MTH4108 Probability 1 – 2009/10 – Exercise Sheet 10

These questions are designed to help you understand cumulative distribution functions and continuous random variables. It is important that you make a serious attempt to do questions Q1-Q2 before the test in week 12. (I only set two questions in order to give you time to revise for your tests.) Questions AQ1-AQ5 are for additional practice. You should attempt them when you have time.

Q1.

(a) A discrete random variable A has the following probability mass function:

n	-1	0	1	2
$P(A=n)$	$1/10$	$3/5$	$1/5$	$1/10$

Find the cumulative distribution function of A.

(b) A discrete random variable B has cumulative distribution function:

$$F_B(r) = \begin{cases} 0 & \text{if } r < 1/2 \\ 1/5 & \text{if } 1/2 \leq r < 3 \\ 3/5 & \text{if } 3 \leq r < 5 \\ 1 & \text{if } 5 \leq r \end{cases}$$

Write down the range of B. Find the probability mass function of B.

Q2. A continuous random variable X has cumulative distribution function:

$$F_X(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ \sqrt{x} & \text{if } 0 < x \leq 1 \\ 1 & \text{if } x > 1 \end{cases}$$

(i) Find the probability density function of X.

(ii) Calculate the expectation and variance of X.

(iii) Calculate the lower quartile of X.

AQ1. A continuous random variable Y has probability density function:

$$f_Y(y) = \begin{cases} 0 & \text{if } y < 1 \\ (y-1) & \text{if } 1 \leq y < 2 \\ (3-y) & \text{if } 2 \leq y < 3 \\ 0 & \text{if } y \geq 3 \end{cases}$$

(i) Find the cumulative distribution function of Y.

(ii) Use the properties of a cumulative distribution function to write down a couple of ways in which you could check that your answer to part (i) is plausible. Perform those checks and revisit part (i) if appropriate.

(iii) Calculate $\mathbb{P}(3/2 < Y < 5/2)$ by evaluating the cumulative distribution function at appropriate places.

(iv) Calculate $\mathbb{P}(3/2 < Y < 5/2)$ by integrating the probability density function and check you get the same answer.

AQ2. Let S be the right-angled triangle with vertices $(0,0),(1,0),(1,1)$. Let a be a point chosen at random from within the triangle with the probability that a is in any fixed region being proportional to the area of the region. Let T be the random variable "the angle that the line from $(0,0)$ to a makes with the x-axis". Find the cumulative distribution function and probability density function of T.

AQ3. Let U be a random variable with the Uniform$[0,1]$ distribution. Show that $U+1$ also has a uniform distribution but that U^2 does not. (Hint: in each case consider the cumulative distribution function of the new random variable.)

AQ4. Let E be a random variable with the Exponential(1) distribution.
Let U be a random variable with the Uniform$[0,1]$ distribution.

(i) Show that for $x, y > 0$, the conditional probability that $E > x + y$ given that $E > x$ is equal to the probability that $E > y$. This is called the memoryless property of the exponential distribution; can you see why?

(ii) Show that for $0 < x < 1$, $0 < y < 1$, the conditional probability that $U > x + y$ given that $U > x$ is strictly less than the probability that $U > y$.

AQ5. Let S be the unit square with vertices $(0,0),(1,0),(0,1),(1,1)$. Let a be a point chosen randomly from within the square with the probability that a is in any fixed region being proportional to the area of the region. Let T be the random variable "the sum of the $x-$coordinate of a and the $y-$coordinate of a". Find the cumulative distribution function and probability density function of T.