Probability I – 2009/10

Solutions to Exercise Sheet 4

Q1*.

Choosing the number involves making an ordered choice of 4 things from 10 with repetition allowed. If we write $X = \{0, 1, 2, \ldots, 9\}$ then the sample space is X^4 (that is the set of all ordered 4-tuples whose entries are elements of X). There are $10^4 = 10000$ of these so $|S| = 10000$ (of course this is just the same as the number of integers between 0 and 9999).

(i) The number of ways of choosing the number to have all even digits is 5^4 (there are 5 even numbers in X so we are making an ordered choice of 4 things from 5 with replacement). Hence the probability is $5^4/10^4 = (1/2)^4 = 1/16 = 0.0625$.

(ii) If we do not allow repetition we have only $10 \times 9 \times 8 \times 7$ choices for the number. Hence the probability of this is $10 \times 9 \times 8 \times 7/10^4 = 63/125 = 0.504$.

(iii) The number is palindromic means it is of the form $xyyx$. So once we have chosen the first 2 digits (which we can do in 10^2 ways) the number is completely determined. Hence the probability is $10^2/10^4 = 1/100 = 0.01$.

(iv) The number of ways of choosing the number to have all digits in $\{0, 1, 2, 3, 4, 5, 6, 7\}$ is 8^4 (there are 8 choices for each of the 4 digits). Hence the probability is $8^4/10^4 = (4/5)^4 = 256/625 = 0.4096$.

(v) If the digits are to be in *strictly* increasing order then there can be no repetitions and each *unordered* selection of 4 distinct digits occurs exactly once. It follows that exactly $\binom{10}{4} = \frac{10!}{4!\,6!} = 10 \times 3 \times 7$ numbers have this property. The probability is therefore $210/10000 = 0.021$.

Q2.

(i) This is unordered selection without replacement. The sample space has cardinality $\binom{18}{11}$ (this is the number of ways of picking 11 things from 18 unordered without replacement). The number of ways of picking 6 batsmen and 5 bowlers is $\binom{10}{6}\binom{8}{5}$ (the first term being the number of ways of choosing the 6 batsmen from the 10 batsmen in the squad, the second term being the number of ways of choosing the 5 bowlers from the 8 bowlers in the squad). So

$$\mathbb{P}(\text{the team contains 6 batsmen and 5 bowlers}) = \frac{\binom{10}{6}\binom{8}{5}}{\binom{18}{11}}.$$

It is reasonable to leave your answer in this form but you might want to work it out numerically. The key thing to remember here is to cancel things as much as possible:

$$\begin{aligned}
\frac{\binom{10}{6}\binom{8}{5}}{\binom{18}{11}} &= \frac{\frac{10!}{6!4!}\frac{8!}{5!3!}}{\frac{18!}{11!7!}} \\
&= \frac{\frac{10\times9\times8\times7}{4\times3\times2}\frac{8\times7\times6}{3\times2}}{\frac{18\times17\times16\times15\times14\times13\times12}{7\times6\times5\times4\times3\times2}} \\
&= \frac{10\times3\times7\times8\times7}{18\times17\times8\times13} \\
&= \frac{5\times7\times7}{3\times17\times13} \\
&= \frac{245}{663}
\end{aligned}$$

(ii) To find this event we use the fact that $\mathbb{P}$(the team contains fewer than 3 bowlers) is equal to the sum of $\mathbb{P}$(the team contains 10 batsmen and 1 bowler) and $\mathbb{P}$(the team contains 9 batsmen and 2 bowlers). Each of these probabilities can be worked out as in part (i). (I'll leave the details to you this time).

$$\mathbb{P}(\text{the team contains fewer than 3 bowlers}) = \frac{\binom{10}{10}\binom{8}{1}}{\binom{18}{11}} + \frac{\binom{10}{9}\binom{8}{2}}{\binom{18}{11}} = \frac{2}{221}$$

Q3.

(i) Choosing a function means choosing a value in the codomain for each element in the domain (with repetition allowed), so in this case it's an ordered choice of 2 things from a 3 element set with replacement. Hence $|S| = 3 \times 3 = 9$.

For an injective function we don't allow repetition. The number of such functions is therefore $3 \times 2 = 6$ and the probability is $6/9 = 2/3$.

(ii) The same argument as for (i) (only this time choosing 3 things from a 2 element set) gives $|S| = 2^3 = 8$.

There are no injections since the domain is larger than the codomain (Proposition 3.1) and so this probability is 0.

(iii) The same argument as for (i) (only this time choosing m things from an n element set) gives $|S| = n^m$

For an injective function we don't allow repetition. If $m > n$ then there are no such functions so the probability is 0. If $m \leq n$ then the number of such functions is $n \times (n-1) \times (n-2) \times \cdots \times (n-m+1) = \frac{n!}{(n-m)!}$. So we have

$$\mathbb{P}(h \text{ is injective}) = \begin{cases} 0 & \text{if } m > n \\ \frac{n\times(n-1)\times(n-2)\times\cdots\times(n-m+1)}{n^m} & \text{if } m \leq n \end{cases}$$

Q4.
The number of subsets of $\{1, 2, \dots, n\}$ of cardinality r is $\binom{n}{r}$. This is therefore the cardinality of our sample space.

(i) To choose a subset containing 1 we must pick 1 and a further $r-1$ elements from the set $\{2, 3, \dots, n\}$. There are $\binom{n-1}{r-1}$ ways to do this. It follows that

$$\mathbb{P}(\text{the set contains 1}) = \frac{\binom{n-1}{r-1}}{\binom{n}{r}}.$$

(ii) To choose a subset which does not contain 1 we must pick r elements from the set $\{2, 3, \dots, n\}$. There are $\binom{n-1}{r}$ ways to do this. It follows that

$$\mathbb{P}(\text{the set does not contain 1}) = \frac{\binom{n-1}{r}}{\binom{n}{r}}.$$

(iii) These are the probabilities of a pair of complementary events so

$$\mathbb{P}(\text{the set contains 1}) + \mathbb{P}(\text{the set does not contain 1}) = 1.$$

That is

$$\frac{\binom{n-1}{r-1}}{\binom{n}{r}} + \frac{\binom{n-1}{r}}{\binom{n}{r}} = 1.$$

Rearranging gives that

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}.$$

AQ1(i) $\binom{30}{6}/\binom{60}{6}$
(ii) $\binom{30}{1}\binom{30}{5}/\binom{60}{6}$
(iii) $\binom{30}{2}\binom{30}{4}/\binom{60}{6}$
(iv) $\binom{30}{3}\binom{30}{3}/\binom{60}{6}$

AQ2(i) $4 \times \binom{13}{5}/\binom{52}{5}$
(ii) $10 \times 4^5/\binom{52}{5}$
(iii) $13 \times 48/\binom{52}{5}$
(iv) $13 \times \binom{4}{3} \times 12 \times \binom{4}{2}/\binom{52}{5}$

AQ3 $1 - (2^m - 1)/3^{m-1}$

Please let me know if you have any comments or corrections