Solutions – exercise 3

1) $\langle M(t)^m \rangle = (M(0))^m \, e^{m\mu t + \frac{1}{2}\sigma^2 t m^2}$ (Q1, coursework 1)

$t < t_D \Rightarrow S(t) = M(t) \Rightarrow \langle S(t)^m \rangle = (S(0))^m \, e^{m\mu t + \frac{1}{2}\sigma^2 t m^2}$

$t \geq t_D \Rightarrow S(t) = (1-f)\, M(t) \Rightarrow \langle S(t)^m \rangle = (S(0))^m (1-f)^m e^{m\mu t + \frac{1}{2}\sigma^2 t m^2}$

2) lectures: $\frac{\partial C}{\partial K} = -e^{-rt}\, \phi(\omega - \sigma\sqrt{t})$

thus $\frac{\partial^2 C}{\partial K^2} = -e^{-rt}\, \phi'(\omega - \sigma\sqrt{t}) \cdot \frac{\partial \omega}{\partial K}$

$\omega = \frac{rt + \frac{1}{2}\sigma^2 t - \log K + \log S}{\sigma\sqrt{t}} \Rightarrow \frac{\partial \omega}{\partial K} = -\frac{1}{\sigma\sqrt{t}} \frac{1}{K}$

$\phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}y^2} dy \Rightarrow \phi'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$

$\Rightarrow \frac{\partial^2 C}{\partial K^2} = \underbrace{e^{-rt}}_{>0} \frac{1}{\sqrt{2\pi}} \underbrace{e^{-\frac{1}{2}(\omega - \sigma\sqrt{t})^2}}_{>0} \cdot \underbrace{\frac{1}{K\sigma\sqrt{t}}}_{>0} > 0 \Rightarrow$ convex

3) $\left.\begin{aligned} S + P_1 - C_1 &= Ke^{-r} \\ S + P_2 - C_2 &= Ke^{-2r} \end{aligned}\right\}$ substract $\Rightarrow P_1 - C_1 - P_2 + C_2 = K(e^{-r} - e^{-2r})$

(put-call option parity formula, see lectures)

Let $x := e^{-r} \Rightarrow \frac{P_1 - C_1 - P_2 + C_2}{K} = x - x^2$

$\Leftrightarrow x^2 - x + \frac{1}{K}(P_1 - C_1 - P_2 + C_2) = 0$

$x_{1/2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{1}{K}(P_1 - C_1 - P_2 + C_2)} = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{2}{100}} = \frac{1}{2} \pm \sqrt{\frac{23}{100}}$

$\Rightarrow r = -\log x = -\log\left(\frac{1}{2} + \sqrt{\frac{23}{100}}\right) = 0.0206 = 2.06\,\%$

↑ reasonable interest rate for + sign

4) e.g. August 2007

closing prices (in pence)	3235.13	3258.33	3223.84	3200.15	...
	1 Aug	2 Aug	3 Aug	6 Aug	

$\Rightarrow$ 20 data points $X_1 = 0.00715$ $X_2 = -0.01064$ $X_3 = -0.00738$ ...

$\bar{X} = \frac{1}{20}\sum_{i=1}^{20} X_i = 0.00039$ $\sigma \simeq \sqrt{\frac{1}{\ell}\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2} \simeq 0.31$

$\ell = \frac{t}{n}$ $t \approx \frac{1}{12}$ $n = 20$ $\mu \simeq \frac{1}{\ell}\bar{X} = 0.094$ $MSE(\sigma^2) = \mathrm{var}\,\sigma^2 \approx \frac{2}{n-1}\sigma^4 \approx 1 \cdot 10^{-}$

Tuesday,29 January 2008, 3:33pm - UK Markets close in 57 minutes.

