3. Geometric Brownian Motion

3.1 Geometric Brownian Motion

Suppose that we are interested in the price of some security as it evolves over time. Let the present time be time 0, and let $S(y)$ denote the price of the security a time y from the present. We say that the collection of prices $S(y)$, $0 \le y < \infty$, follows a *geometric Brownian motion* with drift parameter μ and volatility parameter σ if, for all nonegative values of y and t, the random variable

$$\frac{S(t+y)}{S(y)}$$

is independent of all prices up to time y; and if, in addition,

$$\log\left(\frac{S(t+y)}{S(y)}\right)$$

is a normal random variable with mean μt and variance $t\sigma^2$.

In other words, the series of prices will be a geometric Brownian motion if the ratio of the price a time t in the future to the present price will, independent of the past history of prices, have a lognormal probability distribution with parameters μt and $t\sigma^2$.

It follows that a consequence of assuming a security's prices follow a geometric Brownian motion is that, once μ and σ are determined, it is only the present price – and not the history of past prices – that affects probabilities of future prices. Furthermore, probabilities concerning the ratio of the price a time t in the future to the present price will not depend on the present price. (Thus, for instance, the model implies that the probability a given security doubles in price in the next month is the same no matter whether its present price is 10 or 25.)

It turns out that, for a given initial price $S(0)$, the expected value of the price at time t depends on *both* of the geometric Brownian motion parameters. Specifically, if the initial price is s_0, then

$$E[S(t)] = s_0 e^{t(\mu+\sigma^2/2)}.$$

Thus, under geometric Brownian motion, the expected price grows at the rate $\mu + \sigma^2/2$.

3.2 Geometric Brownian Motion as a Limit of Simpler Models

Let Δ denote a small increment of time and suppose that, every Δ time units, the price of a security either goes up by the factor u with probability p or goes down by the factor d with probability $1-p$, where

$$u = e^{\sigma\sqrt{\Delta}}, \qquad d = e^{-\sigma\sqrt{\Delta}},$$

$$p = \frac{1}{2}\left(1 + \frac{\mu}{\sigma}\sqrt{\Delta}\right).$$

That is, we are supposing that the price of the security changes only at times that are integral multiples of Δ; at these times, it either goes up by the factor u or down by the factor d.

As we take Δ smaller and smaller, so that the price changes occur more and more frequently (though by factors that become closer and closer to 1), the collection of prices becomes a geometric Brownian motion. Consequently, geometric Brownian motion can be approximated by a relatively simple process, one that goes either up or down by fixed factors at regularly specified times.

Let us now verify that the preceding model becomes geometric Brownian motion as we let Δ become smaller and smaller. To begin, let Y_i equal 1 if the price goes up at time $i\Delta$, and let it be 0 if it goes down. Now, the number of times that the security's price goes up in the first n time increments is $\sum_{i=1}^{n} Y_i$, and the number of times it goes down is $n - \sum_{i=1}^{n} Y_i$. Hence, $S(n\Delta)$, its price at the end of this time, can be expressed as

$$S(n\Delta) = S(0)u^{\sum_{i=1}^{n} Y_i} d^{n-\sum_{i=1}^{n} Y_i}$$

or

$$S(n\Delta) = d^n S(0)\left(\frac{u}{d}\right)^{\sum_{i=1}^{n} Y_i}.$$

If we now let $n = t/\Delta$, then the preceding equation can be expressed as

$$\frac{S(t)}{S(0)} = d^{t/\Delta}\left(\frac{u}{d}\right)^{\sum_{i=1}^{t/\Delta} Y_i}.$$

Taking logarithms gives

$$\log\left(\frac{S(t)}{S(0)}\right) = \frac{t}{\Delta}\log(d) + \log\left(\frac{u}{d}\right)\sum_{i=1}^{t/\Delta} Y_i$$
$$= \frac{-t\sigma}{\sqrt{\Delta}} + 2\sigma\sqrt{\Delta}\sum_{i=1}^{t/\Delta} Y_i, \qquad (3.1)$$

where Equation (3.1) used the definitions of u and d. Now, as Δ goes to 0, there are more and more terms in the summation $\sum_{i=1}^{t/\Delta} Y_i$; hence, by the central limit theorem, this sum becomes more and more normal, implying from Equation (3.1) that $\log(S(t)/S(0))$ becomes a normal random variable. Moreover, from Equation (3.1) we obtain that

$$E\left[\log\left(\frac{S(t)}{S(0)}\right)\right] = \frac{-t\sigma}{\sqrt{\Delta}} + 2\sigma\sqrt{\Delta}\sum_{i=1}^{t/\Delta} E[Y_i]$$
$$= \frac{-t\sigma}{\sqrt{\Delta}} + 2\sigma\sqrt{\Delta}\frac{t}{\Delta}p$$
$$= \frac{-t\sigma}{\sqrt{\Delta}} + \frac{t\sigma}{\sqrt{\Delta}}\left(1 + \frac{\mu}{\sigma}\sqrt{\Delta}\right)$$
$$= \mu t.$$

Furthermore, Equation (3.1) yields that

$$\operatorname{Var}\left(\log\left(\frac{S(t)}{S(0)}\right)\right) = 4\sigma^2\Delta\sum_{i=1}^{t/\Delta}\operatorname{Var}(Y_i) \qquad \text{(by independence)}$$
$$= 4\sigma^2 tp(1-p)$$
$$\approx \sigma^2 t \qquad \text{(since, for small } \Delta,\ p \approx 1/2\text{)}.$$

Thus we see that, as Δ becomes smaller and smaller, $\log(S(t)/S(0))$ (and, by the same reasoning, $\log(S(t+y)/S(y))$) becomes a normal random variable with mean μt and variance $t\sigma^2$. In addition, because successive price changes are independent and each has the same probability of being an increase, it follows that $S(t+y)/S(y)$ is independent of earlier price changes before time y. Hence, as Δ goes to 0, both conditions of geometric Brownian motion are met, showing that the model indeed becomes geometric Brownian motion.

3.3 Brownian Motion

Geometric Brownian motion can be considered to be a variant of a long-studied model known as Brownian motion. It is defined as follows.

Definition The collection of prices $S(y)$, $0 \le y < \infty$, is said to follow a *Brownian motion* with drift parameter μ and variance parameter σ^2 if, for all nonegative values of y and t, the random variable

$$S(t+y) - S(y)$$

is independent of all prices up to time y and, in addition, is a normal random variable with mean μt and variance $t\sigma^2$.

Thus, Brownian motion shares with geometric Brownian motion the property that a future price depends on the present and all past prices only through the present price; however, in Brownian motion it is the difference in prices (and not the logarithm of their ratio) that has a normal distribution.

The Brownian motion process has an distinguished scientific pedigree. It is named after the English botanist Robert Brown, who first described (in 1827) the unusual motion exhibited by a small particle that is totally immersed in a liquid or gas. The first explanation of this motion was given by Albert Einstein in 1905. He showed mathematically that Brownian motion could be explained by assuming that the immersed particle was continually being subjected to bombardment by the molecules of the surrounding medium. A mathematically concise definition, as well as an elucidation of some of the mathematical properties of Brownian motion, was given by the American applied mathematician Norbert Wiener in a series of papers originating in 1918.

Interestingly, Brownian motion was independently introduced in 1900 by the French mathematician Bachelier, who used it in his doctoral dissertation to model the price movements of stocks and commodities. However, Brownian motion appears to have two major flaws when used to model stock or commodity prices. First, since the price of a stock is a normal random variable, it can theoretically become negative. Second, the assumption that a price *difference* over an interval of fixed length has the same normal distribution no matter what the price at the beginning of the interval does not seem totally reasonable. For instance, many people might not think that the probability a stock presently selling at \$20 would drop to \$15 (a loss of 25%) in one month would be the same as the probability that when the stock is at \$10 it would drop to \$5 (a loss of 50%) in one month.

The geometric Brownian motion model, on the other hand, possesses neither of these flaws. Since it is now the *logarithm* of the stock's price that is a normal random variable, the model does not allow for negative stock prices. In addition, since it is the ratios of prices separated by a fixed length of time that have the same distribution, geometric Brownian motion makes what many feel is the more reasonable assumption that it is the *percentage* change in price, and not the absolute change, whose probabilities do not depend on the present price. However, it should be noted that – in both of these models – once the model parameters μ and σ are determined, the only information that is needed for predicting future prices is the present price; information about past prices is irrelevant.

3.4 Exercises

Exercise 3.1 Suppose that $S(y)$, $y \geq 0$, is a geometric Brownian motion with drift parameter $\mu = .01$ and volatility parameter $\sigma = .2$. If $S(0) = 100$, find:

(a) $E[S(10)]$;
(b) $P\{S(10) > 100\}$;
(c) $P\{S(10) < 110\}$.

Exercise 3.2 Repeat Exercise 3.1 when the volatility parameter is equal to .4.

Exercise 3.3 Repeat Exercise 3.2 when the volatility parameter is equal to .6.

Exercise 3.4 It can be shown that if X is a normal random variable with mean m and variance v^2, then

$$E[e^X] = e^{m+v^2/2}.$$

Use this result to verify the formula for $E[S(t)]$ given in Section 3.1.

Exercise 3.5 Use the result of the preceding exercise to find $\text{Var}(S(t))$ when $S(0) = s_0$.

Hint: Use the identity

$$\text{Var}(X) = E[X^2] - (E[X])^2.$$

REFERENCES

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4. Interest Rates and Present Value Analysis

4.1 Interest Rates

If you borrow the amount P (called the principal), which must be repaid after a time T along with simple interest at rate r per time T, then the amount to be repaid at time T is

$$P + rP = P(1 + r).$$

That is, you must repay both the principal P and the interest, equal to the principal times the interest rate. For instance, if you borrow \$100 to be repaid after one year with a simple interest rate of 5% per year (i.e., $r = .05$), then you will have to repay \$105 at the end of the year.

Example 4.1a Suppose that you borrow the amount P, to be repaid after one year along with interest at a rate r per year *compounded* semiannually. What does this mean? How much is owed in a year?

Solution. In order to solve this example, you must realize that having your interest compounded semiannually means that after half a year you are to be charged simple interest at the rate of $r/2$ per half-year, and that interest is then added on to your principal, which is again charged interest at rate $r/2$ for the second half-year period. In other words, after six months you owe

$$P(1 + r/2).$$

This is then regarded as the new principal for another six-month loan at interest rate $r/2$; hence, at the end of the year you will owe

$$P(1 + r/2)(1 + r/2) = P(1 + r/2)^2. \qquad \square$$

Example 4.1b If you borrow \$1,000 for one year at an interest rate of 8% per year compounded quarterly, how much do you owe at the end of the year?

Solution. An interest rate of 8% that is compounded quarterly is equivalent to paying simple interest at 2% per quarter-year, with each successive quarter charging interest not only on the original principal but also on the interest that has accrued up to that point. Thus, after one quarter you owe

$$1{,}000(1 + .02);$$

after two quarters you owe

$$1{,}000(1 + .02)(1 + .02) = 1{,}000(1 + .02)^2;$$

after three quarters you owe

$$1{,}000(1 + .02)^2(1 + .02) = 1{,}000(1 + .02)^3;$$

and after four quarters you owe

$$1{,}000(1 + .02)^3(1 + .02) = 1{,}000(1 + .02)^4 = \$1{,}082.40. \qquad \square$$

Example 4.1c Many credit-card companies charge interest at a yearly rate of 18% compounded monthly. If the amount P is charged at the beginning of a year, how much is owed at the end of the year if no previous payments have been made?

Solution. Such a compounding is equivalent to paying simple interest every month at a rate of $18/12 = 1.5\%$ per month, with the accrued interest then added to the principal owed during the next month. Hence, after one year you will owe

$$P(1 + .015)^{12} = 1.1956P. \qquad \square$$

If the interest rate r is compounded then, as we have seen in Examples 4.1b and 4.1c, the amount of interest actually paid is greater than if we were paying simple interest at rate r. The reason, of course, is that in compounding we are being charged interest on the interest that has already been computed in previous compoundings. In these cases, we call r the *nominal* interest rate, and we define the *effective interest rate*, call it r_{eff}, by

$$r_{\text{eff}} = \frac{\text{amount repaid at the end of a year} - P}{P}.$$

For instance, if the loan is for one year at a nominal interest rate r that is to be compounded quarterly, then the effective interest rate for the year is

$$r_{\text{eff}} = (1 + r/4)^4 - 1.$$

Thus, in Example 4.1b the effective interest rate is 8.24% whereas in Example 4.1c it is 19.56%. Since

$$P(1 + r_{\text{eff}}) = \text{amount repaid at the end of a year},$$

the payment made in a one-year loan with compound interest is the same as if the loan called for simple interest at rate r_{eff} per year.

Example 4.1d *The Doubling Rule* If you put funds into an account that pays interest at rate r compounded annually, how many years does it take for your funds to double?

