Solutions – exercise 4

1) lectures: payoff of index option: $G_{index} = \left(\sum_{j=1}^{n} w_j \left(S_j(t) - K_j \right) \right)^+$

payoff of option on individual shares: $G_{indi} = \sum_{j=1}^{n} w_j \left(S_j(t) - K_j \right)^+$

case 1: $S_j(t) - K > 0 \quad \forall j$ $\qquad (w_j > 0)$

$$\Rightarrow G_{index} = \sum_{j=1}^{n} w_j \left(S_j(t) - K_j \right) = G_{indi}$$

case 2: $S_j(t) - K \leq 0 \quad \forall j$

$$\Rightarrow G_{index} = 0 = G_{indi} \qquad \blacksquare$$

2) Consider two plain options, one with strike price K, the other with strike price $K+B$.

$G_1 = (S(t) - K)^+$ $\qquad$ $G_2 = (S(t) - K - B)^+$

$\uparrow$ payoff of 1st option $\qquad$ $\uparrow$ payoff of 2nd option

$$G_1 - G_2 = \left\{ (S(t) - K)^+ - (S(t) - K - B)^+ \right\}$$

$$= \begin{cases} 0 & \text{if } S(t) < K \\ S(t) - K & \text{if } K < S(t) < K+B \\ B & \text{if } S(t) > K+B \end{cases}$$

Thus no arbitrage cost of capped option is

$$C_B = C(s, t, K, \sigma, r) - C(s, t, K+B, \sigma, r)$$

3) $P - C = K e^{-rt} - S$ $\qquad$ (lectures, put-call option parity formula)

Current value of BP shares $\quad S \approx 5.60$ £ $\quad \uparrow$ (fluctuates a bit!)

Current interest rate $\quad r = 0.5\%$

expiration time $\quad t = \frac{1}{2}$

$$\Rightarrow P - C \approx 9 \cdot e^{-0.005 \cdot \frac{1}{2}} - 5.60 \approx 3.38$$

Mr X gains $50 \cdot (P - C) =$ £ 169

So even if he pays you dinner for £ 60, he still makes a profit of £ 109 $\Rightarrow$ don't accept!!

4) *

a) $\langle N\rangle = \sum_{n=0}^{\infty} p_t(n)\cdot n = \sum_{n=1}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{(n-1)!} = e^{-\lambda t}\cdot\lambda t \sum_{n=1}^{\infty}\frac{(\lambda t)^{n-1}}{(n-1)!}$

$= e^{-\lambda t}\cdot\lambda t\cdot e^{+\lambda t} = \lambda t$ (10)

b) $\langle N(N-1)\rangle = \sum_{n=0}^{\infty} p_t(n)\cdot n(n-1) = \sum_{n=2}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{(n-2)!}$

$= e^{-\lambda t}(\lambda t)^2\sum_{n=2}^{\infty}\frac{(\lambda t)^{n-2}}{(n-2)!} = e^{-\lambda t}(\lambda t)^2\sum_{j=0}^{\infty}\frac{(\lambda t)^j}{j!}$

$= e^{-\lambda t}(\lambda t)^2 e^{+\lambda t} = (\lambda t)^2$ (20)

c) $\langle e^N\rangle = \sum_{n=0}^{\infty} p_t(n)\, e^n = \sum_{n=0}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!}\cdot e^n$

$= e^{-\lambda t}\sum_{n=0}^{\infty}\frac{(e\lambda t)^n}{n!} = e^{-\lambda t}e^{e\lambda t} = e^{\lambda t(e-1)}$ (20)

d) $\langle N^2\rangle - \langle N\rangle^2 = \langle N^2\rangle \underbrace{-\langle N\rangle + \langle N\rangle}_{0} - \langle N\rangle^2$

$= \langle N(N-1)\rangle + \langle N\rangle - \langle N\rangle^2$

$\overset{a),b)}{=} (\lambda t)^2 + \lambda t - (\lambda t)^2 = \lambda t$ (20)

e) $p_t(n) = \frac{1}{n!}e^{-\lambda t}(\lambda t)^n$

max from $\frac{\partial}{\partial t}p_t(n) = 0$ (10)

$\frac{\partial}{\partial t}p_t(n) = \frac{1}{n!}\left(-\lambda e^{-\lambda t}(\lambda t)^n + e^{-\lambda t}n(\lambda t)^{n-1}\lambda\right)$

$= \frac{e^{-\lambda t}(\lambda t)^{n-1}}{n!}\left(-\lambda\cdot(\lambda t) + n\lambda\right) \overset{!}{=} 0$

$\Rightarrow$ maximum obtained for $t^* = \frac{n}{\lambda}$ (10)

$\Rightarrow p_{t^*}(n) = \frac{1}{n!}e^{-\lambda t^*}(\lambda t^*)^n = \frac{1}{n!}e^{-n}n^n = \max_t p_t(n)$ (10)