Probability I – 2009/10

Solutions to Exercise Sheet 5

Q1.

(i) If we record the outcome by listing the children in the order they were born, writing b for boy and g for girl, then the sample space is

$$\{gg, gbg, gbbg, gbbb, bgg, bgbg, bgbb, bbgg, bbgb, bbbg, bbbb\}.$$

(ii) By independence and the fact that each child is equally likely to be a boy or girl we have for example

$$\mathbb{P}(gg) = \left(\frac{1}{2}\right)^2 = \frac{1}{4}.$$

Similarly,

$$\mathbb{P}(gbg) = \left(\frac{1}{2}\right)^3 = \frac{1}{8}.$$

So $\mathbb{P}(gg) \neq \mathbb{P}(gbg)$ and so not all elements are equally likely.

Note that to answer this part we don't need to calculate the probability of *every* element of the sample space. It suffices just to find two which have different probabilities. (Of course, you could have chosen a different pair from me).

(iii)

$$\mathbb{P}(2\text{ girls}) = \mathbb{P}(\{gg, gbg, gbbg, bgg, bgbg, bbgg\}) = \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{8} + \frac{1}{16} + \frac{1}{16} = \frac{11}{16}.$$

Here we have used the same method as part (ii) to calculate the probability of each of the outcomes making up the event.

Q2. Clearly, $|S| = 52$, $|A| = 4$, $|R| = |M| = 26$, $|A \cap R| = |A \cap M| = 2$, $|R \cap M| = 13$ and $|A \cap R \cap M| = 1$. Since the deck is thoroughly shuffled we may assume that each card is equally likely to be chosen and so

$$\mathbb{P}(A) = \frac{4}{52} = \frac{1}{13},$$
$$\mathbb{P}(R) = \mathbb{P}(M) = \frac{26}{52} = \frac{1}{2},$$
$$\mathbb{P}(A \cap R) = \mathbb{P}(A \cap M) = \frac{2}{52} = \frac{1}{26},$$
$$\mathbb{P}(R \cap M) = \frac{13}{52} = \frac{1}{4},$$
$$\mathbb{P}(A \cap R \cap M) = \frac{1}{52}.$$

It follows that

$$\begin{aligned}
\mathbb{P}(A)\mathbb{P}(R) &= \frac{1}{13} \times \frac{1}{2} = \frac{1}{26} = \mathbb{P}(A \cap R) \\
\mathbb{P}(A)\mathbb{P}(M) &= \frac{1}{13} \times \frac{1}{2} = \frac{1}{26} = \mathbb{P}(A \cap M) \\
\mathbb{P}(R)\mathbb{P}(M) &= \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} = \mathbb{P}(R \cap M) \\
\mathbb{P}(A)\mathbb{P}(R)\mathbb{P}(M) &= \frac{1}{13} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{52} = \mathbb{P}(A \cap R \cap M)
\end{aligned}$$

which are precisely the conditions we need for A, R, M to be mutually independent.

Q3. Here we have $|S| = 36$, $|E| = |O| = 18$, $|Q| = |\{1, 4, 9, 16, 25, 26\}| = 6$, $|E \cap O| = 0$, $|Q \cap E| = |\{4, 16, 36\}| = 3$, $|Q \cap O| = |\{1, 9, 25\}| = 3$. All choices are equally likely so the probability of any event X is $\frac{|X|}{|S|}$.

(i) $\mathbb{P}(E) = \frac{18}{36} = \frac{1}{2}$, $\mathbb{P}(O) = \frac{18}{36} = \frac{1}{2}$, $\mathbb{P}(E \cap O) = \frac{0}{36} = 0$. It follows that

$$\mathbb{P}(E)\mathbb{P}(O) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} \neq 0 = \mathbb{P}(E \cap O)$$

and so E and O are not independent.

(ii) $\mathbb{P}(E) = \frac{18}{36} = \frac{1}{2}$, $\mathbb{P}(Q) = \frac{6}{36} = \frac{1}{6}$, $\mathbb{P}(E \cap Q) = \frac{3}{36} = \frac{1}{12}$. It follows that

$$\mathbb{P}(E)\mathbb{P}(Q) = \frac{1}{2} \times \frac{1}{6} = \frac{1}{12} = \mathbb{P}(E \cap Q)$$

and so E and Q are independent.

(iii) $\mathbb{P}(O) = \frac{18}{36} = \frac{1}{2}$, $\mathbb{P}(Q) = \frac{6}{36} = \frac{1}{6}$, $\mathbb{P}(O \cap Q) = \frac{3}{36} = \frac{1}{12}$. It follows that

$$\mathbb{P}(O)\mathbb{P}(Q) = \frac{1}{2} \times \frac{1}{6} = \frac{1}{12} = \mathbb{P}(O \cap Q)$$

and so O and Q are independent.

Q4. I can travel from A to B unless both roads are closed. By independence the probability that both roads are closed is p^2. So if I write X for the event "I cannot travel from A to B" then

$$\mathbb{P}(X) = p^2.$$

Similarly, if Y is the event "I cannot travel from B to C" then

$$\mathbb{P}(Y) = p^2.$$

I cannot travel from A to C if either (or both) of X and Y occur so we want $\mathbb{P}(X \cup Y)$ to be at most $1/2$. Now,

$$\mathbb{P}(X \cup Y) = \mathbb{P}(X) + \mathbb{P}(Y) - \mathbb{P}(X \cap Y).$$

The event $X \cap Y$ occurs precisely when all 4 roads are closed and the probability of this is p^4 (using independence again). So we require that

$$p^2 + p^2 - p^4 \leq \frac{1}{2}$$

This is satisfied if

$$2p^4 - 4p^2 + 1 \geq 0$$

Solving this quadratic in p^2 gives the condition

$$p^2 \leq 1 - \frac{\sqrt{2}}{2}$$

So we need that $p \leq 0.541$.

Q5. We have

$$\begin{aligned}
\mathbb{P}(A^c \cap B) &= \mathbb{P}(B) - \mathbb{P}(A \cap B) \quad \text{(by axiom 3)} \\
&= \mathbb{P}(B) - \mathbb{P}(A)\mathbb{P}(B) \quad \text{(by independence of } A \text{ and } B\text{)} \\
&= \mathbb{P}(B)(1 - \mathbb{P}(A)) \\
&= \mathbb{P}(A^c)\mathbb{P}(B) \quad \text{(by Proposition 3.1)},
\end{aligned}$$

and so A^c and B are independent.

This is illustrated by parts (ii) and (iii) of question 3. There we have that $O = E^c$. In part (ii) you showed that E and Q are independent. It follows by the result we have just proved that E^c and Q (that is O and Q) are also independent. This is verified in part (iii).

AQ1 $p = 1 - (0.1)^{1/3} \sim 0.5358$

AQ2 0.916

AQ3 $(3 - \sqrt{5})/2$

Please let me know if you have any comments or corrections