Solution. Since your initial deposit of D will be worth $D(1 + r)^n$ after n years, we need to find the value of n such that

$$(1 + r)^n = 2.$$

Now,

$$(1 + r)^n = \left(1 + \frac{nr}{n}\right)^n \approx e^{nr},$$

where the approximation is fairly precise provided that n is not too small. Therefore,

$$e^{nr} \approx 2,$$

implying that

$$n \approx \frac{\log(2)}{r} = \frac{.693}{r}.$$

Thus, it will take n years for your funds to double when

$$n \approx \frac{.7}{r}.$$

For instance, if the interest rate is 1% ($r = .01$) then it will take approximately 70 years for your funds to double; if $r = .02$, it will take about 35 years; if $r = .03$, it will take about $23\frac{1}{3}$ years; if $r = .05$, it will take about 14 years; if $r = .07$, it will take about 10 years; and if $r = .10$, it will take about 7 years.

As a check on the preceding approximations, note that (to three-decimal-place accuracy):

$$(1.01)^{70} = 2.007,$$
$$(1.02)^{35} = 2.000,$$
$$(1.03)^{23.33} = 1.993,$$
$$(1.05)^{14} = 1.980,$$
$$(1.07)^{10} = 1.967,$$
$$(1.10)^7 = 1.949.$$

□

Suppose now that we borrow the principal P for one year at a nominal interest rate of r per year, compounded *continuously*. Now, how much is owed at the end of the year? Of course, to answer this we must first decide on an appropriate definition of "continuous" compounding. To do so, note that if the loan is compounded at n equal intervals in the year, then the amount owed at the end of the year is $P(1 + r/n)^n$. As it is reasonable to suppose that continuous compounding refers to the limit of this process as n grows larger and larger, the amount owed at time 1 is

$$P \lim_{n\to\infty} (1 + r/n)^n = Pe^r.$$

Example 4.1e If a bank offers interest at a nominal rate of 5% compounded continuously, what is the effective interest rate per year?

Solution. The effective interest rate is

$$r_{\text{eff}} = \frac{Pe^{.05} - P}{P} = e^{.05} - 1 \approx .05127.$$

That is, the effective interest rate is 5.127% per year. □

If the amount P is borrowed for t years at a nominal interest rate of r per year compounded continuously, then the amount owed at time t is Pe^{rt}. This follows because if interest is compounded n times during the year, then there would have been nt compoundings by time t, giving a debt level of $P(1+r/n)^{nt}$. Consequently, under continuous compounding the debt at time t would be

$$P \lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{nt} = P\left(\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{n}\right)^{t} = Pe^{rt}.$$

It follows from the preceding that continuous compounded interest at rate r per unit time can be interpreted as being a continuous compounding of a nominal interest rate of rt per (unit of time) t.

4.2 Present Value Analysis

Suppose that one can both borrow and loan money at a nominal rate r per period that is compounded periodically. Under these conditions, what is the present worth of a payment of v dollars that will be made at the end of period i? Since a bank loan of $v(1+r)^{-i}$ would require a payoff of v at period i, it follows that the *present value* of a payoff of v to be made at time period i is $v(1+r)^{-i}$.

The concept of present value enables us to compare different income streams to see which is preferable.

Example 4.2a Suppose that you are to receive payments (in thousands of dollars) at the end of each of the next five years. Which of the following three payment sequences is preferable?

A. 12, 14, 16, 18, 20;
B. 16, 16, 15, 15, 15;
C. 20, 16, 14, 12, 10.

Solution. If the nominal interest rate is r compounded yearly, then the present value of the sequence of payments x_i $(i = 1, 2, 3, 4, 5)$ is

$$\sum_{i=1}^{5}(1+r)^{-i}x_i;$$

the sequence having the largest present value is preferred. It thus follows that the superior sequence of payments depends on the interest rate.

Table 4.1: *Present Values*

	Payment Sequence		
r	**A**	**B**	**C**
.1	59.21	58.60	56.33
.2	45.70	46.39	45.69
.3	36.49	37.89	38.12

If r is small, then the sequence **A** is best since its sum of payments is the highest. For a somewhat larger value of r, the sequence **B** would be best because – although the total of its payments (77) is less than that of **A** (80) – its earlier payments are larger than are those of **A**. For an even larger value of r, the sequence **C**, whose earlier payments are higher than those of either **A** or **B**, would be best. Table 4.1 gives the present values of these payment streams for three different values of r.

It should be noted that the payment sequences can be compared according to their values at any specified time. For instance, to compare them in terms of their time-5 values, we would determine which sequence of payments yields the largest value of

$$\sum_{i=1}^{5}(1+r)^{5-i}x_i = (1+r)^5\sum_{i=1}^{5}(1+r)^{-i}x_i.$$

Consequently, we obtain the same preference ordering as a function of interest rate as before. □

Remark. Let the given interest rate be r, compounded yearly. Any cash flow stream $\mathbf{a} = a_1, a_2, \dots, a_n$ that returns you a_i dollars at the end of year i (for each $i = 1, \dots, n$) can be replicated by depositing

$$\text{PV}(\mathbf{a}) = \frac{a_1}{1+r} + \frac{a_2}{(1+r)^2} + \cdots + \frac{a_n}{(1+r)^n}$$

in a bank at time 0 and then making the successive withdrawals $a_1, a_2, \dots, a_n$. To verify this claim, note that withdrawing a_1 at the end of year 1 would leave you with

$$(1+r)\left[\frac{a_1}{1+r} + \frac{a_2}{(1+r)^2} + \cdots + \frac{a_n}{(1+r)^n}\right] - a_1 = \frac{a_2}{(1+r)} + \cdots + \frac{a_n}{(1+r)^{n-1}}$$

on deposit. Thus, after withdrawing a_2 at the end of year 2 you would have

$$(1+r)\left[\frac{a_2}{1+r} + \cdots + \frac{a_n}{(1+r)^{n-1}}\right] - a_2 = \frac{a_3}{(1+r)} + \cdots + \frac{a_n}{(1+r)^{n-2}}.$$

Continuing, it follows that withdrawing a_i at the end of year i $(i < n)$ would leave you with

$$\frac{a_{i+1}}{(1+r)} + \cdots + \frac{a_n}{(1+r)^{n-i}}$$

on deposit. Consequently, you would have $a_n/(1+r)$ on deposit after withdrawing a_{n-1}, and this is just enough to cover your next withdrawal of a_n at the end of the following year.

In a similar manner, the cash flow sequence $a_1, a_2, \dots, a_n$ can be transformed into the initial capital PV(**a**) by borrowing this amount from a bank and then using the cash flow to pay off this debt. Therefore, any cash flow sequence is equivalent to an initial reception of the present value of the cash flow sequence, thus showing that one cash flow sequence is preferable to another whenever the former has a larger present value than the latter. □

Example 4.2b A company needs a certain type of machine for the next five years. They presently own such a machine, which is now worth \$6,000 but will lose \$2,000 in value in each of the next three years, after which it will be worthless and unuseable. The (beginning-of-the-year) value of its yearly operating cost is \$9,000, with this amount expected to increase by \$2,000 in each subsequent year that it is used. A new machine can be purchased at the beginning of any year for a fixed cost of \$22,000. The lifetime of a new machine is six years, and its value decreases by \$3,000 in each of its first two years of use and then by \$4,000 in each following year. The operating cost of a new machine is \$6,000 in its first year, with an increase of \$1,000 in each subsequent year. If the interest rate is 10%, when should the company purchase a new machine?

Solution. The company can purchase a new machine at the beginning of year 1, 2, 3, or 4, with the following six-year cash flows (in units of \$1,000) as a result:

- buy at beginning of year 1: 22, 7, 8, 9, 10, −4;
- buy at beginning of year 2: 9, 24, 7, 8, 9, −8;
- buy at beginning of year 3: 9, 11, 26, 7, 8, −12;
- buy at beginning of year 4: 9, 11, 13, 28, 7, −16.

To see why this listing is correct, suppose that the company will buy a new machine at the beginning of year 3. Then its year-1 cost is the \$9,000 operating cost of the old machine; its year-2 cost is the \$11,000 operating cost of this machine; its year-3 cost is the \$22,000 cost of a new machine, plus the \$6,000 operating cost of this machine, minus the \$2,000 obtained for the replaced machine; its year-4 cost is the \$7,000 operating cost; its year-5 cost is the \$8,000 operating cost; and its year-6 cost is −\$12, 000, the negative of the value of the 3-year-old machine that it no longer needs. The other cash flow sequences are similarly argued.

With the yearly interest rate $r = .10$, the present value of the first cost-flow sequence is

$$22 + \frac{7}{1.1} + \frac{8}{(1.1)^2} + \frac{9}{(1.1)^3} + \frac{10}{(1.1)^4} - \frac{4}{(1.1)^5} = 46.083.$$

The present values of the other cash flows are similarly determined, and the four present values are

$$46.083, \quad 43.794, \quad 43.760, \quad 45.627.$$

Therefore, the company should purchase a new machine two years from now. □

Example 4.2c An individual who plans to retire in 20 years has decided to put an amount A in the bank at the beginning of each of the next 240 months, after which she will withdraw \$1,000 at the beginning of each of the following 360 months. Assuming a nominal yearly interest rate of of 6% compounded monthly, how large does A need to be?

Solution. Let $r = .06/12 = .005$ be the monthly interest rate. With $\beta = \frac{1}{1+r}$, the present value of all her deposits is

$$A + A\beta + A\beta^2 + \cdots + A\beta^{239} = A\frac{1-\beta^{240}}{1-\beta}.$$

Similarly, if W is the amount withdrawn in the following 360 months, then the present value of all these withdrawals is

$$W\beta^{240} + W\beta^{241} + \cdots + W\beta^{599} = W\beta^{240}\frac{1-\beta^{360}}{1-\beta}.$$

Thus she will be able to fund all withdrawals (and have no money left in her account) if

$$A\frac{1-\beta^{240}}{1-\beta} = W\beta^{240}\frac{1-\beta^{360}}{1-\beta}.$$

With $W = 1{,}000$, and $\beta = 1/1.005$, this gives

$$A = 360.99.$$

That is, saving \$361 a month for 240 months will enable her to withdraw \$1,000 a month for the succeeding 360 months.

Remark. In this example we have made use of the algebraic identity

$$1 + b + b^2 + \cdots + b^n = \frac{1-b^{n+1}}{1-b}.$$

We can prove this identity by letting

$$x = 1 + b + b^2 + \cdots + b^n$$

and then noting that

$$\begin{aligned} x - 1 &= b + b^2 + \cdots + b^n \\ &= b(1 + b + \cdots + b^{n-1}) \\ &= b(x - b^n). \end{aligned}$$

Therefore,

$$(1-b)x = 1 - b^{n+1},$$

which yields the identity.

It can be shown by the same technique, or by letting n go to infinity, that when $|b| < 1$ we have

$$1 + b + b^2 + \cdots = \frac{1}{1-b}.$$ □

Example 4.2d A perpetuity entitles its holder to be paid the constant amount c at the end of each of an infinite sequence of years. That is, it pays its holder c at the end of year i for each $i = 1, 2, \ldots$. If the interest rate is r, compounded yearly, then what is the present value of such a cash flow sequence?

Solution. Because such a cash flow could be replicated by initially putting the principle c/r in the bank and then withdrawing the interest earned (leaving the principal intact) at the end of each period, whereas it could not be replicated by putting any smaller amount in the bank, it would seem that the present value of the infinite flow is c/r. This intuition is easily checked mathematically by

$$\begin{aligned} \text{PV} &= \frac{c}{1+r} + \frac{c}{(1+r)^2} + \frac{c}{(1+r)^3} + \cdots \\ &= \frac{c}{1+r}\left[1 + \frac{1}{1+r} + \frac{1}{(1+r)^2} + \cdots\right] \\ &= \frac{c}{1+r}\frac{1}{1-\frac{1}{1+r}} \\ &= \frac{c}{r}. \end{aligned}$$ □

Example 4.2e Suppose you have just spoken to a bank about borrowing \$100,000 to purchase a house, and the loan officer has told you that a \$100,000 loan, to be repaid in monthly installments over 15 years with an interest rate of .6% per month, could be arranged. If the bank charges a loan initiation fee of \$600, a house inspection fee of \$400, and 1 "point," what is the effective annual interest rate of the loan being offered?

Solution. To begin, let us determine the monthly mortgage payment, call it A, of such a loan. Since \$100,000 is to be repaid in 180 monthly payments at an interest rate of .6% per month, it follows that

Solution. The rate of return will be the solution to

$$100 = \frac{60}{1+r} + \frac{60}{(1+r)^2}.$$

Letting $x = 1/(1+r)$, the preceding can be written as

$$60x^2 + 60x - 100 = 0,$$

which yields that

$$x = \frac{-60 \pm \sqrt{60^2 + 4(60)(100)}}{120}.$$

Since $-1 < r$ implies that $x > 0$, we obtain the solution

$$x = \frac{\sqrt{27{,}600} - 60}{120} \approx .8844.$$

Hence, the rate of return r^* is such that

$$1 + r^* \approx \frac{1}{.8844} \approx 1.131.$$

That is, the investment yields a rate of return of approximately 13.1% per period. □

The rate of return of investments whose string of payments spans more than two periods will usually have to be numerically determined. Because of the monotonicity of $P(r)$, a trial-and-error approach is usually quite efficient.

