Solutions – exercise 5

1) i) $u'(x) = \frac{1}{x+a} + 1 \quad u''(x) = -\frac{1}{(x+a)^2} < 0 \quad \forall a \Rightarrow$ strictly concave

ii) $u'(x) = b e^{-bx} \quad u''(x) = -b^2 e^{-bx} \leq 0$ if $b \neq 0$ strictly concave

iii) $u'(x) = x^{-q} \quad u''(x) = -q x^{-q-1} < 0$ for $q > 0$ only

$\Rightarrow$ strictly concave for $q > 0$

2) lectures: $\langle J \rangle = e^{\mu_0 + \frac{1}{2}\sigma_0^2} = e^{\frac{3}{20}}$

risk-neutral $\Rightarrow \mu + \frac{1}{2}\sigma^2 - \lambda(1 - \langle J \rangle) = r$

$$\Rightarrow r(n) = r + \lambda(1 - \langle J \rangle) + \frac{n}{t}\log\langle J \rangle = \mu + \frac{1}{2}\sigma^2 + \frac{n}{t}\log\langle J \rangle = \frac{3}{20}\left(1 + \frac{n}{t}\right)$$

$$\sigma^2(n) = \sigma^2 + \frac{n}{t}\sigma_0^2 = \frac{1}{10}\left(1 + \frac{n}{t}\right)$$

$$C_J = \sum_{n=0}^{\infty} e^{-\lambda t \langle J \rangle} \frac{(\lambda t \langle J \rangle)^n}{n!} C(s, t, K, \sigma(n), r(n))$$

$$\approx e^{-\lambda t \langle J \rangle}\left(C(s, t, K, \sigma(0), r(0)) + \lambda t \langle J \rangle \, C(s, t, K, \sigma(1), r(1)) \right)$$

$\lambda = \frac{1}{100} \quad t = 1, \; s = 1 \quad K = 2 \Rightarrow C_J \approx \left(1 - \frac{1}{100} e^{\frac{3}{20}}\right) \cdot \left(C\left(1, 1, 2, \frac{1}{\sqrt{10}}, \frac{3}{20}\right) + \frac{1}{100} e^{\frac{3}{20}} C\left(1, 1, 2, \sqrt{\frac{1}{5}}, \frac{3}{10}\right)\right)$

3) lecture:

$$C_J = \sum_{n=0}^{\infty} e^{-\lambda t \langle J \rangle} \frac{(\lambda t \langle J \rangle)^n}{n!} C(s, t, K, \sigma(n), r(n))$$

We have $\sigma^2(n) = \sigma^2 + n\frac{\sigma_0^2}{t} \geq \sigma^2$ (since $n\frac{\sigma_0^2}{t} \geq 0$)

and $r(n) = r + \lambda(1 - \langle J \rangle) + \underbrace{\frac{n}{t}\log\langle J \rangle}_{\geq 0 \text{ since } \langle J \rangle > 1}$

$$\Rightarrow r(n) \geq r + \lambda(1 - \langle J \rangle)$$

Thus, since $\frac{\partial}{\partial \sigma} C \geq 0$ and $\frac{\partial}{\partial r} C \geq 0$,

we get $C(s, t, K, \sigma(n), r(n)) \geq C(s, t, K, \sigma, \overbrace{r + \lambda(1 - \langle J \rangle)}^{\tilde{r}})$

$$\Rightarrow C_J \geq \sum_{n=0}^{\infty} e^{-\lambda t \langle J \rangle} \frac{(\lambda t \langle J \rangle)^n}{n!} C(s, t, K, \sigma, \tilde{r})$$

$\uparrow$ independent of n

$$= e^{-\lambda t \langle J \rangle} C(s, t, K, \sigma, \tilde{r}) \cdot \underbrace{\sum_{n=0}^{\infty} \frac{(\lambda t \langle J \rangle)^n}{n!}}_{e^{\lambda t \langle J \rangle}}$$

$$= C(s, t, K, \sigma, \tilde{r})$$

4) *

In the lecture notes, the particular example

$$S = 100 \leftarrow \text{initial share price}$$

evolving to three possible share prices

$$S_1 = 200$$
$$S_2 = 100$$
$$S_3 = 50$$

was done in detail, with strike price $K = 150$. The result for the allowed range of option price was $C \in (0, \frac{50}{3})$.

The task is to generalize this example to general S, S_i, K:

$$S \begin{array}{l} \nearrow S_1 \quad \text{case } 1 \\ \rightarrow S_2 \quad \text{case } 2 \\ \searrow S_3 \quad \text{case } 3 \end{array}$$

G_S : gain from buying 1 share
G_O : " " " 1 option
c : option cost

$$G_S = \begin{cases} S_1 - S & \text{case } 1 \\ 0 & \text{case } 2 \\ S_3 - S & \text{case } 3 \end{cases} \quad 10 \quad \Rightarrow \langle G_S \rangle = p_1 (S_1 - S) + p_3 (S_3 - S) \overset{!}{=} 0 \quad 20$$

$$\Rightarrow p_1 = \frac{S - S_3}{S_1 - S} p_3$$

$$G_O = \begin{cases} S_1 - K - c & \text{case } 1 \\ -c & \text{case } 2 \,\&\, 3 \end{cases} \quad 10 \quad \Rightarrow \langle G_O \rangle = p_1 (S_1 - K - c) - (p_2 + p_3) \cdot c$$

$$= p_1 (S_1 - K) - c \underbrace{(p_1 + p_2 + p_3)}_{1} \overset{!}{=} 0 \quad 20$$

$$\Rightarrow c = p_1 (S_1 - K)$$

$$p_1 + p_2 + p_3 = 1 \quad 10 \quad \Leftrightarrow \quad p_1 + p_2 + \frac{S_1 - S}{S - S_3} p_1 = 1 \Rightarrow p_1 = \frac{1 - p_2}{1 + \frac{S_1 - S}{S - S_3}}$$

$$p_2 \in (0, 1) \Rightarrow \quad p_1 \in \left(0, \frac{1}{1 + \frac{S_1 - S}{S - S_3}}\right) = \left(0, \frac{S - S_3}{S_1 - S_3}\right)$$

$$\Rightarrow c \in \left(0, \frac{S - S_3}{S_1 - S_3} (S_1 - K)\right) \quad 30$$