\mathbb{P}(A|B)$. By definition

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

We can find $\mathbb{P}(A \cap B)$ in a similar way to part (i)

$$\mathbb{P}(A \cap B) = \mathbb{P}(\{bgg, bgbg, bbgg\}) = \frac{1}{8} + \frac{2}{16} = \frac{1}{4}.$$

So

$$\mathbb{P}(A|B) = \frac{1/4}{1/2} = \frac{1}{2}.$$

(iii) This time we want $\mathbb{P}(B|A)$. By definition

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)}.$$

We worked out $\mathbb{P}(A \cap B)$ and $\mathbb{P}(A)$ earlier and so

$$\mathbb{P}(B|A) = \frac{1/4}{11/16} = \frac{4}{11}.$$

(iv) The event "their first child is a girl" is B^c. The questions asks for $\mathbb{P}(B^c|A)$ which by definition is given by

$$\mathbb{P}(B^c|A) = \frac{\mathbb{P}(A \cap B^c)}{\mathbb{P}(A)}.$$

Now

$$\mathbb{P}(A \cap B^c) = \mathbb{P}(\{gg, gbg, gbbg\} = \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{7}{16}$$

and so

$$\mathbb{P}(B^c|A) = \frac{7/16}{11/16} = \frac{7}{11}.$$

Q2.

(i)

$$\begin{aligned}
\mathbb{P}(A^c|B) &= \frac{\mathbb{P}(A^c \cap B)}{\mathbb{P}(B)} \quad \text{(by definition)} \\
&= \frac{\mathbb{P}(B) - \mathbb{P}(A \cap B)}{\mathbb{P}(B)} \quad \text{(by axiom 3)} \\
&= 1 - \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} \\
&= 1 - \mathbb{P}(A|B) \quad \text{(by definition)}
\end{aligned}$$

(ii) If $\mathbb{P}(A|B) < \mathbb{P}(A)$ then by definition

$$\frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} < \mathbb{P}(A).$$

Rearranging (using that $\mathbb{P}(A) > 0$ and $\mathbb{P}(B) > 0$) gives

$$\frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} < \mathbb{P}(B).$$

That is $\mathbb{P}(B|A) < \mathbb{P}(B)$.

In fact more is true. Either of these two inequalities is equivalent to the condition $\mathbb{P}(A \cap B) < \mathbb{P}(A)\mathbb{P}(B)$. This is rather like the relation between independence and equalities involving conditional probabilities described by Theorem 9.1. Events A, B satisfying $\mathbb{P}(A \cap B) < \mathbb{P}(A)\mathbb{P}(B)$ are said to be *negatively correlated*. Similarly events satisfying $\mathbb{P}(A \cap B) > \mathbb{P}(A)\mathbb{P}(B)$ are said to be *positively correlated*.

(iii) In question Q1(iii) you showed that $\mathbb{P}(B|A) = 4/11$. It follows from question Q2(i) that $\mathbb{P}(B^c|A) = 1 - 4/11 = 7/11$ and this is verified in question Q1(iv).

In question Q1(ii) you showed that $\mathbb{P}(A|B) = 1/2 < \mathbb{P}(A) = 11/16$. It follows from question Q2(ii) that $\mathbb{P}(B|A)$ must be less that $\mathbb{P}(B)$ and indeed in question Q1(iii) it is shown that $\mathbb{P}(B|A) = 4/11$ which is less that $1/2$.

Q3. This question illustrates an amusing and counterintuitve property of conditional probability.

Write A for the event "the patient receives treatment A", B for the event "the patient receives treatment B", M for the event "the patient is a man" and R for the event "the patient recovers". With these definitions you should be able to see that the question is asking you to calculate various conditional probabilities.

(i) For treatment A, 160 patients are treated of which 60 recover so

$$\mathbb{P}(R|A) = \frac{60}{160} = \frac{3}{8}.$$

(Of course you could equally well write $\frac{60/390}{160/390}$ for this and get the same answer). For treatment B, 230 patients are treated of which 65 recover so

$$\mathbb{P}(R|B) = \frac{65}{230} = \frac{13}{46} < \frac{3}{8}.$$

So there is a higher probability of recovery with treatment A. In terms of conditional probabilities

$$\mathbb{P}(R|A) > \mathbb{P}(R|B).$$

(ii) For treatment A, 100 men are treated of which 20 recover so

$$\mathbb{P}(R|A \cap M) = \frac{20}{100} = \frac{1}{5}.$$

For treatment B, 210 men are treated of which 50 recover so

$$\mathbb{P}(R|B \cap M) = \frac{50}{210} = \frac{5}{21} > \frac{1}{5}.$$

So for the men there is a higher probability of recovery with treatment B. In terms of conditional probabilities

$$\mathbb{P}(R|A \cap M) < \mathbb{P}(R|B \cap M).$$

(iii) For treatment A, 60 women are treated of which 40 recover so

$$\mathbb{P}(R|A \cap M^c) = \frac{40}{60} = \frac{2}{3}.$$

For treatment B, 20 women are treated of which 15 recover so

$$\mathbb{P}(R|B \cap M^c) = \frac{15}{20} = \frac{3}{4} > \frac{2}{3}.$$

So for the women there is a higher probability of recovery with treatment B. In terms of conditional probabilities

$$\mathbb{P}(R|A \cap M^c) < \mathbb{P}(R|B \cap M^c).$$

(iv) I think the answers to the previous parts are counterintuitive. In particular they show that if A, B, M and R are events with $\mathbb{P}(R|A \cap M) < \mathbb{P}(R|B \cap M)$ and $\mathbb{P}(R|A \cap M^c) < \mathbb{P}(R|B \cap M^c)$ then it does not necessarily follow that $\mathbb{P}(R|A) < \mathbb{P}(R|B)$. This is an example of a phenomenon called Simpson's paradox.

The paradox is caused because the experiment is badly designed. We can avoid it by making the ratio of patients given treatments A and B to be the same for both men and women. (You will learn more about the design of experiments if you take the level 6 module with the same title, MTH6116.)

AQ1(i) 1/2
AQ1(ii) 1/6

Please let me know if you have any comments or corrections