Remarks. (1) If we interpret the cash flow sequence by supposing that $b_1, \dots, b_n$ represent the successive periodic payments made to a lender who loans a to a borrower, then the lender's periodic rate of return r^* is exactly the effective interest rate per period paid by the borrower.

(2) The quantity r^* is also sometimes called the *internal rate of return*.

Consider now a more general investment cash flow sequence $c_0, c_1, \dots, c_n$. Here, if $c_i \geq 0$ then the amount c_i is received by the investor at the end of period i, and if $c_i < 0$ then the amount $-c_i$ must be paid by the investor at the end of period i. If we let

$$P(r) = \sum_{i=0}^{n} c_i(1+r)^{-i}$$

be the present value of this cash flow when the interest rate is r per period, then in general there will not necessarily be a unique solution of the equation

$$P(r) = 0$$

in the region $r > -1$. As a result, the rate-of-return concept is unclear in the case of more general cash flows than the ones considered here. In addition, even in cases where we can show that the preceding equation has a unique solution r^*, it may result that $P(r)$ is not a monotone function of r; consequently, we could *not* assert that the investment yields a positive present value return when the interest rate is on one side of r^* and a negative present value return when it is on the other side.

One general situation for which we can prove that there is a unique solution is when the cash flow sequence starts out negative (resp. positive), eventually becomes positive (negative), and then remains nonnegative (nonpositive) from that point on. In other words, the sequence $c_0, c_1, \dots, c_n$ has a single sign change. It then follows – upon using Descartes' rule of sign, along with the known existence of at least one solution – that there is a unique solution of the equation $P(r) = 0$ in the region $r > -1$.

4.4 Continuously Varying Interest Rates

Suppose that interest is continuously compounded but with a rate that is changing in time. Let the present time be time 0, and let $r(s)$ denote the interest rate at time s. Thus, if you put x in a bank at time s, then the

$$\text{amount in your account at time } s+h \approx x(1+r(s)h) \quad (h \text{ small}).$$

The quantity $r(s)$ is called the *spot* or the *instantaneous* interest rate at time s.

Let $D(t)$ be the amount that you will have on account at time t if you deposit 1 at time 0. In order to determine $D(t)$ in terms of the interest rates $r(s)$, $0 \leq s \leq t$, note that (for h small) we have

$$D(s+h) \approx D(s)(1+r(s)h)$$

or

$$D(s+h) - D(s) \approx D(s)r(s)h$$

or

$$\frac{D(s+h) - D(s)}{h} \approx D(s)r(s).$$

The preceding approximation becomes exact as h becomes smaller and smaller. Hence, taking the limit as $h \to 0$, it follows that

$$D'(s) = D(s)r(s)$$

or

$$\frac{D'(s)}{D(s)} = r(s),$$

implying that

$$\int_0^t \frac{D'(s)}{D(s)}\,ds = \int_0^t r(s)\,ds$$

or

$$\log(D(t)) - \log(D(0)) = \int_0^t r(s)\,ds.$$

Since $D(0) = 1$, we obtain from the preceding equation that

$$D(t) = \exp\left\{\int_0^t r(s)\,ds\right\}.$$

Now let $P(t)$ denote the present (i.e. time-0) value of the amount 1 that is to be received at time t ($P(t)$ would be the cost of a bond that yields a return of 1 at time t; it would equal e^{-rt} if the interest rate were always equal to r). Because a deposit of $1/D(t)$ at time 0 will be worth 1 at time t, we see that

$$P(t) = \frac{1}{D(t)} = \exp\left\{-\int_0^t r(s)\,ds\right\}. \tag{4.3}$$

Let $\bar{r}(t)$ denote the average of the spot interest rates up to time t; that is,

$$\bar{r}(t) = \frac{1}{t}\int_0^t r(s)\,ds.$$

The function $\bar{r}(t)$, $t \geq 0$, is called the *yield curve*.

Example 4.4a Find the yield curve and the present value function if

$$r(s) = \frac{1}{1+s}r_1 + \frac{s}{1+s}r_2.$$

Solution. Rewriting $r(s)$ as

$$r(s) = r_2 + \frac{r_1 - r_2}{1+s}, \quad s \geq 0,$$

shows that the yield curve is given by

$$\bar{r}(t) = \frac{1}{t}\int_0^t \left(r_2 + \frac{r_1 - r_2}{1+s}\right) ds = r_2 + \frac{r_1 - r_2}{t}\log(1+t).$$

Consequently, the present value function is

$$P(t) = \exp\{-t\bar{r}(t)\} = \exp\{-r_2 t\}\exp\{-\log((1+t)^{r_1 - r_2})\} = \exp\{-r_2 t\}(1+t)^{r_2 - r_1}.$$

□

4.5 Exercises

Exercise 4.1 What is the effective interest rate when the nominal interest rate of 10% is

(a) compounded semiannually;
(b) compounded quarterly;
(c) compounded continuously?

Exercise 4.2 Suppose that you deposit your money in a bank that pays interest at a nominal rate of 10% per year. How long will it take for your money to double if the interest is compounded continuously?

Exercise 4.3 If you receive 5% interest compounded yearly, approximately how many years will it take for your money to quadruple? What if you were earning only 4%?

Exercise 5.22 A (K_1, t_1, K_2, t_2) *double call option* is one that can be exercised either at time t_1 with strike price K_1 or at time t_2 $(t_2 > t_1)$ with strike price K_2. Argue that you would never exercise at time t_1 if $K_1 > e^{-r(t_2-t_1)}K_2$.

Exercise 5.23 In a *capped call option*, the return is capped at a certain specified value A. That is, if the option has strike price K and expiration time t, then the payoff at time t is

$$\min(A, (S(t) - K)^+),$$

where $S(t)$ is the price of the security at time t. Show that an equivalent way of defining such an option is to let

$$\max(K, S(t) - A)$$

be the strike price when the call is exercised at time t.

Exercise 5.24 Argue that an American capped call option should be exercised early only when the price of the security is at least $K + A$.

Exercise 5.25 A function $f(x)$ is said to be *concave* if, for all x, y and $0 < \lambda < 1$,

$$f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda)f(y).$$

(a) Give a geometrical interpretation of when a function is concave.
(b) Argue that $f(x)$ is concave if and only if $g(x) = -f(x)$ is convex.

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6. The Arbitrage Theorem

6.1 The Arbitrage Theorem

Consider an experiment whose set of possible outcomes is $\{1, 2, \ldots, m\}$, and suppose that n wagers concerning this experiment are available. If the amount x is bet on wager i, then $xr_i(j)$ is received if the outcome of the experiment is j $(j = 1, \ldots, m)$. In other words, $r_i(\cdot)$ is the *return function* for a unit bet on wager i. The amount bet on a wager is allowed to be positive, negative, or zero.

A betting strategy is a vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, with the interpretation that x_1 is bet on wager 1, x_2 is bet on wager 2, ..., x_n is bet on wager n. If the outcome of the experiment is j, then the return from the betting strategy $\mathbf{x}$ is given by

$$\text{return from } \mathbf{x} = \sum_{i=1}^{n} x_i r_i(j).$$

The following result, known as the *arbitrage theorem*, states that either there exists a probability vector $\mathbf{p} = (p_1, p_2, \ldots, p_m)$ on the set of possible outcomes of the experiment under which the expected return of each wager is equal to zero, or else there exists a betting strategy that yields a positive win for each outcome of the experiment.

Theorem 6.1.1 (The Arbitrage Theorem) *Exactly one of the following is true: Either*

(a) *there is a probability vector* $\mathbf{p} = (p_1, p_2, \ldots, p_m)$ *for which*

$$\sum_{j=1}^{m} p_j r_i(j) = 0 \quad \textit{for all } i = 1, \ldots, n,$$

or else

(b) *there is a betting strategy* $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ *for which*

$$\sum_{i=1}^{n} x_i r_i(j) > 0 \quad \textit{for all } j = 1, \ldots, m.$$

Proof. See Section 6.3.

If X is the outcome of the experiment, then the arbitrage theorem states that either there is a set of probabilities $(p_1, p_2, \ldots, p_m)$ such that if

$$P\{X = j\} = p_j \quad \text{for all } j = 1, \ldots, m$$

then

$$E[r_i(X)] = 0 \quad \text{for all } i = 1, \ldots, n,$$

or else there is a betting strategy that leads to a sure win. In other words, either there is a probability vector on the outcomes of the experiment that results in all bets being fair, or else there is a betting scheme that guarantees a win.

Definition Probabilities on the set of outcomes of the experiment that result in all bets being fair are called *risk-neutral probabilities*.

Example 6.1a In some situations, the only type of wagers allowed are ones that choose one of the outcomes i $(i = 1, \ldots, m)$ and then bet that i is the outcome of the experiment. The return from such a bet is often quoted in terms of *odds*. If the odds against outcome i are o_i (often expressed as "o_i to 1"), then a one-unit bet will return either o_i if i is the outcome of the experiment or -1 if i is not the outcome. That is, a one-unit bet on i will either win o_i or lose 1. The return function for such a bet is given by

$$r_i(j) = \begin{cases} o_i & \text{if } j = i, \\ -1 & \text{if } j \neq i. \end{cases}$$

Suppose that the odds $o_1, o_2, \ldots, o_m$ are quoted. In order for there not to be a sure win, there must be a probability vector $\mathbf{p} = (p_1, p_2, \ldots, p_m)$ such that, for each i $(i = 1, \ldots, m)$,

$$0 = E_{\mathbf{p}}[r_i(X)] = o_i p_i - (1 - p_i).$$

That is, we must have

$$p_i = \frac{1}{1 + o_i}.$$

Since the p_i must sum to 1, this means that the condition for there not to be an arbitrage is that

$$\sum_{i=1}^{m} \frac{1}{1 + o_i} = 1.$$

That is, if $\sum_{i=1}^{m} (1 + o_i)^{-1} \neq 1$, then a sure win is possible. For instance, suppose there are three possible outcomes and the quoted odds are as follows.

Outcome	Odds
1	1
2	2
3	3

That is, the odds against outcome 1 are 1 to 1; they are 2 to 1 against outcome 2; and they are 3 to 1 against outcome 3. Since

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{13}{12} \neq 1,$$

a sure win is possible. One possibility is to bet -1 on outcome 1 (so you either win 1 if the outcome is not 1 or you lose 1 if the outcome is 1) and bet $-.7$ on outcome 2 (so you either win .7 if the outcome is not 2 or you lose 1.4 if it is 2), and $-.5$ on outcome 3 (so you either win .5 if the outcome is not 3 or you lose 1.5 if it is 3). If the experiment results in outcome 1, you win $-1 + .7 + .5 = .2$; if it results in outcome 2, you win $1 - 1.4 + .5 = .1$; if it results in outcome 3, you win $1 + .7 - 1.5 = .2$. Hence, in all cases you win a positive amount. □

Example 6.1b Let us reconsider the option pricing example of Section 5.1, where the initial price of a stock is 100 and the price after one period is assumed to be either 200 or 50. At a cost of C per share, we can purchase at time 0 the option to buy the stock at time 1 for the price of 150. For what value of C is no sure win possible?

Solution. In the context of this section, the outcome of the experiment is the value of the stock at time 1; thus, there are two possible outcomes. There are also two different wagers: to buy (or sell) the stock, and to buy (or sell) the option. By the arbitrage theorem, there will be no sure win if there are probabilities $(p, 1-p)$ on the outcomes that make the expected present value return equal to zero for both wagers.

The present value return from purchasing one share of the stock is

$$\text{return} = \begin{cases} 200(1+r)^{-1} - 100 & \text{if the price is 200 at time 1,} \\ 50(1+r)^{-1} - 100 & \text{if the price is 50 at time 1.} \end{cases}$$

Hence, if p is the probability that the price is 200 at time 1, then

$$E[\text{return}] = p\left[\frac{200}{1+r} - 100\right] + (1-p)\left[\frac{50}{1+r} - 100\right]$$
$$= p\frac{150}{1+r} + \frac{50}{1+r} - 100.$$

Setting this equal to zero yields that

$$p = \frac{1+2r}{3}.$$

Therefore, the only probability vector $(p, 1-p)$ that results in a zero expected return for the wager of purchasing the stock has $p = (1+2r)/3$.

