MTH4108 Probability 1 – 2009/10 – Exercise Sheet 11

These questions are designed to help you understand joint distributions. You should discuss them in your week 12 exercise class. It is important that you make a serious attempt to do questions Q1-Q3 before the second semester begins. Question AQ1 concerns the hypergeometric distribution which is not examinable, but attempting this question will aid your understanding of joint distribution. I will put solutions on the module web page after Christmas.

Q1. Two fair standard dice are rolled. Let A be the number of 1s seen and B be the number of 2s seen in the outcome.

(i) Find the joint distribution of A and B.

(ii) Find the covariance of A and B and the correlation coefficient of A and B.

(iii) Are A and B independent?

Q2. Suppose that X, Y, Z are random variables with $X \sim \text{Bin}(7, 1/6)$, $Y \sim \text{Geom}(1/2)$, $Z \sim \text{Poisson}(6)$. Suppose further that X and Y are independent but that X and Z are not independent. Which of the following can be determined from this information? Find the value of those which can be determined.

(i) $\mathbb{E}(X + Y)$

(ii) $\mathbb{E}(X + Z)$

(iii) $\mathbb{E}(X + 2Y + 3Z)$

(iv) $\text{Var}(X + Y)$

(v) $\text{Var}(X + Z)$

(vi) $\text{Var}(X + 2Y + 3Z)$.

Q3. Let X be the number of fish caught by a fisherman and Y be the number of fish caught by a second fisherman in one afternoon of fishing. Suppose that X is distributed Poisson(λ) and Y is distributed Poisson(μ). Suppose further that X and Y are independent random variables.

(i) Show that

$$\mathbb{P}(X + Y = n) = \sum_{k=0}^{n} e^{-\lambda} \frac{\lambda^k}{k!} e^{-\mu} \frac{\mu^{n-k}}{(n-k)!}.$$

(ii) Hence find the probability distribution of the total number of fish caught i.e. determine which of the special distributions it belongs to.

AQ1 Suppose that a bag contains n balls, m of which are red and the remaining $n-m$ of which are blue. We make a selection of r balls without replacement. Let X be the number of red balls among the r balls chosen. Recall that the random variable X has the *hypergeometric distribution* (see Section 12.5 in the notes on the module web page). This question leads you through the calculation of the expectation and variance of X. The method is similar to the way we used joint distributions to calculate the expectation and variance of a binomial random variable.

(i) Regard the selection as being made with order and define random variables $X_1, \ldots, X_r$ by

$$X_i = \begin{cases} 0 & \text{if the } i\text{th ball chosen is blue} \\ 1 & \text{if the } i\text{th ball chosen is red} \end{cases}$$

Find the probability mass function of each X_i (hint: they each have the same probability distribution) and express X in terms of the X_i.

(ii) Hence show that the expectation of X is $r\frac{m}{n}$.

(iii) What is the probability mass function of X_i^2?

(iv) Determine the probability mass function of $X_i X_j$ for $i \neq j$.

(v) Hence find $\mathbb{E}(X^2)$ and deduce that

$$\mathrm{Var}(X) = r\frac{m}{n}\left(1 - \frac{m}{n}\right)\frac{n-r}{n-1}.$$