Probability I – 2009/10

Solutions to Exercise Sheet 11

Q1. (i) Each of A and B is a random variable taking values in $\{0, 1, 2\}$. To find the joint distribution we need to find the probability

$$\mathbb{P}(A = i \text{ and } B = j)$$

for each pair $i, j \in \{0, 1, 2\}$.

$$\mathbb{P}(A = 1, B = 0) = \mathbb{P}(\{13, 14, 15, 16, 31, 41, 51, 61\}) = 8/36 = 2/9,$$

where, as usual we are expressing the outcome "1st die shows i, 2nd die shows j" by ij.)
Working this out similarly for all pairs we get the joint distribution:

		A = 0	A = 1	A = 2
B	0	4/9	2/9	1/36
	1	2/9	1/18	0
	2	1/36	0	0

(ii) To find the covariance it is probably simplest to use the formula from Proposition 15.6(a) in the notes. That is

$$\text{Cov}(A, B) = \mathbb{E}(AB) - \mathbb{E}(A)\mathbb{E}(B).$$

The only outcome with positive probability for which AB is non-zero is $A = 1, B = 1$; this outcome has probability $1/18$ and so $\mathbb{E}(AB) = 1 \times 1 \times 1/18 = 1/18$. You can work out $\mathbb{E}(A)$ by finding the probability mass function of A (given by the column sums in the above table).

x	0	1	2
$P(A = x)$	25/36	5/18	1/36

and so $\mathbb{E}(A) = 1 \times 5/18 + 2 \times 1/36 = 1/3$. Similarly $\mathbb{E}(B) = 1/3$. It follows that,

$$\text{Cov}(A, B) = 1/18 - 1/3 \times 1/3 = -1/18.$$

The correlation coefficient is defined to be

$$\frac{\text{Cov}(A, B)}{\sqrt{\text{Var}(A)\text{Var}(B)}}.$$

In our case, we can find Var(A) = 5/18, Var(B) = 5/18 (using the marginal distributions) and so

$$\text{corr}(A,B) = \frac{-1/18}{\sqrt{5/18 \times 5/18}} = -1/5.$$

(iii) A and B are not independent since $\text{Cov}(A,B) \neq 0$.

Q2. Firstly, you can work out the expectation and variance of each of X, Y and Z (this is an exercise in what you remember about distributions).

$$\begin{aligned} \mathbb{E}(X) &= 7/6 \quad \text{Var}(X) = 35/36 \\ \mathbb{E}(Y) &= 2 \quad \text{Var}(Y) = 2 \\ \mathbb{E}(Z) &= 6 \quad \text{Var}(Z) = 6 \end{aligned}$$

(i) $\mathbb{E}(X+Y) = \mathbb{E}(X) + \mathbb{E}(Y) = 7/6 + 2 = 19/6$.

(ii) $\mathbb{E}(X+Z) = \mathbb{E}(X) + \mathbb{E}(Z) = 7/6 + 6 = 43/6$ (It doesn't matter that X and Z are not independent the expectation of their sum is still the sum of their expectations).

(iii) $\mathbb{E}(X + 2Z + 3Z) = \mathbb{E}(X) + 2\mathbb{E}(Y) + 3\mathbb{E}(Z) = 7/6 + 4 + 18 = 139/6$ (Again it doesn't matter that we don't have independence.......)

(iv) ... but here we do need independence). Because X and Y are independent, $\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y) = 35/36 + 2 = 107/36$.

(v) $\text{Var}(X+Z)$ cannot be determined.

(vi) $\text{Var}(X + 2Y + 3Z)$ cannot be determined.

Q3.

(i) To have $X + Y = n$ we need $X = k$ and $Y = n - k$ for some k with $0 \le k \le n$. Hence

$$\mathbb{P}(X+Y=n) = \sum_{k=0}^{n} \mathbb{P}(X = k \text{ and } Y = n-k).$$

By independence

$$\mathbb{P}(X+Y=n) = \sum_{k=0}^{n} \mathbb{P}(X=k)\mathbb{P}(Y=n-k).$$

Now substituting in the pmf of a Poisson random variable

$$\mathbb{P}(X+Y=n) = \sum_{k=0}^{n} e^{-\lambda}\frac{\lambda^k}{k!} e^{-\mu}\frac{\mu^{n-k}}{(n-k)!}.$$