In addition, the present value return from purchasing one option is

$$\text{return} = \begin{cases} 50(1+r)^{-1} - C & \text{if the price is 200 at time 1,} \\ -C & \text{if the price is 50 at time 1.} \end{cases}$$

Hence, when $p = (1+2r)/3$, the expected return of purchasing one option is

$$E[\text{return}] = \frac{1+2r}{3}\frac{50}{1+r} - C.$$

It thus follows from the arbitrage theorem that the only value of C for which there will not be a sure win is

$$C = \frac{1+2r}{3}\frac{50}{1+r};$$

that is, when

$$C = \frac{50+100r}{3(1+r)},$$

which is in accord with the result of Section 5.1. □

6.2 The Multiperiod Binomial Model

Let us now consider a stock option scenario in which there are n periods and where the nominal interest rate is r per period. Let $S(0)$ be the initial price of the stock, and for $i = 1, \dots, n$ let $S(i)$ be its price at i time periods later. Suppose that $S(i)$ is either $uS(i-1)$ or $dS(i-1)$, where $d < 1+r < u$. That is, going from one time period to the next, the price either goes up by the factor u or down by the factor d. Furthermore, suppose that at time 0 an option may be purchased that enables one to buy the stock after n periods have passed for the amount K. In addition, the stock may be purchased and sold anytime within these n time periods.

Let X_i equal 1 if the stock's price goes up by the factor u from period $i-1$ to i, and let it equal 0 if that price goes down by the factor d. That is,

$$X_i = \begin{cases} 1 & \text{if } S(i) = uS(i-1), \\ 0 & \text{if } S(i) = dS(i-1). \end{cases}$$

The outcome of the experiment can now be regarded as the value of the vector $(X_1, X_2, \dots, X_n)$. It follows from the arbitrage theorem that, in order for there not to be an arbitrage opportunity, there must be probabilities on these outcomes that make all bets fair. That is, there must be a set of probabilities

$$P\{X_1 = x_1, \dots, X_n = x_n\}, \quad x_i = 0, 1, \ i = 1, \dots, n,$$

that make all bets fair.

Now consider the following type of bet: First choose a value of i ($i = 1, \dots, n$) and a vector $(x_1, \dots, x_{i-1})$ of zeros and ones, and then observe the first $i-1$ changes. If $X_j = x_j$ for each $j = 1, \dots, i-1$, immediately buy one unit of stock and then sell it back the next period. If the stock is purchased, then its cost at time $i-1$ is $S(i-1)$; the time-$(i-1)$ value of the amount obtained when it is then sold at time i is either $(1+r)^{-1}uS(i-1)$ if the stock goes up or $(1+r)^{-1}dS(i-1)$ if it goes down. Therefore, if we let

$$\alpha = P\{X_1 = x_1, \dots, X_{i-1} = x_{i-1}\}$$

denote the probability that the stock is purchased, and let

$$p = P\{X_i = 1 \mid X_1 = x_1, \dots, X_{i-1} = x_{i-1}\}$$

(a) What is the no-arbitrage cost (at time 0) of this option?
(b) Is the cost of part (a) unique? Briefly explain.
(c) If each price change is equally likely to be an up or a down movement, what is the expected amount that an option holder receives at the time of expiration?

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7. The Black–Scholes Formula

7.1 Introduction

In this chapter we derive the celebrated Black–Scholes formula, which gives – under the assumption that the price of a security evolves according to a geometric Brownian motion – the unique no-arbitrage cost of a call option on this security. Section 7.2 gives the derivation of the no-arbitrage cost, which is a function of five variables, and Section 7.3 discusses some of the properties of this function. Section 7.4 gives the strategy that can, in theory, be used to obtain an abitrage when the cost of the security is not as specified by the formula. Section 7.5, which is more theoretical than other sections of the text, presents simplified derivations of (1) the computational form of the Black–Scholes formula and (2) the partial derivatives of the no-arbitrage cost with respect to each of its five parameters.

7.2 The Black–Scholes Formula

Consider a call option having strike price K and expiration time t. That is, the option allows one to purchase a single unit of an underlying security at time t for the price K. Suppose further that the nominal interest rate is r, compounded continuously, and also that the price of the security follows a geometric Brownian motion with drift parameter μ and volatility parameter σ. Under these assumptions, we will find the unique cost of the option that does not give rise to an arbitrage.

To begin, let $S(y)$ denote the price of the security at time y. Because $\{S(y),\ 0 \le y \le t\}$ follows a geometric Brownian motion with volatility parameter σ and drift parameter μ, the n-stage approximation of this model supposes that, every t/n time units, the price changes; its new value is equal to its old value multiplied either by the factor

$$u = e^{\sigma\sqrt{t/n}} \quad \text{with probability } \frac{1}{2}\left(1 + \frac{\mu}{\sigma}\sqrt{t/n}\right)$$

or by the factor

$$d = e^{-\sigma\sqrt{t/n}} \quad \text{with probability } \frac{1}{2}\left(1 - \frac{\mu}{\sigma}\sqrt{t/n}\right).$$

Thus, the n-stage approximation model is an n-stage binomial model in which the price at each time interval t/n either goes up by a multiplicative factor u or down by a multiplicative factor d. Therefore, if we let

$$X_i = \begin{cases} 1 & \text{if } S(it/n) = uS((i-1)t/n), \\ 0 & \text{if } S(it/n) = dS((i-1)t/n), \end{cases}$$

then it follows from the results of Section 6.2 that the only probability law on $X_1, \dots, X_n$ that makes all security buying bets fair in the n-stage approximation model is the one that takes the X_i to be independent with

$$\begin{aligned} p &\equiv P\{X_i = 1\} \\ &= \frac{1 + rt/n - d}{u - d} \\ &= \frac{1 - e^{-\sigma\sqrt{t/n}} + rt/n}{e^{\sigma\sqrt{t/n}} - e^{-\sigma\sqrt{t/n}}}. \end{aligned}$$

Using the first three terms of the Taylor series expansion about 0 of the function e^x shows that

$$e^{-\sigma\sqrt{t/n}} \approx 1 - \sigma\sqrt{t/n} + \sigma^2 t/2n,$$
$$e^{\sigma\sqrt{t/n}} \approx 1 + \sigma\sqrt{t/n} + \sigma^2 t/2n.$$

Therefore,

$$\begin{aligned} p &\approx \frac{\sigma\sqrt{t/n} - \sigma^2 t/2n + rt/n}{2\sigma\sqrt{t/n}} \\ &= \frac{1}{2} + \frac{r\sqrt{t/n}}{2\sigma} - \frac{\sigma\sqrt{t/n}}{4} \\ &= \frac{1}{2}\left(1 + \frac{r - \sigma^2/2}{\sigma}\sqrt{t/n}\right). \end{aligned}$$

That is, the unique risk-neutral probabilities on the n-stage approximation model result from supposing that, in each period, the price either

goes up by the factor $e^{\sigma\sqrt{t/n}}$ with probability p or goes down by the factor $e^{-\sigma\sqrt{t/n}}$ with probability $1 - p$. But, from Section 3.2, it follows that as $n \to \infty$ this risk-neutral probability law converges to geometric Brownian motion with drift coefficient $r - \sigma^2/2$ and volatility parameter σ. Because the n-stage approximation model becomes the geometric Brownian motion as n becomes larger, it is reasonable to suppose (and can be rigorously proven) that this risk-neutral geometric Brownian motion is the only probability law on the evolution of prices over time that makes all security buying bets fair. (In other words, we have just argued that if the underlying price of a security follows a geometric Brownian motion with volatility parameter σ, then the only probability law on the sequence of prices that results in all security buying bets being fair is that of a geometric Brownian motion with drift parameter $r - \sigma^2/2$ and volatility parameter σ.) Consequently, by the arbitrage theorem, either options are priced to be fair bets according to the risk-neutral geometric Brownian motion probability law or else there will be an arbitrage.

Now, under the risk-neutral geometric Brownian motion, $S(t)/S(0)$ is a lognormal random variable with mean parameter $(r - \sigma^2/2)t$ and variance parameter $\sigma^2 t$. Hence C, the unique no-arbitrage cost of a call option to purchase the security at time t for the specified price K, is

$$\begin{aligned} C &= e^{-rt}E[(S(t) - K)^+] \\ &= e^{-rt}E[(S(0)e^W - K)^+], \end{aligned} \tag{7.1}$$

where W is a normal random variable with mean $(r - \sigma^2/2)t$ and variance $\sigma^2 t$.

The right side of Equation (7.1) can be explicitly evaluated (see Section 7.4 for the derivation) to give the following expression, known as the *Black–Scholes option pricing formula*:

$$C = S(0)\Phi(\omega) - Ke^{-rt}\Phi(\omega - \sigma\sqrt{t}), \tag{7.2}$$

where

$$\omega = \frac{rt + \sigma^2 t/2 - \log(K/S(0))}{\sigma\sqrt{t}}$$

and where $\Phi(x)$ is the standard normal distribution function.

Example 7.1a Suppose that a security is presently selling for a price of 30, the nominal interest rate is 8% (with the unit of time being one year), and the security's volatility is .20. Find the no-arbitrage cost of a call option that expires in three months and has a strike price of 34.

Solution. The parameters are

$$t = .25, \quad r = .08, \quad \sigma = .20, \quad K = 34, \quad S(0) = 30,$$

so we have

$$\omega = \frac{.02 + .005 - \log(34/30)}{(.2)(.5)} \approx -1.0016.$$

Therefore,

$$\begin{aligned} C &= 30\Phi(-1.0016) - 34e^{-.02}\Phi(-1.1016) \\ &= 30(.15827) - 34(.9802)(.13532) \\ &\approx .2383. \end{aligned}$$

The appropriate price of the option is thus 24 cents. □

Remarks. 1. Another way to derive the no-arbitrage option cost C is to consider the unique no-arbitrage cost of an option in the n-period approximation model and then let n go to infinity.

2. Let $C(s, t, K)$ be the no-arbitrage cost of an option having strike price K and exercise time t when the initial price of the security is s. That is, $C(s, t, K)$ is the C of the Black–Scholes formula having $S(0) = s$. If the price of the underlying security at time y $(0 < y < t)$ is $S(y) = s_y$, then $C(s_y, t - y, K)$ is the unique no-arbitrage cost of the option at time y. This is because, at time y, the option will expire after an additional time $t - y$ with the same exercise price K, and for the next $t - y$ units of time the security will follow a geometric Brownian motion with initial value s_y.

3. It follows from the put–call option parity formula given in Proposition 5.2.2 that the no-arbitrage cost of a European put option with initial price s, strike price K, and exercise time t – call it $P(s, t, K)$ – is given by

$$P(s, t, K) = C(s, t, K) + Ke^{-rt} - s,$$

where $C(s, t, K)$ is the no-arbitrage cost of a call option on the same stock.

4. Because the risk-neutral geometric Brownian motion depends only on σ and not on μ, it follows that the no-arbitrage cost of the option depends on the underlying Brownian motion only through its volatility parameter σ and not its drift parameter.

5. The no-arbitrage option cost is unchanged if the security's price over time is assumed to follow a geometric Brownian motion with a fixed volatility σ but with a drift that varies over time. Because the n-stage approximation model for the price history up to time t of the time-varying drift process is still a binomial up–down model with $u = e^{\sigma\sqrt{t/n}}$ and $d = e^{-\sigma\sqrt{t/n}}$, it has the same unique risk-neutral probability law as when the drift parameter is unchanging, and thus it will give rise to the same unique no-arbitrage option cost. (The only way that a changing drift parameter would affect our derivation of the Black–Scholes formula is by leading to different probabilities for up moves in the different time periods, but these probabilities have no effect on the the risk-neutral probabilities.)

7.3 Properties of the Black–Scholes Option Cost

The no-arbitrage option cost $C = C(s, t, K, \sigma, r)$ is a function of five variables: the security's initial price s; the expiration time t of the option; the strike price K; the security's volatility parameter σ; and the interest rate r. To see what happens to the cost as a function of each of these variables, we use Equation (7.1):

$$C(s, t, K, \sigma, r) = e^{-rt}E[(se^W - K)^+],$$

where W is a normal random variable with mean $(r - \sigma^2/2)t$ and variance $\sigma^2 t$.

Properties of $C = C(s, t, K, \sigma, r)$

1. *C is an increasing, convex function of s.*

This means that if the other four variables remain the same, then the no-arbitrage cost of the option is an increasing function of the security's initial price as well as a convex function of the security's initial price. These results (the first of which is very intuitive) follow from Equation (7.1). To see why, first note (see Figure 7.1) that, for any positive constant a, the function $e^{-rt}(sa - K)^+$ is an increasing, convex function

Figure 7.1: The Increasing, Convex Function $f(s) = e^{-rt}(sa - K)^+$

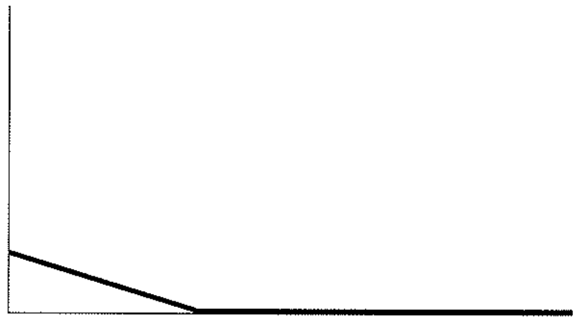

Figure 7.2: The Decreasing, Convex Function $f(K) = e^{-rt}(a - K)^+$

of s. Consequently, because the probability distribution of W does not depend on s, the quantity $e^{-rt}(se^W - K)^+$ is, for all W, increasing and convex in s, and thus so is its expected value.