FTSE ALL-SHARE (^FTAS)

At 3:33PM : 2,986.64 ↑ 35.31 (1.20%)

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Example
August 2007

$X_i^t = \log \frac{\text{new closing price}}{\text{old closing price}}$

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PRICES

Date	Open	High	Low	Close	Volume	Adj Close*	
31-Aug-07	3,211.58	3,262.05	3,211.58	3,260.48	0	3,260.48	$X_{20}^t = 0.01511$
30-Aug-07	3,168.18	3,215.10	3,166.52	3,211.58	0	3,211.58	$X_{19}^t = 0.01361$
29-Aug-07	3,151.43	3,170.80	3,127.60	3,168.18	0	3,168.18	$X_{18}^t = 0.00530$
28-Aug-07	3,209.46	3,209.46	3,143.74	3,151.43	0	3,151.43	$X_{17}^t = -0.01825$
[illegible]-Aug-07	3,198.06	3,215.60	3,190.83	3,209.46	0	3,209.46	$X_{16}^t = 0.00356$
23-Aug-07	3,195.66	3,245.13	3,195.66	3,198.06	0	3,198.06	$X_{15}^t = 0.00075$
22-Aug-07	3,136.36	3,196.34	3,136.36	3,195.66	0	3,195.66	$X_{14}^t = 0.01873$
21-Aug-07	3,136.05	3,155.63	3,113.32	3,136.36	0	3,136.36	$X_{13}^t = 0.00010$
20-Aug-07	3,128.97	3,177.96	3,128.97	3,136.04	0	3,136.04	$X_{12}^t = 0.00226$
17-Aug-07	3,031.91	3,160.75	3,014.01	3,128.97	0	3,128.97	$X_{11}^t = 0.03151$
16-Aug-07	3,159.24	3,159.24	3,031.91	3,031.91	0	3,031.91	$X_{10}^t = -0.04114$
15-Aug-07	3,179.17	3,179.17	3,127.80	3,159.24	0	3,159.24	$X_9^t = -0.01875$
13-Aug-07	3,129.62	3,226.14	3,129.62	3,219.05	0	3,219.05	$X_8^t = 0.02817$
10-Aug-07	3,244.62	3,244.62	3,128.72	3,129.62	0	3,129.62	$X_7^t = -0.03609$
9-Aug-07	3,307.94	3,307.94	3,222.84	3,244.62	0	3,244.62	$X_6^t = -0.01932$
8-Aug-07	3,261.71	3,313.96	3,261.71	3,307.93	0	3,307.93	$X_5^t = 0.01407$

⋮ ⋮

31 Aug ⋮ 3260.48 ⋮ ⋮

7-Aug-07	3,200.15	3,261.72	3,200.15	3,261.72	0	3,261.72
6-Aug-07	3,223.84	3,228.25	3,190.21	3,200.15	0	3,200.15
3-Aug-07	3,258.33	3,270.38	3,217.21	3,223.84	0	3,223.84
2-Aug-07	3,235.13	3,266.87	3,235.13	3,258.33	0	3,258.33
1-Aug-07	3,289.12	3,289.12	3,202.45	3,235.13	0	3,235.13

$X_4 = 0.01906$

$X_3 = -0.00738$

$X_2 = -0.01064$

$X_1 = 0.00715$

* Close price adjusted for dividends and splits.

$$\bar{X} = \frac{1}{20}\sum_{i=1}^{20} X_i = 0.00039$$

$$\text{sample variance} = \frac{1}{19}\sum_{i=1}^{20}(X_i - \bar{X})^2 = 4.02 \cdot 10^{-4}$$

$$l = \frac{t}{n} \qquad t \approx \frac{1}{12} \quad (1 \text{ month}) \qquad n = 20$$

$\uparrow$ number of data points used

$$\Rightarrow l \approx 0.00417$$

$$\mu \approx \frac{1}{l}\bar{X} = 0.094$$

$$\sigma^2 \approx \frac{1}{l}\,\text{sample variance} = 0.0964 \Rightarrow \sigma \simeq 0.31$$

$$MSE(\sigma^2) = \text{var}\,\sigma^2 \underset{\uparrow \text{lectures}}{\approx} \frac{2}{n-1}\sigma^4 \approx 1 \cdot 10^{-3}$$

This is an example only.

The numbers

$\mu, \sigma, \text{var}(\sigma^2)$ may strongly fluctuate for the different months.