Solutions – exercise 3

1) $\langle M(t)^m \rangle = (M(0))^m \, e^{m\mu t + \frac{1}{2}\sigma^2 t m^2}$ (Q1, coursework 1)

$t < t_D \Rightarrow S(t) = M(t) \Rightarrow \langle S(t)^m \rangle = (S(0))^m \, e^{m\mu t + \frac{1}{2}\sigma^2 t m^2}$

$t \geq t_D \Rightarrow S(t) = (1-F)\, M(t) \Rightarrow \langle S(t)^m \rangle = (S(0))^m (1-F)^m e^{m\mu t + \frac{1}{2}\sigma^2 t m^2}$

2) lectures: $\frac{\partial C}{\partial K} = -e^{-rt}\, \phi(\omega - \sigma\sqrt{t})$

thus $\frac{\partial^2 C}{\partial K^2} = -e^{-rt}\, \phi'(\omega - \sigma\sqrt{t}) \cdot \frac{\partial \omega}{\partial K}$

$\omega = \frac{rt + \frac{1}{2}\sigma^2 t - \log K - \log S}{\sigma\sqrt{t}} \Rightarrow \frac{\partial \omega}{\partial K} = -\frac{1}{\sigma\sqrt{t}} \frac{1}{K}$

$\phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}y^2} dy \Rightarrow \phi'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$

$\Rightarrow \frac{\partial^2 C}{\partial K^2} = \underbrace{e^{-rt}}_{>0} \frac{1}{\sqrt{2\pi}} \underbrace{e^{-\frac{1}{2}(\omega - \sigma\sqrt{t})^2}}_{>0} \cdot \underbrace{\frac{1}{K\sigma\sqrt{t}}}_{>0} > 0 \Rightarrow$ convex

3) $\left.\begin{array}{l} S + P_1 - C_1 = Ke^{-r} \\ S + P_2 - C_2 = Ke^{-2r} \end{array}\right\}$ substract $\Rightarrow P_1 - C_1 - P_2 + C_2 = K(e^{-r} - e^{-2r})$

(put-call option parity)

Let $x := e^{-r} \Rightarrow \frac{P_1 - C_1 - P_2 + C_2}{K} = x - x^2$

$\Leftrightarrow x^2 - x + \frac{1}{K}(P_1 - C_1 - P_2 + C_2) = 0$

$x_{1/2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{1}{K}(P_1 - C_1 - P_2 + C_2)} = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{2}{100}} = \frac{1}{2} \pm \sqrt{\frac{23}{100}}$

$\Rightarrow r = -\log x = -\log\left(\frac{1}{2} + \sqrt{\frac{23}{100}}\right) = 0.0206 = 2.06\,\%$

↑ reasonable interest rate for + sign

4)* e.g. August 2007

closing prices (in pence): 3235.13 (1 Aug), 3258,33 (2 Aug), 3223.84 (3 Aug), 3200.15 (6 Aug) ...

$\Rightarrow$ 20 data points $X_1 = 0.00715$ $X_2 = -0.01064$ $X_3 = -0.00738$ ...

$\bar{X} = \frac{1}{20}\sum_{i=1}^{20} X_i = 0.00039$ $\sigma \simeq \sqrt{\frac{1}{\ell}\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2} \simeq 0.31$

$\ell = \frac{t}{n}$ $t \approx \frac{1}{12}$ $n = 20$ $\mu \simeq \frac{1}{\ell}\bar{X} = 0.094$ $MSE(\sigma^2) = \mathrm{var}\,\sigma^2 \simeq \frac{2}{n-1}\sigma^4 \approx 1 \cdot 10^{-3}$

31 Aug   ⋮   3260.48   ⋮

⋮			⋮			⋮
7-Aug-07	3,200.15	3,261.72	3,200.15	3,261.72	0	3,261.72
6-Aug-07	3,223.84	3,228.25	3,190.21	3,200.15	0	3,200.15
3-Aug-07	3,258.33	3,270.38	3,217.21	3,223.84	0	3,223.84
2-Aug-07	3,235.13	3,266.87	3,235.13	3,258.33	0	3,258.33
1-Aug-07	3,289.12	3,289.12	3,202.45	3,235.13	0	3,235.13

$X_4^t = 0.01906$

$X_3^t = -0.00738$

$X_2^t = -0.01064$

$X_1^t = 0.00715$

* Close price adjusted for dividends and splits.

First | Prev | Next | Last

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Marking scheme

printed list	(10)
list of X_i^t	(20)
μ	(20)
σ	(25)
MSE	(25)

$$\overline{X^t} \approx \frac{1}{20} \sum_{i=1}^{20} X_i^t = 0.00039$$

$$\text{sample variance} = \frac{1}{19} \sum_{i=1}^{20} (X_i^t - \overline{X^t})^2 = 4.02 \cdot 10^{-4}$$

$$l = \frac{t}{n} \qquad t \approx \frac{1}{12} \quad (1 \text{ month}) \qquad n = 20$$

↑ number of data points used

$$\Rightarrow l \approx 0.00417$$

$$\mu \approx \frac{1}{l} \overline{X^t} = 0.094$$

$$\sigma^2 \approx \frac{1}{l} \text{ sample variance} = 0.0964 \Rightarrow \sigma \simeq 0.31$$

$$MSE(\sigma^2) = \operatorname{var} \sigma^2 \underset{\uparrow \text{lectures}}{\approx} \frac{2}{n-1}\sigma^4 \approx 1 \cdot 10^{-3}$$

This is an example only.

The numbers $\mu, \sigma, \operatorname{var}(\sigma^2)$ may strongly fluctuate for the different months.