2. *C is a decreasing, convex function of K.*

This follows from the fact that $e^{-rt}(se^W - K)^+$ is, for all W, decreasing and convex in K (see Figure 7.2), and thus so is its expectation. (This is in agreement with the more general arbitrage argument made in Section 5.2, which did not assume a model for the security's price evolution.)

3. *C is increasing in t.*

Although a mathematical argument can be given (see Section 7.4), a simpler and more intuitive argument is obtained by noting that it is immediate that the option cost would be increasing in t if the option were an *American* call option (for any additional time to exercise could not hurt, since one could always elect not to use it). Because the value of a European call option is the same as that of an American call option (Proposition 5.2.1), the result follows.

4. *C is increasing in σ.*

Because an option holder will greatly benefit from very large prices at the exercise time, while any additional price decrease below the exercise price will not cause any additional loss, this result seems at first sight to be quite intuitive. However, it is more subtle than it appears, because an increase in σ results not only in an increase in the variance of the logarithm of the final price under the risk-neutral valuation but also in a *decrease* in the mean (since $E[\log(S(t)/S(0))] = (r - \sigma^2/2)t$). Nevertheless, the result is true and will be shown mathematically in Section 7.4.

5. *C is increasing in r.*

To verify this property, note that we can express W, a normal random variable with mean $(r - \sigma^2/2)t$ and variance $t\sigma^2$, as

$$W = rt - \sigma^2 t/2 + \sigma\sqrt{t}Z,$$

where Z is a standard normal random variable with mean 0 and variance 1. Hence, from Equation (7.1) we have that

$$C = E[(se^{-\sigma^2 t/2 + \sigma\sqrt{t}Z} - Ke^{-rt})^+].$$

The result now follows because $(se^{-\sigma^2 t/2 + \sigma\sqrt{t}Z} - Ke^{-rt})^+$, and thus its expected value, is increasing in r. Indeed, it follows from the preceding that, under the no-arbitrage geometric Brownian motion model, the only effect of an increased interest rate is that it reduces the present value of the amount to be paid if the option is exercised, thus increasing the value of the option.

The rate of change in the value of the call option as a function of a change in the price of the underlying security is described by the quantity *delta,* denoted as Δ. Formally, if $C(s, t, K, \sigma, r)$ is the Black–Scholes cost valuation of the option, then Δ is its partial derivative with respect to s; that is,

$$\Delta = \frac{\partial}{\partial s} C(s, t, K, \sigma, r).$$

In Section 7.4 we will show that

$$\Delta = \Phi(\omega)$$

where, as given in Equation (7.2),

$$\omega = \frac{rt + \sigma^2 t/2 - \log(K/S(0))}{\sigma\sqrt{t}}.$$

Delta can be used to construct investment portfolios that hedge against risk. For instance, suppose that an investor feels that a call option is underpriced and consequently buys the call. To protect himself against a decrease in its price, he can simultaneously sell a certain number of shares of the security. To determine how many shares he should sell, note that if the price of the security decreases by the small amount h then the worth of the option will decrease by the amount $h\Delta$, implying that the investor would be covered if he sold Δ shares of the security. Therefore, a reasonable hedge might be to sell Δ shares of the security for each option purchased. This heuristic argument will be made precise in the next section, where we present the delta hedging arbitrage strategy – a strategy that can, in theory, be used to construct an arbitrage if a call option is not priced according to the Black–Scholes formula.

7.4 The Delta Hedging Arbitrage Strategy

In this section we show how the payoff from an option can be replicated by a fixed initial payment (divided into an initial purchasing of shares and an initial bank deposit, where either might be negative) and a continual readjustment of funds. We first present it for the finite-stage approximation model and then for the geometric Brownian motion model for the security's price evolution.

To begin, consider a security whose initial price is s and suppose that, after each time period, its price changes either by the multiple u or by the multiple d. Let us determine the amount of money x that you must

have at time 0 in order to meet a payment, at time 1, of a if the price of the stock is us at time 1 or of b if the price at time 1 is ds. To determine x, and the investment that enables you to meet the payment, suppose that you purchase y shares of the stock and then either put the remaining $x - ys$ in the bank if $x - ys \geq 0$ or borrow $ys - x$ from the bank if $x - ys < 0$. Then, for the initial cost of x, you will have a return at time 1 given by

$$\text{return at time } 1 = \begin{cases} yus + (x - ys)(1 + r) & \text{if } S(1) = us, \\ yds + (x - ys)(1 + r) & \text{if } S(1) = ds, \end{cases}$$

where $S(1)$ is the price of the security at time 1 and r is the interest rate per period. Thus, if we choose x and y such that

$$yus + (x - ys)(1 + r) = a,$$
$$yds + (x - ys)(1 + r) = b,$$

then after taking our money out of the bank (or meeting our loan payment) we will have the desired amount. Subtracting the second equation from the first gives that

$$y = \frac{a - b}{s(u - d)}.$$

Substituting the preceding expression for y into the first equation yields

$$\frac{a - b}{u - d}[u - (1 + r)] + x(1 + r) = a$$

or

$$\begin{aligned} x &= \frac{1}{1 + r}\left(a\left[1 - \frac{u - (1 + r)}{u - d}\right] + b\frac{u - 1 - r}{u - d}\right) \\ &= \frac{1}{1 + r}\left(a\frac{1 + r - d}{u - d} + b\frac{u - 1 - r}{u - d}\right) \\ &= p\frac{a}{1 + r} + (1 - p)\frac{b}{1 + r}, \end{aligned}$$

where

$$p = \frac{1 + r - d}{u - d}.$$

7.5 Some Derivations

In Section 7.5.1 we give the derivation of Equation (7.2), the computational form of the Black–Scholes formula. In Section 7.5.2 we derive the partial derivative of $C(s, t, K, \sigma, r)$ with respect to each of the quantities s, t, K, σ, and r.

7.5.1 *The Black–Scholes Formula*

Let

$$C(s, t, K, \sigma, r) = E[e^{-rt}(S(t) - K)^+]$$

be the risk-neutral cost of a call option with strike price K and expiration time t when the interest rate is r and the underlying security, whose initial price is s, follows a geometric Brownian motion with volatility parameter σ. To derive the Black–Scholes option pricing formula as well as the partial derivatives of C, we will use the fact that, under the risk-neutral probabilities, $S(t)$ can be expressed as

$$S(t) = s\exp\{(r - \sigma^2/2)t + \sigma\sqrt{t}Z\}, \tag{7.3}$$

where Z is a standard normal random variable.

Let I be the indicator random variable for the event that the option finishes in the money. That is,

$$I = \begin{cases} 1 & \text{if } S(t) > K, \\ 0 & \text{if } S(t) \le K. \end{cases} \tag{7.4}$$

We will use the following lemmas.

Lemma 7.5.1 *Using the representations* (7.3) *and* (7.4),

$$I = \begin{cases} 1 & \text{if } Z > \sigma\sqrt{t} - \omega, \\ 0 & \text{otherwise}, \end{cases}$$

where

$$\omega = \frac{rt + \sigma^2 t/2 - \log(K/s)}{\sigma\sqrt{t}}.$$

Proof.

$$\begin{aligned} S(t) > K &\iff \exp\{(r - \sigma^2/2)t + \sigma\sqrt{t}Z\} > K/s \\ &\iff Z > \frac{\log(K/s) - (r - \sigma^2/2)t}{\sigma\sqrt{t}} \\ &\iff Z > \sigma\sqrt{t} - \omega. \end{aligned}$$

□

Lemma 7.5.2

$$E[I] = P\{S(t) > K\} = \Phi(\omega - \sigma\sqrt{t}),$$

where Φ *is the standard normal distribution function.*

Proof. It follows from its definition that

$$\begin{aligned} E[I] &= P\{S(t) > K\} \\ &= P\{Z > \sigma\sqrt{t} - \omega\} \qquad \text{(from Lemma 7.5.1)} \\ &= P\{Z < \omega - \sigma\sqrt{t}\} \\ &= \Phi(\omega - \sigma\sqrt{t}). \end{aligned}$$

□

Lemma 7.5.3

$$e^{-rt}E[IS(t)] = s\Phi(\omega).$$

Proof. With $c = \sigma\sqrt{t} - \omega$, it follows from the representation (7.3) and Lemma 7.5.1 that

$$\begin{aligned} E[IS(t)] &= \int_c^\infty s\exp\{(r - \sigma^2/2)t + \sigma\sqrt{t}x\}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\,dx \\ &= \frac{1}{\sqrt{2\pi}}s\exp\{(r - \sigma^2/2)t\}\int_c^\infty \exp\{-(x^2 - 2\sigma\sqrt{t}x)/2\}\,dx \\ &= \frac{1}{\sqrt{2\pi}}se^{rt}\int_c^\infty \exp\{-(x - \sigma\sqrt{t})^2/2\}\,dx \\ &= se^{rt}\frac{1}{\sqrt{2\pi}}\int_{-\omega}^\infty e^{-y^2/2}\,dy \qquad \text{(by letting } y = x - \sigma\sqrt{t}\text{)} \\ &= se^{rt}P\{Z > -\omega\} \\ &= se^{rt}\Phi(\omega). \end{aligned}$$

□

Theorem 7.5.1 (The Black–Scholes Pricing Formula)

$$C(s, t, K, \sigma, r) = s\Phi(\omega) - Ke^{-rt}\Phi(\omega - \sigma\sqrt{t}).$$

Proof.

$$\begin{aligned} C(s, t, K, \sigma, r) &= e^{-rt}E[(S(t) - K)^+] \\ &= e^{-rt}E[I(S(t) - K)] \\ &= e^{-rt}E[I(S(t)] - Ke^{-rt}E[I], \end{aligned}$$

and the result follows from Lemmas 7.5.2 and 7.5.3. □

7.5.2 *The Partial Derivatives*

Let

$$C = C(s, t, K, \sigma, r) = E[e^{-rt}(S(t) - K)^+] = E[e^{-rt}I(S(t) - K)]$$

be the Black–Scholes call option formula, where I is defined by (7.4). Let x be one of the five parameters s, t, K, σ, r. To determine

$$\frac{\partial C}{\partial x} = \frac{\partial}{\partial x}E[e^{-rt}I(S(t) - K)],$$

we will make use of the fact that the partial derivative and the expectation operation can be interchanged. This gives

$$\frac{\partial C}{\partial x} = E\left[\frac{\partial}{\partial x}e^{-rt}I(S(t) - K)\right].$$

Because

$$\frac{\partial I}{\partial x} = 0 \quad \text{if } S(t) \neq K,$$

we see, on using the chain rule for the derivative of a product, that

$$\frac{\partial}{\partial x}e^{-rt}I(S(t) - K) = I\frac{\partial}{\partial x}e^{-rt}(S(t) - K) \quad \text{if } S(t) \neq K.$$

Because $P\{S(t) = K\} = 0$, we can conclude from the preceding that

$$\frac{\partial C}{\partial x} = E\left[I\frac{\partial}{\partial x}e^{-rt}(S(t) - K)\right]. \tag{7.5}$$

We will now derive the partial derivatives of C with respect to K, s, and r.

Proposition 7.5.1

$$\frac{\partial C}{\partial K} = -e^{-rt}\Phi(\omega - \sigma\sqrt{t}).$$

Proof. Because $S(t)$ does not depend on K,

$$\frac{\partial}{\partial K}e^{-rt}(S(t) - K) = -e^{-rt}.$$

Using Equation (7.5), this gives

$$\begin{aligned} \frac{\partial C}{\partial K} &= E[-Ie^{-rt}] \\ &= -e^{-rt}E[I] \\ &= -e^{-rt}\Phi(\omega - \sigma\sqrt{t}), \end{aligned}$$

where the final equality used Lemma 7.5.2. □

As noted previously, $\frac{\partial C}{\partial s}$ is called *delta*.

Proposition 7.5.2

$$\frac{\partial C}{\partial s} = \Phi(\omega).$$

Proof. Using the representation of Equation (7.3), we see that

$$\frac{\partial}{\partial s}e^{-rt}(S(t) - K) = e^{-rt}\frac{\partial S(t)}{\partial s} = \frac{S(t)}{s}e^{-rt}.$$

Hence, by Equation (7.5),

$$\begin{aligned} \frac{\partial C}{\partial s} &= \frac{e^{-rt}}{s}E[IS(t)] \\ &= \Phi(\omega), \end{aligned}$$

where the final equality used Lemma 7.5.3. □

The partial derivative of C with respect to r is called *rho*.

