MTH4108 Probability 1 – 2009/10

Exercise Sheet 3

These questions are designed to help you understand the material covered in week 3 lectures. You should write up your solution to the starred question, Q2,* **clearly** *and hand it in during your week 4 exercise class for feedback. Put your* **full name and student number** *on the top of your solution. It is important that you make a serious attempt to do* **all** *of questions Q1-Q3 before week 4 lectures begin. Questions AQ1-AQ3 are for additional practice. You should attempt them when you have time.*
This first question recalls the notion of events as subsets of the sample space from week 1. The remainder of the sheet covers the axioms and basic properties of probability.

Q1. One student is chosen at random from a set S of students. Let F be the event "the chosen student speaks French", G be the event "the chosen student speaks German", and H be the event "the chosen student speaks Hungarian".

(a) Express the following events in symbols:

(i) the chosen student speaks both German and Hungarian,

(ii) the chosen student speaks French and German but not Hungarian,

(iii) the chosen student speaks exactly two of the languages.

(b) Express the following events in words without using any mathematical symbols or terminology:

(i) H^c,

(ii) $F^c \cap G^c \cap H^c$,

(iii) $(F \cup G \cup H)^c$.

(c) You are told that there are 100 students in total and that exactly 30 of them speak German. You are also told that no student speaks both French and German and that all Hungarian speakers also speak German. Express each of these statements using mathematical symbols.

Q2*.

(a) Let A and B be events with $\mathbb{P}(A) = 2/5$, $\mathbb{P}(B) = 1/2$, $\mathbb{P}(A \cap B) = 3/20$. Calculate the following probabilities.

(i) $\mathbb{P}(A^c)$

(ii) $\mathbb{P}(A \cup B)$

(iii) $\mathbb{P}(A^c \cap B)$

(iv) $\mathbb{P}(B \setminus A)$

Justify each answer with either an axiom or a result from lectures.

(b) Of the books in a public library 40% are non-fiction books, 50% are hardback books, and 15% are hardback non-fiction books. A book is chosen at random from the library with all choices equally likely. What are the probabilities that it is

(i) a fiction book,

(ii) either a hardback book or non-fiction,

(iii) a hardback fiction book.

Q3.

(i) Let A, B, C be events. Explain why

$$\mathbb{P}(A \cup B \cup C) = \mathbb{P}(A) + \mathbb{P}(B \setminus A) + \mathbb{P}(C \setminus (A \cup B)).$$

(ii) Deduce that

$$\mathbb{P}(A \cup B \cup C) \leq \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C).$$

(iii) Formulate and prove a version of part (ii) for n events.

AQ1. Let $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and for $A \subseteq S$ define a probability by

$$\mathbb{P}(A) = \frac{1}{12} \left(|A \cap \{1, 2, 3, 4\}| + 2|A \cap \{5, 6, 7, 8\}| \right).$$

(i) Verify that this satisfies the axioms for probability.

(ii) Give an example of a physical situation which you would expect the mathematics above to describe.

AQ2. Prove that if A and B are events then

(i) $\mathbb{P}(A \cap B) \geq \mathbb{P}(A) + \mathbb{P}(B) - 1$

(ii) $\mathbb{P}(A \triangle B) = \mathbb{P}(A) + \mathbb{P}(B) - 2\mathbb{P}(A \cap B)$.

Justify each step in your proofs with either an axiom or a result from lectures.

AQ3. (harder) Try to work out what the inclusion-exclusion formula for n events is (that is find an expression for $\mathbb{P}(A_1 \cup \cdots \cup A_n)$ in terms of the probabilities of various intersections of $A_1, \ldots, A_n$). (It is often a good idea to think about the simplest unknown case of a problem. We saw the cases $n = 2$ and $n = 3$ in lectures so look at $n = 4$ first.)