Probability I – 2009/10

Solutions to Exercise Sheet 3

Q1. The key here is to remember how to interpret the set operations ($\cup, \cap$ etc) as "or", "and" etc.

a) (i) $G \cap H$

(ii) $(F \cap G) \setminus H$ (or $F \cap G \cap H^c$)

(iii) $(F \cap G \cap H^c) \cup (F \cap G^c \cap H) \cup (F^c \cap G \cap H)$ (there are other possibilities),

b) (i) This is the event "the chosen student does not speak Hungarian".

(ii) This is the event "the chosen student speaks none of the languages"

(iii) This is also the event "the chosen student speaks none of the languages" (notice how the same event can be expressed in ways that look different).

c) The first sentence means that $|S| = 100$ and $|G| = 30$. The second sentence means that $F \cap G = \emptyset$ and $H \subset G$. Again there are equivalent ways of expressing this, for example $F \subset G^c$ and $H \cap G^c = \emptyset$

Q2*.

(a) (i) $\mathbb{P}(A^c) = 1 - \mathbb{P}(A) = 3/5$ (by Proposition 5.1)

(ii) For this question we need the inclusion-exclusion formula for two events (Proposition 5.6).

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) = 2/5 + 1/2 - 3/20 = 3/4$$

(iii)

$$\begin{aligned} \mathbb{P}(A^c \cap B) &= \mathbb{P}(B) - \mathbb{P}(A \cap B) \quad \text{(by Axiom 3 rearranged)} \\ &= 1/2 - 3/20 = 7/20 \end{aligned}$$

(iv) $B \setminus A = A^c \cap B$ and so this probability is the same as that calculated in part (iii). So $\mathbb{P}(B \setminus A) = 7/20$.

(b) This is part (a) in disguise. If we write A for the event "the book is non-fiction" and B for the event "the book is a hardback" then the probabilities and events are the same as in part (a).

(i) This asks for $\mathbb{P}(A^c)$. We worked this out as 3/5 in the part (a).

(ii) Here we want $\mathbb{P}(A \cup B)$. We worked this out to be 3/4.

(iii) This is $\mathbb{P}(A^c \cap B)$ or $\mathbb{P}(B \setminus A)$ so it's 7/20.

Q3.

(i) If we let $E_1 = A$, $E_2 = B \setminus A$ and $E_3 = C \setminus (A \cup B)$ then E_1, E_2, E_3 are pairwise disjoint and $E_1 \cup E_2 \cup E_3 = A \cup B \cup C$ (draw a Venn diagram if you're not convinced). So

$$\begin{aligned} \mathbb{P}(A \cup B \cup C) &= \mathbb{P}(E_1 \cup E_2 \cup E_3) \\ &= \mathbb{P}(E_1) + \mathbb{P}(E_2) + \mathbb{P}(E_3) \quad \text{(by Axiom 3)} \\ &= \mathbb{P}(A) + \mathbb{P}(B \setminus A) + \mathbb{P}(C \setminus (A \cup B)). \end{aligned}$$

(ii) We have that $E_2 \subset B$ and $E_3 \subset C$ and so by Proposition 5.4 $\mathbb{P}(E_2) \leq \mathbb{P}(B)$ and $\mathbb{P}(E_3) \leq \mathbb{P}(C)$. Putting this into part (i) gives

$$\mathbb{P}(A \cup B \cup C) \leq \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C).$$

(iii) If $A_1, A_2, \ldots, A_n$ are events and for $1 \leq i \leq n$ we let

$$E_i = A_i \setminus (A_1 \cup A_2 \cup \cdots \cup A_{i-1})$$

then as before $E_1, E_2, \ldots, E_n$ are pairwise disjoint and

$$E_1 \cup E_2 \cup \cdots \cup E_n = A_1 \cup A_2 \cup \cdots \cup A_n.$$

So

$$\begin{aligned} \mathbb{P}(A_1 \cup A_2 \cup \cdots \cup A_n) &= \mathbb{P}(E_1 \cup E_2 \cup \cdots \cup E_n) \\ &= \mathbb{P}(E_1) + \mathbb{P}(E_2) + \cdots + \mathbb{P}(E_n) \quad \text{(by Axiom 3).} \end{aligned}$$

Now, as before $E_i \subset A_i$ and so $\mathbb{P}(E_i) \leq \mathbb{P}(A_i)$. It follows that

$$\mathbb{P}(A_1 \cup A_2 \cup \cdots \cup A_n) \leq \mathbb{P}(A_1) + \mathbb{P}(A_2) + \cdots + \mathbb{P}(A_n).$$

This generalises (ii) to n events.

Please let me know if you have any comments or corrections