Proposition 7.5.3

$$\frac{\partial C}{\partial r} = Kte^{-rt}\Phi(\omega - \sigma\sqrt{t}).$$

Proof.

$$\begin{aligned}\frac{\partial}{\partial r}[e^{-rt}(S(t) - K)] &= -te^{-rt}(S(t) - K) + e^{-rt}\frac{\partial S(t)}{\partial r}\\ &= -te^{-rt}(S(t) - K) + e^{-rt}tS(t) \qquad \text{(from (7.3))}\\ &= Kte^{-rt}.\end{aligned}$$

Therefore, by Equation (7.5) and Lemma 7.5.2,

$$\frac{\partial C}{\partial r} = Kte^{-rt}E[I] = Kte^{-rt}\Phi(\omega - \sigma\sqrt{t}).$$ □

In order to determine the other partial derivatives, we need an additional lemma, whose proof is similar to that of Lemma 7.5.3.

Lemma 7.5.4 *With $S(t)$ as given by Equation* (7.3),

$$e^{-rt}E[IS(t)Z] = s(\Phi'(\omega) + \sigma\sqrt{t}\Phi(\omega)).$$

Proof. With $c = \sigma\sqrt{t} - \omega$, it follows from Lemma 7.5.1 that

$$\begin{aligned}&E[IZS(t)]\\ &= \int_c^\infty xs\exp\{(r - \sigma^2/2)t + \sigma\sqrt{t}x\}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\,dx\\ &= \frac{1}{\sqrt{2\pi}}s\exp\{(r - \sigma^2/2)t\}\int_c^\infty x\exp\{-(x^2 - 2\sigma\sqrt{t}x)/2\}\,dx\\ &= \frac{1}{\sqrt{2\pi}}se^{rt}\int_c^\infty x\exp\{-(x - \sigma\sqrt{t})^2/2\}\,dx\\ &= \frac{1}{\sqrt{2\pi}}se^{rt}\int_{-\omega}^\infty (y + \sigma\sqrt{t})e^{-y^2/2}\,dy \qquad \text{(by letting } y = x - \sigma\sqrt{t})\\ &= se^{rt}\left[\int_{-\omega}^\infty \frac{1}{\sqrt{2\pi}}ye^{-y^2/2}\,dy + \sigma\sqrt{t}\frac{1}{\sqrt{2\pi}}\int_{-\omega}^\infty e^{-y^2/2}\,dy\right]\end{aligned}$$

The partial derivative of C with respect to σ is called *vega*.

Proposition 7.5.4

$$\frac{\partial C}{\partial \sigma} = s\sqrt{t}\Phi'(\omega).$$

Proof. Equation (7.3) yields that

$$\frac{\partial}{\partial \sigma}[e^{-rt}(S(t) - K)] = e^{-rt}S(t)(-t\sigma + \sqrt{t}Z).$$

Hence, by Equation (7.5),

$$\begin{aligned}\frac{\partial C}{\partial \sigma} &= E[e^{-rt}IS(t)(-t\sigma + \sqrt{t}Z)]\\ &= -t\sigma e^{-rt}E[IS(t)] + \sqrt{t}e^{-rt}E[IS(t)Z]\\ &= -t\sigma s\Phi(\omega) + s\sqrt{t}(\Phi'(\omega) + \sigma\sqrt{t}\Phi(\omega))\\ &= s\sqrt{t}\Phi'(\omega),\end{aligned}$$

where the next-to-last equality used Lemmas 7.5.3 and 7.5.4. □

The negative of the partial derivative of C with respect to t is called *theta*.

Proposition 7.5.5

$$\frac{\partial C}{\partial t} = \frac{\sigma}{2\sqrt{t}}s\Phi'(\omega) + Kre^{-rt}\Phi(\omega - \sigma\sqrt{t}).$$

Proof.

$$\begin{aligned}\frac{\partial}{\partial t}[e^{-rt}(S(t) - K)] &= e^{-rt}\frac{\partial S(t)}{\partial t} - re^{-rt}S(t) + Kre^{-rt}\\ &= e^{-rt}S(t)\left(r - \frac{\sigma^2}{2} + \frac{\sigma}{2\sqrt{t}}Z\right)\\ &\quad - re^{-rt}S(t) + Kre^{-rt}\\ &= e^{-rt}S(t)\left(\frac{-\sigma^2}{2} + \frac{\sigma}{2\sqrt{t}}Z\right) + Kre^{-rt}.\end{aligned}$$

8. Additional Results on Options

8.1 Introduction

In this chapter we look at some extensions of the basic call option model. In Section 8.2 we consider European call options on dividend-paying securities under three different scenarios for how the dividend is paid. In Section 8.2.1 we suppose that the dividend for each share owned is paid continuously in time at a rate equal to a fixed fraction of the price of the security. In Sections 8.2.2 and 8.2.3 we suppose that the dividend is to be paid at a specified time, with the amount paid equal to a fixed fraction of the price of the security (Section 8.2.2) or to a fixed amount (Section 8.2.3). In Section 8.3 we show how to determine the no-arbitrage price of an American put option. In Section 8.4 we introduce a model that allows for the possibilities of jumps in the price of a security. This model supposes that the security's price changes according to a geometric Brownian motion, with the exception that at random times the price is assumed to change by a random multiplicative factor. In Section 8.4.1 we derive an exact formula for the no-arbitrage cost of a call option when the multiplicative jumps have a lognormal probability distribution. In Section 8.4.2 we suppose that the multiplicative jumps have an arbitrary probability distribution; we show that the no-arbitrage cost is always at least as large as the Black–Scholes formula when there are no jumps, and we then present an approximation for the no-arbitrage cost. In Section 8.5 we describe a variety of different techniques for estimating the volatility parameter. Section 8.6 consists of comments regarding the results obtained in this and the previous chapter.

8.2 Call Options on Dividend-Paying Securities

In this section we determine the no-arbitrage price for a European call option on a stock that pays a dividend. We consider three cases that correspond to different types of dividend payments.

8.2.1 *The Dividend for Each Share of the Security Is Paid Continuously in Time at a Rate Equal to a Fixed Fraction f of the Price of the Security*

For instance, if the stock's price is presently S, then in the next dt time units the dividend payment per share of stock owned will be approximately $fS\,dt$ when dt is small.

To begin, we need a model for the evolution of the price of the security over time. One way to obtain a reasonable model is to suppose that all dividends are reinvested in the purchase of additional shares of the stock. Thus, we would be continuously adding additional shares at the rate f times the number of shares we presently own. Consequently, our number of shares is growing by a continuously compounded rate f. Therefore, if we purchased a single share at time 0, then at time t we would have e^{ft} shares with a total market value of

$$M(t) = e^{ft}S(t).$$

It seems reasonable to suppose that $M(t)$ follows a geometric Brownian motion with volatility given by, say, σ. The risk-neutral probabilities on $M(t)$ are those of a geometric Brownian motion with volatility σ and drift $r - \sigma^2/2$. Consequently, for there not to be an arbitrage, all options must be priced to be fair bets under the assumption that $e^{fy}S(y)$ ($y \geq 0$) follows such a risk-neutral geometric Brownian motion.

Consider a European option to purchase the security at time t for the price K. Under the risk-neutral probabilities on $M(t)$, we have

$$\frac{S(t)}{S(0)} = \frac{e^{-ft}M(t)}{M(0)} = e^{-ft}e^{W},$$

where W is a normal random variable with mean $(r - \sigma^2/2)t$ and variance $t\sigma^2$. Thus, under the risk-neutral probabilities,

$$S(t) = S(0)e^{-ft}e^{W}.$$

Therefore, by the arbitrage theorem, we see that if $S(0) = s$ then the

$$\begin{aligned}\text{no-arbitrage cost of } (K,t) \text{ option} &= e^{-rt}E[(S(t) - K)^+] \\ &= e^{-rt}E[(se^{-ft}e^{W} - K)^+] \\ &= C(se^{-ft}, t, K, \sigma, r),\end{aligned}$$

where $C(s, t, K, \sigma, r)$ is the Black–Scholes formula. In other words, the no-arbitrage cost of the European (K, t) call option, when the initial price is s, is exactly what its cost would be if there were no dividends but the inital price were se^{-ft}.

8.2.2 *For Each Share Owned, a Single Payment of $fS(t_d)$ Is Made at Time t_d*

It is usual to suppose that, at the moment the dividend is paid, the price of a share instantaneously decreases by the amount of the dividend. (If one assumes that the price never drops by at least the amount of the dividend, then buying immediately before and selling immediately after the payment of the dividend would result in an arbitrage; hence, there must be some possibility of a drop in price of at least the amount of the dividend, and the usual assumption – which is roughly in agreement with actual data – is that the price decreases by exactly the dividend paid.) Because of this downward price jump at the moment at which the dividend is paid, it is clear that we cannot model the price of the security as a geometric Brownian motion (which has no discontinuities). However, if we again suppose that the dividend payment at time t_d is used to purchase additional shares, then we can model the market value of our shares by a geometric Brownian motion. Because the price of a share immediately after the dividend is paid is $S(t_d) - fS(t_d) = (1-f)S(t_d)$, the dividend $fS(t_d)$ from a single share can be used to purchase $f/(1-f)$ additional shares. Hence, starting with a single share at time 0, the market value of our portfolio at time y, call it $M(y)$, is

$$M(y) = \begin{cases} S(y) & \text{if } y < t_d, \\ \frac{1}{1-f} S(y) & \text{if } y \geq t_d. \end{cases}$$

Let us take as our model that $M(y)$ $(y \geq 0)$ follows a geometric Brownian motion with volatility parameter σ. The risk-neutral probabilities for this process are that of a geometric Brownian motion with volatility parameter σ and drift parameter $r - \sigma^2/2$. For $y < t_d$, $M(y) = S(y)$; thus, when $t < t_d$, the unique no-arbitrage cost of a (K, t) option on the security is just the usual Black–Scholes cost. For $t > t_d$, note that

$$\frac{S(t)}{S(0)} = (1-f)\frac{M(t)}{M(0)}, \quad t > t_d.$$

Thus, under the risk-neutral probabilities,

$$\frac{1}{1-f}\frac{S(t)}{S(0)} = \frac{M(t)}{M(0)} = e^W, \quad t > t_d,$$

where W is a normal random variable with mean $(r - \sigma^2/2)t$ and variance $t\sigma^2$. Thus, again under the risk-neutral probabilities,

$$S(t) = (1-f)S(0)e^W, \quad t > t_d.$$

When $t > t_d$, it follows by the arbitrage theorem that the unique no-arbitrage cost of a European (K, t) call option, when the initial price of the security is s, is exactly what its cost would be if there were no dividends but the inital price of the security were $s(1-f)$. That is, for $t > t_d$, the

$$\begin{aligned} \text{no-arbitrage cost of } (K, t) \text{ option} &= e^{-rt}E[(S(t) - K)^+] \\ &= e^{-rt}E[(s(1-f)e^W - K)^+] \\ &= C(s(1-f), t, K, \sigma, r), \end{aligned}$$

where $C(s, t, K, \sigma, r)$ is the Black–Scholes formula.

8.2.3 *For Each Share Owned, a Fixed Amount D Is to Be Paid at Time t_d*

As in the previous cases, we must first determine an appropriate model for $S(y)$ $(y \geq 0)$, the price evolution of the security. To begin, note that the known dividend payment D to be made to shareholders at the known time t_d necessitates that the price of the security at time $y < t_d$ must be at least $De^{-r(t_d - y)}$. This is true because, if $S(y) < De^{-r(t_d - y)}$ for some $y < t_d$, then an arbitrage can be effected by borrowing $S(y)$ at time y and using this amount to purchase the security; the security is held through time t_d and the loan is paid off immediately after the dividend is received. Consequently, we cannot model $S(y)$ $(0 \leq y \leq t_d)$ as a geometric Brownian motion.

