

Probability I – 2009/10

Solutions to Exercise Sheet 2

Q1.

- (i) This is not a function from A to B since $f(1) = 0 \notin B$.
- (ii) This is a function from A to B since the given rule defines a unique element of B for each element of A .
- (iii) This is a function, each real number is mapped to a single real number. Note that this is just one function even though it is defined by three different expressions depending on which part of A we are in.
- (iv) This is not a function since it doesn't assign a single value to 1 (the second line of the definition suggests that $f(1)$ should be 1, the third line suggests it should be 0). We say in this case that f is not well-defined at 1. Note that there is no such problem at 0; the first and second lines of the definition both define $f(0)$ to be 0.

Q2. Note that a single violation of the definition is enough to show that a function is *not* injective/surjective but a short argument is needed to show that a function is injective/surjective.

- (i) This is injective (if $f(x) = f(y)$ then $3x + 4 = 3y + 4$ and so $x = y$).
It is not surjective (no element of \mathbb{N} maps to 1).
It is not bijective and so does not have an inverse.
- (ii) This is injective (again if $f(x) = f(y)$ then $3x + 4 = 3y + 4$ and so $x = y$).
It is surjective (for any $x \in \mathbb{R}$ we have $\frac{x-4}{3} \in \mathbb{R}$ and $f(\frac{x-4}{3}) = x$).
It is bijective. Since it is bijective it has an inverse function, g , given by $g(x) = \frac{x-4}{3}$.

(iii) This is not injective ($f(0) = f(\pi)$).

It is surjective ($\sin(x)$ takes all values in the interval $[-1, 1]$).

It is not bijective and so does not have an inverse.

(iv) Drawing a quick sketch of this is probably helpful if you haven't done so yet. Notice that $f(x) \geq 0$ if and only if $x \geq 0$.

The function is injective (if $f(x) = f(y) \geq 0$ then $x^2 = y^2$ and $x, y \geq 0$ so $x = y$, if $f(x) = f(y) < 0$ then $-x^2 = -y^2$ and $x, y < 0$ and so $x = y$).

It is surjective (any non-negative $x \in \mathbb{R}$ is the value of f at $+\sqrt{x}$, any negative $x \in \mathbb{R}$ is the value of f at $-\sqrt{-x}$).

It is bijective. Since it is bijective it has an inverse function, g , given by

$$g : x \mapsto \begin{cases} +\sqrt{x} & \text{if } x \geq 0 \\ -\sqrt{-x} & \text{if } x < 0 \end{cases}$$

Q3.

- (i) For t to be a function we need that every course is taught by some lecturer in the department (otherwise the value of t at that course may not be defined or not be in L) and that no course is taught by more than one lecturer (otherwise the value of t at a course will not be well-defined – as in question 1iv)
- (ii) If the function is injective then no lecturer teaches more than one course.
- (iii) If the function is surjective then every lecturer teaches at least one course.

Q4.

- (i) One possibility is $f : \mathbb{N} \rightarrow \mathbb{N}$, $f(x) = x + 1$. Another possibility would be $f(x) = x^2$ and another would be $f(x) = 2x$. There are lots of other such functions so your answer may well be correct even if it is not one of these.
- (ii) One possibility is $f : \mathbb{N} \rightarrow \mathbb{N}$, $f(1) = 1$, $f(x) = x - 1$ for all $x \geq 2$. Again there are many other possibilities.

(iii) It is not possible to find such a function which is injective but not surjective. Indeed, if f is injective it takes a different value at each element of A . So the set of values it takes contains $|A|$ elements and hence is the whole of A . It follows that f is surjective.

Similarly, if f is surjective then it takes $|A|$ different values. It follows that it must take a different value at each element of A . Hence f is injective.

Q5.

(i) There are 8 such triples

$(0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 0), (1, 0, 1), (0, 1, 1), (1, 1, 1)$.

(ii) There are 8 such functions (have a go at writing them out).

The answers to these parts are the same since there is a ‘natural bijection’ between the two sets. Let’s see how a triple in (i) corresponds to a function in (ii).

Given a triple (x, y, z) we can define a function f by putting $f(1) = x$, $f(2) = y$ and $f(3) = z$. So for example the triple $(1, 0, 1)$ corresponds to the function $f : \{1, 2, 3\} \rightarrow \{0, 1\}$ given by $f(1) = 1$, $f(2) = 0$ and $f(3) = 1$.

This correspondence gives a bijection between the set of triples and the set of functions. As we saw in lectures if there is a bijection between two finite sets then they have the same cardinality.

Please let me know if you have any comments or corrections