(ii)

$$\begin{aligned}\mathbb{P}(X+Y=n) &= \sum_{k=0}^{n} e^{-\lambda}\frac{\lambda^k}{k!}e^{-\mu}\frac{\mu^{n-k}}{(n-k)!} \\ &= e^{-(\lambda+\mu)}\sum_{k=0}^{n}\frac{1}{n!}\binom{n}{k}\lambda^k\mu^{n-k} \\ &= e^{-(\lambda+\mu)}\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}\lambda^k\mu^{n-k} \\ &= e^{-(\lambda+\mu)}\frac{1}{n!}(\lambda+\mu)^n.\end{aligned}$$

where the last line is an application of the binomial theorem.

This is the pmf of a Poisson$(\lambda+\mu)$ distribution and so

$$X+Y \sim \text{Poisson}(\lambda+\mu).$$

AQ1.

(i) With the X_i defined as on the sheet we have that each is a Bernoulli random variable where the probability of success is the probability that the ith ball is red.

There are m choices for the ith ball and $(n-1)\times(n-2)\times\cdots\times(n-i+1)$ for the others and so this probability is

$$\frac{m\times(n-1)\times(n-2)\times\cdots\times(n-i+1)}{n\times(n-1)\times\cdots\times(n-i+1)} = \frac{m}{n}.$$

So $X_i \sim \text{Bernoulli}(m/n)$. That is the pmf is

x	0	1
$P(X_i = x)$	$(1-m/n)$	m/n

Since X is the number of red balls we have that

$$X = X_1 + X_2 + \cdots + X_r.$$

Notice that this is very similar to the way we wrote a Binomial random variable as the sum of n indicator variables. An important difference is that in the Binomial case the indicator variables are independent while here they are not.

(ii) Even though the indicator variables are not independent we still have

$$\mathbb{E}(X) = \mathbb{E}(X_1+X_2+\cdots+X_r) = \mathbb{E}(X_1)+\mathbb{E}(X_2)+\cdots+\mathbb{E}(X_r) = r\frac{m}{n}.$$

(iii) Since X_i takes values 0 and 1 only, $X_i^2 = X_i$. So X_i^2 is also a Bernoulli(m/n) random variable.

(iv) The random variable X_iX_j takes the value 1 if the ith and jth balls are both red, and 0 otherwise. So it is another Bernoulli random variable and the probability of success is the probability that both the ith and jth balls are red. This probability is

$$\frac{m \times (m-1) \times (n-2) \times (n-3) \times \cdots \times (n-i+1)}{n \times (n-1) \times \cdots \times (n-i+1)} = \frac{m(m-1)}{n(n-1)}.$$

So if $i \neq j$ then $X_iX_j \sim$ Bernoullli $\left(\frac{m(m-1)}{n(n-1)}\right)$.

(v)

$$X^2 = (X_1 + X_2 + \cdots + X_r)^2 = \sum_{i=1}^{r} X_i^2 + \sum_{i \neq j} X_iX_j$$

Using (iii) we have $\mathbb{E}(X_i^2) = \frac{m}{n}$ for each of the r values of i and, using (iv), $\mathbb{E}(X_iX_j) = \frac{m(m-1)}{n(n-1)}$ for each of the $r(r-1)$ values of $i \neq j$ and so

$$\mathbb{E}(X^2) = r\frac{m}{n} + r(r-1)\frac{m(m-1)}{n(n-1)}$$

Finally

$$\begin{aligned}
\mathrm{Var}(X) &= \mathbb{E}(X^2) - (\mathbb{E}(X))^2 \\
&= r\frac{m}{n} + r(r-1)\frac{m(m-1)}{n(n-1)} - r^2\frac{m^2}{n^2} \\
&= r\frac{m}{n}\left(1 + \frac{(r-1)(m-1)}{n-1} - \frac{rm}{n}\right) \\
&= r\frac{m}{n}\left(\frac{n^2 - n + rmn - rn - mn + n - rmn + rm}{n(n-1)}\right) \\
&= r\frac{m}{n}\frac{(n-m)(n-r)}{n(n-1)} \\
&= r\frac{m}{n}\left(1 - \frac{m}{n}\right)\frac{n-r}{n-1}
\end{aligned}$$

Please let me know if you have any comments or corrections