To model the price evolution up to time t_d, it is best to separate the price of the security into two parts of which one is riskless and results from the fixed payment at time t_d. That is, let

$$S^*(y) = S(y) - De^{-r(t_d - y)}, \quad y < t_d,$$

and write

$$S(y) = De^{-r(t_d - y)} + S^*(y), \quad y < t_d.$$

It is reasonable to model $S^*(y)$, $y < t_d$, as a geometric Brownian motion, with its volatility parameter denoted by σ. Because the riskless part of the price is increasing at rate r, it is intuitive that risk-neutral probabilities would result when the drift parameter of $S^*(y)$, $y < t_d$, is $r - \sigma^2/2$. To check that this assumption on the drift would result in all bets being fair, note that under it the expected present value return from purchasing the security at time 0 and then selling at time $t < t_d$ is

$$\begin{aligned} e^{-rt}E[S(t)] &= e^{-rt}De^{-r(t_d - t)} + e^{-rt}E[S^*(t)] \\ &= De^{-rt_d} + S^*(0) \\ &= S(0). \end{aligned}$$

Suppose now that we want to find the no-arbitrage cost of a European call option with strike price K and expiration time $t < t_d$ when the initial price of the security is s. If $K < De^{-r(t_d - t)}$, then the option will definitely be exercised (because $S(t) \geq De^{-r(t_d - t)}$). Consequently, purchasing the option in this case is equivalent to purchasing the security. By the law of one price, the cost of the option plus the present value of the strike price must therefore equal the cost of the security. That is, if $t < t_d$ and $K < De^{-r(t_d - t)}$ then the

$$\text{no-arbitrage cost of option} = s - Ke^{-rt}.$$

Suppose now that the option expires at time $t < t_d$ and its strike price K satisfies $K \geq De^{-r(t_d - t)}$. Because $S^*(y)$ is geometric Brownian motion, we can use the risk-neutral representation

$$S^*(t) = S^*(0)e^W = (s - De^{-rt_d})e^W,$$

where W is a normal random variable with mean $(r - \sigma^2/2)t$ and variance $t\sigma^2$. The arbitrage theorem yields that the

$$\begin{aligned} \text{no-arbitrage cost of option} &= e^{-rt}E[(S(t) - K)^+] \\ &= e^{-rt}E[(S^*(t) + De^{-r(t_d - t)} - K)^+] \\ &= e^{-rt}E[((s - De^{-rt_d})e^W \\ &\qquad - (K - De^{-r(t_d - t)}))^+] \end{aligned}$$

In other words, if the dividend is to be paid after the expiration date of the option, then the no-arbitrage cost of the option is given by the Black–Scholes formula for a call option on a security whose initial price is $s - De^{-rt_d}$ and whose strike price is $K - De^{-r(t_d - t)}$.

Now consider a European call option with strike price K that expires at time $t > t_d$. Suppose the initial price of the security is s. Because the price of the security will immediately drop by the dividend amount D at time t_d, we have that

$$S(t) = S^*(t), \quad t \geq t_d.$$

Hence, assuming that the volatility of the geometric Brownian motion process $S^*(y)$ remains unchanged after time t_d, we see that the risk-neutral cost of a (K, t) call option is

$$\begin{aligned} e^{-rt}E[(S(t) - K)^+] &= e^{-rt}E[(S^*(t) - K)^+] \\ &= e^{-rt}E[(S^*(0)e^W - K)^+] \\ &= e^{-rt}E[((s - De^{-rt_d})e^W - K)^+]. \end{aligned}$$

Because the right side of the preceding equation is the Black–Scholes cost of a call option with strike price K and expiration time t, when the initial price of the security is $s - De^{-rt_d}$ we obtain that the

$$\text{risk-neutral cost of option} = C(s - De^{-rt_d}, t, K, \sigma, r).$$

In other words, if the dividend is to be paid during the life of the option, then the no-arbitrage cost of the option is given by the Black–Scholes formula – except that the initial price of the security is reduced by the present value of the dividend.

8.3 Pricing American Put Options

There is no difficulty in determining the risk-neutral prices of European put options. The put–call option parity formula gives that

$$P(s, t, K, \sigma, r) = C(s, t, K, \sigma, r) + Ke^{-rt} - s,$$

where $P(s, t, K, \sigma, r)$ is the risk-neutral price of a European put having strike price K at exercise time t, given that the price at time 0 is

$$V_5(0) = 3.565,$$
$$V_5(1) = 2.641,$$
$$V_5(2) = 1.584,$$
$$V_5(3) = 0.375,$$
$$V_5(i) = 0 \quad (i = 4, 5).$$

Since $9u^2d^2 = 9$, Equation (8.1) gives

$$V_4(2) = \max(1,\ \beta p V_5(3) + \beta(1-p)V_5(2)) = 1,$$

which shows that it is optimal to exercise the option at time t_4 when the price is 9. From Remark 1(b) it follows that the option should also be exercised at this time at any lower price, so

$$V_4(1) = 10 - 9ud^3 = 2.130$$

and

$$V_4(0) = 10 - 9d^4 = 3.119.$$

As $9u^3d = 10.293$, Equation (8.1) gives

$$V_4(3) = \beta p V_5(4) + \beta(1-p)V_5(3) = 0.181.$$

Similarly,

$$V_4(4) = \beta p V_5(5) + \beta(1-p)V_5(4) = 0.$$

Continuing, we obtain

$$V_3(0) = \max(2.641,\ \beta p V_4(1) + \beta(1-p)V_4(0)) = 2.641,$$
$$V_3(1) = \max(1.584,\ \beta p V_4(2) + \beta(1-p)V_4(1)) = 1.584,$$
$$V_3(2) = \max(0.375,\ \beta p V_4(3) + \beta(1-p)V_4(2)) = 0.584,$$
$$V_3(3) = \beta p V_4(4) + \beta(1-p)V_4(3) = 0.089.$$

Similarly,

$$V_2(0) = \max(2.130,\ \beta p V_3(1) + \beta(1-p)V_3(0)) = 2.130,$$
$$V_2(1) = \max(1,\ \beta p V_3(2) + \beta(1-p)V_3(1)) = 1.075,$$
$$V_2(2) = \beta p V_3(3) + \beta(1-p)V_3(2) = 0.333,$$

and

$$V_1(0) = \max(1.584,\ \beta p V_2(1) + \beta(1-p)V_2(0)) = 1.592,$$
$$V_1(1) = \max(0.375,\ \beta p V_2(2) + \beta(1-p)V_2(1)) = 0.698,$$

which gives the result

$$V_0(0) = \max(1,\ \beta p V_1(1) + \beta(1-p)V_1(0)) = 1.137.$$

That is, the risk-neutral price of the put option is approximately 1.137. (The exact answer, to three decimal places, is 1.126, indicating a very respectable approximation given the small value of n that was used.) □

8.4 Adding Jumps to Geometric Brownian Motion

One of the drawbacks of using geometric Brownian motion as a model for a security's price over time is that it does not allow for the possibility of a discontinuous price jump in either the up or down direction. (Under geometric Brownian motion, the probability of having a jump would, in theory, equal 0.) Because such jumps do occur in practice, it is advantageous to consider a model for price evolution that superimposes random jumps on a geometric Brownian motion. We now consider such a model.

Let us begin by considering the times at which the jumps occur. We will suppose, for some positive constant λ, that in any time interval of length h there will be a jump with probability approximately equal to λh when h is very small. Moreover, we will assume that this probability is unchanged by any information about earlier jumps. If we let $N(t)$ denote the number of jumps that occur by time t then, under the preceding assumptions, $N(t)$, $t \geq 0$, is called a *Poisson process,* and it can be shown that

$$P\{N(t) = n\} = e^{-\lambda t}\frac{(\lambda t)^n}{n!}, \quad n = 0, 1, \dots.$$

Let us also suppose that, when the ith jump occurs, the price of the security is multiplied by the amount J_i, where $J_1, J_2, \dots$ are independent random variables having a common specified probability distribution. Further, this sequence is assumed to be independent of the times at which the jumps occur.

To complete our description of the price evolution, let $S(t)$ denote the price of the security at time t, and suppose that

$$S(t) = S^*(t) \prod_{i=1}^{N(t)} J_i, \quad t \geq 0,$$

where $S^*(t)$, $t \geq 0$, is a geometric Brownian motion, say with volatility parameter σ and drift parameter μ, that is independent of the J_i and of the times at which the jumps occur, and where $\prod_{i=1}^{N(t)} J_i$ is defined to equal 1 when $N(t) = 0$.

To find the risk-neutral probabilities for the price evolution, let

$$J(t) = \prod_{i=1}^{N(t)} J_i.$$

It will be shown in Section 8.7 that

$$E[J(t)] = e^{-\lambda t(1-E[J])}, \tag{8.2}$$

where $E[J] = E[J_i]$ is the expected value of a multiplicative jump. Because $S^*(t)$, $t \geq 0$, is a geometric Brownian motion with parameters μ and σ, we have

$$E[S^*(t)] = S^*(0)e^{(\mu+\sigma^2/2)t}.$$

Therefore,

$$\begin{aligned} E[S(t)] &= E[S^*(t)J(t)] \\ &= E[S^*(t)]E[J(t)] \qquad \text{(by independence)} \\ &= S^*(0)e^{(\mu+\sigma^2/2-\lambda(1-E[J]))t}. \end{aligned}$$

Consequently, security-buying bets will be fair bets (i.e., $E[S(t)] = S(0)e^{rt}$) provided that

$$\mu + \sigma^2/2 - \lambda(1 - E[J]) = r.$$

In other words, risk-neutral probabilities for the security's price evolution will result when μ, the drift parameter of the geometric Brownian motion $S^*(t)$, $t \geq 0$, is given by

$$\mu = r - \sigma^2/2 + \lambda - \lambda E[J].$$

By the arbitrage theorem, if all options are priced to be fair bets with respect to the preceding risk-neutral probabilities, then no arbitrage is possible. For instance, the no-arbitrage cost of a European call option having strike price K and expiration time t is given by

$$\begin{aligned} \text{no-arbitrage cost} &= E[e^{-rt}(S(t) - K)^+] \\ &= e^{-rt}E[(J(t)S^*(t) - K)^+] \\ &= e^{-rt}E[(J(t)se^W - K)^+], \end{aligned} \tag{8.3}$$

where $s = S^*(0)$ is the initial price of the security and W is a normal random variable with mean $(r - \sigma^2/2 + \lambda - \lambda E[J])t$ and variance $t\sigma^2$.

In Section 8.4.1 we explicitly evaluate Equation (8.3) when the J_i are lognormal random variables, and in Section 8.4.2 we derive an approximation in the case of a general jump distribution. As always, $C(s, t, K, \sigma, r)$ will be the Black–Scholes formula.

8.4.1 *When the Jump Distribution Is Lognormal*

If the jumps J_i have a lognormal distribution with mean parameter μ_0 and variance parameter σ_0^2, then

$$E[J] = \exp\{\mu_0 + \sigma_0^2/2\}.$$

If we let

$$X_i = \log(J_i), \quad i \geq 1,$$

then the X_i are independent normal random variables with mean μ_0 and variance σ_0^2. Also,

$$J(t) = \prod_{i=1}^{N(t)} J_i = \prod_{i=1}^{N(t)} e^{X_i} = \exp\left\{\sum_{i=1}^{N(t)} X_i\right\}.$$

Consequently, using Equation (8.3), we see that the no-arbitrage cost of a European call option having strike price K and expiration time t is

$$\text{no-arbitrage cost} = e^{-rt}E\left[\left(s\exp\left\{W + \sum_{i=1}^{N(t)} X_i\right\} - K\right)^+\right], \tag{8.4}$$

where s is the initial price of the security. Now suppose that there were a total of n jumps by time t. That is, suppose it were known that $N(t) = n$.

Then $W + \sum_{i=1}^{N(t)} X_i$ would be a normal random variable with mean and variance given by

$$E\left[W + \sum_{i=1}^{N(t)} X_i \mid N(t) = n\right] = (r - \sigma^2/2 + \lambda - \lambda E[J])t + n\mu_0,$$

$$\mathrm{Var}\left(W + \sum_{i=1}^{N(t)} X_i \mid N(t) = n\right) = t\sigma^2 + n\sigma_0^2.$$

Therefore, if we let

$$\sigma^2(n) = \sigma^2 + n\sigma_0^2/t$$

and let

$$\begin{aligned} r(n) &= r - \sigma^2/2 + \lambda - \lambda E[J] + \frac{n\mu_0}{t} + \sigma^2(n)/2 \\ &= r + \lambda - \lambda E[J] + \frac{n}{t}(\mu_0 + \sigma_0^2/2) \\ &= r + \lambda - \lambda E[J] + \frac{n}{t}\log(E[J]), \end{aligned} \tag{8.5}$$

then it follows, when $N(t) = n$, that $W + \sum_{i=1}^{N(t)} X_i$ is a normal random variable with variance $t\sigma^2(n)$ and mean $(r(n) - \sigma^2(n)/2)t$. But this implies that, when $N(t) = n$,

$$e^{-r(n)t} E\left[\left(s\exp\left\{W + \sum_{i=1}^{N(t)} X_i\right\} - K\right)^+ \mid N(t) = n\right]$$
$$= C(s, t, K, \sigma(n), r(n)).$$

Multiplying both sides of the preceding equation by $e^{(r(n)-r)t}$ gives

$$e^{-rt} E\left[\left(s\exp\left\{W + \sum_{i=1}^{N(t)} X_i\right\} - K\right)^+ \mid N(t) = n\right]$$
$$= e^{(r(n)-r)t} C(s, t, K, \sigma(n), r(n)).$$

Equation (8.4) shows that the preceding expression is the desired expected value if we are given that there are n jumps by time t. Consequently, it is reasonable (and can be shown to be correct) that the unconditional expected value should be a weighted average of these quantities, with the weight given to the quantity indexed by n equal to the probability that $N(t) = n$. That is,

no-arbitrage cost

$$\begin{aligned} &= \sum_{n=0}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!} e^{(r(n)-r)t} C(s, t, K, \sigma(n), r(n)) \\ &= \sum_{n=0}^{\infty} e^{-\lambda t E[J]}(E[J])^n \frac{(\lambda t)^n}{n!} C(s, t, K, \sigma(n), r(n)) \qquad \text{(from (8.5))} \\ &= \sum_{n=0}^{\infty} e^{-\lambda t E[J]} \frac{(\lambda t E[J])^n}{n!} C(s, t, K, \sigma(n), r(n)). \end{aligned}$$

Summing up, we have proved the following.

Theorem 8.4.1 *If the jumps have a lognormal distribution with mean parameter μ_0 and variance parameter σ_0^2, then the no-arbitrage cost of a European call option having strike price K and expiration time t is as follows:*

$$\textit{no-arbitrage cost} = \sum_{n=0}^{\infty} e^{-\lambda t E[J]} \frac{(\lambda t E[J])^n}{n!} C(s, t, K, \sigma(n), r(n)),$$

where

$$\sigma^2(n) = \sigma^2 + n\sigma_0^2/t,$$
$$r(n) = r + \lambda(1 - E[J]) + \frac{n}{t}\log(E[J]),$$

and

$$E[J] = \exp\{\mu_0 + \sigma_0^2/2\}.$$

Remark. Although Theorem 8.4.1 involves an infinite series, in most applications λ – the rate at which jumps occur – will be quite small and thus the sum will converge rapidly.

8.4.2 *When the Jump Distribution Is General*

We start with Equation (8.3), which states that the no-arbitrage cost of a European call option having strike price K and expiration time t is as follows:

$$\text{no-arbitrage cost} = e^{-rt}E[(J(t)se^{W} - K)^{+}],$$

where s is the price of the security at time 0 and W is a normal random variable with mean $(r - \sigma^2/2 + \lambda - \lambda E[J])t$ and variance $t\sigma^2$. If we let

$$W^* = W - \lambda t(1 - E[J])$$

and

$$s_t = se^{\lambda t(1-E[J])} = \frac{s}{E[J(t)]},$$

then we can write

$$\text{no-arbitrage cost} = E[e^{-rt}(s_t J(t)e^{W^*} - K)^{+}].$$

Because W^* is a normal random variable with mean $(r - \sigma^2/2)t$ and variance $t\sigma^2$, it follows that

$$\text{no-arbitrage cost} = E[C(s_t J(t), t, K, \sigma, r)]. \tag{8.6}$$

Because $C(s, t, K, \sigma, r)$ is a convex function of s, it follows from a result known as Jensen's inequality (see Section 9.2) that

$$E[C(s_t J(t), t, K, \sigma, r)] \geq C(E[s_t J(t)], t, K, \sigma, r) = C(s, t, K, \sigma, r),$$

thus showing that the no-arbitrage cost in the jump model is not less than it is in the same model excluding jumps. (Actually, it will be strictly larger in the jump model provided that $P\{J_i = 1\} \neq 1$.)

An approximation for the no-arbitrage cost can be obtained by regarding $C(x) = C(x, t, K, \sigma, r)$ solely as a function of x (by keeping the other variables fixed), expanding it in a Taylor series about some value x_0, and then ignoring all terms beyond the third to obtain

$$C(x) \approx C(x_0) + C'(x_0)(x - x_0) + C''(x_0)(x - x_0)^2/2.$$

Therefore, for any nonnegative random variable X, we have

$$C(X) \approx C(x_0) + C'(x_0)(X - x_0) + C''(x_0)(X - x_0)^2/2.$$

Letting $x_0 = E[X]$ and taking expectations of both sides of the preceding yields that

$$E[C(X)] \approx C(E[X]) + C''(E[X])\,\text{Var}(X)/2.$$

$$X = s_t J(t), \qquad E[X] = s$$

gives that

$$E[C(s_t J(t))] \approx C(s) + C''(s)s_t^2\,\text{Var}(J(t))/2.$$

It can now be shown (see Section 8.7) that

$$\text{Var}(J(t)) = e^{-\lambda t(1-E[J^2])} - e^{-2\lambda t(1-E[J])}, \tag{8.7}$$

where J has the probability distribution of the J_i. Therefore, using the formula derived in Section 7.5 for $C''(s)$ (which is called gamma in that section) leads to the approximation given in the following theorem, which sums up the results of this subsection.

Theorem 8.4.2 *Assuming a general distribution for the size of a jump, the*

$$\textit{no-arbitrage option cost} = E[C(s_t J(t), t, K, \sigma, r)]$$
$$\geq C(s, t, K, \sigma, r).$$

Moreover,

no-arbitrage option cost

$$\approx C(s, t, K, \sigma, r) + s_t^2[e^{-\lambda t(1-E[J^2])} - e^{-2\lambda t(1-E[J])}]\frac{1}{2s\sigma\sqrt{2\pi t}}e^{-\omega^2/2}$$

$$= C(s, t, K, \sigma, r) + s^2(e^{\lambda t(1-2E[J]+E[J^2])} - 1)\frac{1}{2s\sigma\sqrt{2\pi t}}e^{-\omega^2/2},$$

where

$$s_t = se^{\lambda t(1-E[J])}$$

and

$$\omega = \frac{rt + \sigma^2 t/2 - \log(K/s)}{\sigma\sqrt{t}}.$$

8.5 Estimating the Volatility Parameter

Whereas four of the five parameters needed to evaluate the Black–Scholes formula – namely, s, t, K, and r – are known quantities, the value of σ has to be estimated. One approach is to use historical data. Section 8.5.1 gives the standard approach for estimating a population variof σ based on closing prices of the security over successive days; Section 8.5.3 gives an improved estimator based on both daily closing and opening prices; and Section 8.5.4 gives a more sophisticated estimator that uses daily high and low prices as well as daily opening and closing prices.

8.5.1 *Estimating a Population Mean and Variance*

Suppose that $X_1, \dots, X_n$ are independent random variables having a common probability distribution with mean μ_0 and variance σ_0^2. The average of these data values,

$$\bar{X} = \frac{\sum_{i=1}^n X_i}{n},$$

is the usual estimator of the mean. Because

$$\sigma_0^2 = \text{Var}(X_i) = E[(X_i - \mu_0)^2],$$

it would appear that σ_0^2 could be estimated by

$$\frac{\sum_{i=1}^n (X_i - \mu_0)^2}{n}.$$

However, this estimator cannot be directly utilized when the mean μ_0 is unknown. To use it, we must first replace the unknown μ_0 by its estimator $\bar{X}$. If we then replace n by $n-1$, we obtain the *sample variance* S^2, defined by

$$S^2 = \frac{\sum_{i=1}^n (X_i - \bar{X})^2}{n-1}.$$

The sample variance is the standard estimator of the variance σ_0^2. It is an *unbiased* estimator of σ_0^2, meaning that

$$E[S^2] = \sigma_0^2.$$

(It is because we wanted the estimator to be unbiased that we changed its denominator from n to $n-1$.) The effectiveness of S^2 as an estimator of the variance can be measured by its mean square error (MSE), defined as

$$\text{MSE} = E[(S^2 - \sigma_0^2)^2]$$

When the X_i come from a normal distribution, it can be shown that

$$\text{Var}(S^2) = \frac{2\sigma_0^4}{n-1}. \tag{8.8}$$

8.5.2 *The Standard Estimator of Volatility*

Suppose that we want to estimate σ using t time units of historical data, which we will suppose run from time 0 to time t. That is, suppose that the present time is t and that we have the historical price data $S(y)$, $0 \le y \le t$. Fix a positive integer n, let $\ell = t/n$, and define the random variables

$$X_1 = \log\left(\frac{S(\ell)}{S(0)}\right),$$

$$X_2 = \log\left(\frac{S(2\ell)}{S(\ell)}\right),$$

$$X_3 = \log\left(\frac{S(3\ell)}{S(2\ell)}\right),$$

$$\vdots$$

$$X_n = \log\left(\frac{S(n\ell)}{S((n-1)\ell)}\right).$$

Under the assumption that the price evolution follows a geometric Brownian motion with parameters μ and σ, it follows that $X_1, \dots, X_n$ are independent normal random variables with mean $\ell\mu$ and variance $\ell\sigma^2$. From Section 8.5.1, it follows that we can use $\sum_{i=1}^n (X_i - \bar{X})^2/(n-1)$ to estimate $\ell\sigma^2$. Therefore, we can estimate σ^2 by

$$\widehat{\sigma^2} = \frac{1}{\ell} \frac{\sum_{i=1}^n (X_i - \bar{X})^2}{n-1}.$$

Moreover, it follows from Equation (8.8) that

$$\text{Var}(\widehat{\sigma^2}) = \frac{1}{\ell^2} \frac{2(\ell\sigma^2)^2}{n-1} = \frac{2\sigma^4}{n-1}. \tag{8.9}$$

It follows from Equation (8.9) that we can use price data history over any time interval to obtain an arbitrarily precise estimator of σ^2. That is, in an unbiased estimator of σ^2 having an arbitrarily small variance. The difficulty with this approach, however, is that it strongly depends on the assumption that the logarithms of price ratios $S(i\ell)/S((i-1)\ell)$ are independent with a common distribution, even when the time lag ℓ is arbitrarily small. Indeed, even assuming that a security's price history resembles a geometric Brownian motion process, it is unlikely to look like one under a microscope. That is, while successive daily closing prices might appear to be consistent with a geometric Brownian motion, it is unlikely that this would be true for hourly (or more frequent) prices. For this reason we recommend that the preceding procedure be used with ℓ equal to one day. Because the unit of time is one year and there are approximately 252 trading days in a year, $\ell = 1/252$.

To use this method to estimate σ, consider n successive daily closing prices $C_1, \dots, C_n$, where C_i is the closing price on trading day i. Let C_0 be the closing price of the security immediately before these n days, and set

$$X_i = \log\left(\frac{C_i}{C_{i-1}}\right) = \log(C_i) - \log(C_{i-1}).$$

The sample variance of these data values,

$$S^2 = \frac{\sum_{i=1}^n (X_i - \bar{X})^2}{n-1},$$

can be taken as the estimator of $\sigma^2/252$; $S\sqrt{252}$ can be used to estimate σ.

Remark. If μ and σ are the drift and volatility parameters of the geometric Brownian motion, then

$$E\left[\log\left(\frac{C_i}{C_{i-1}}\right)\right] = \frac{\mu}{252}, \qquad \sqrt{\operatorname{Var}\left(\log\left(\frac{C_i}{C_{i-1}}\right)\right)} = \frac{\sigma}{\sqrt{252}}.$$

Because μ will typically have a value close to 0 whereas σ is typically greater than .2, it follows that the mean of $X_i = \log(C_i/C_{i-1})$ is negligible with respect to its standard deviation. Therefore, we could approximate μ by 0 and, with very small loss of efficiency, use

$$\frac{\sum_{i=1}^n X_i^2}{n}$$

as the estimator of $\sigma^2/252$. It is important to note that this estimator can be used even when the geometric Brownian motion has a time-varying drift parameter. (Recall that the Black–Scholes formula yields the unique no-arbitrage cost even in the case of a time-varying drift parameter.)

8.5.3 *Using Opening and Closing Data*

Let C_i denote the (closing) price of a security at the end of trading day i. Under the assumption that the security's price follows a geometric Brownian motion, $\log(C_i/C_{i-1})$ is a normal random variable whose mean is approximately 0 and whose variance is $\sigma^2/252$. Letting O_i be the opening price of the security at the beginning of trading day i, we can write

$$\log\left(\frac{C_i}{C_{i-1}}\right) = \log\left(\frac{C_i}{O_i}\frac{O_i}{C_{i-1}}\right) = \log\left(\frac{C_i}{O_i}\right) + \log\left(\frac{O_i}{C_{i-1}}\right).$$

Assuming that C_i/O_i and O_i/C_{i-1} are independent – that is, assuming that the ratio price change during a trading day is independent of the ratio price change that occurred while the market was closed – it follows that

$$\begin{aligned}\operatorname{Var}(\log(C_i/C_{i-1})) &= \operatorname{Var}(\log(C_i/O_i)) + \operatorname{Var}(\log(O_i/C_{i-1})) \\ &= \operatorname{Var}(C_i^* - O_i^*) + \operatorname{Var}(O_i^* - C_{i-1}^*), \qquad (8.10)\end{aligned}$$

where

$$C_j^* = \log(C_j), \qquad O_j^* = \log(O_j).$$

Because $C_i^* - O_i^*$ and $O_i^* - C_{i-1}^*$ both have a mean of approximately 0, we can estimate $\sigma^2/252 = \operatorname{Var}(\log(C_i/C_{i-1}))$ by

$$\frac{\sum_{i=1}^n (C_i^* - O_i^*)^2}{n} + \frac{\sum_{i=1}^n (O_i^* - C_{i-1}^*)^2}{n}.$$

This yields the estimator $\hat{\sigma}$ of the volatility parameter σ:

$$\hat{\sigma} = \sqrt{\frac{252}{n}\sum_{i=1}^n [(C_i^* - O_i^*)^2 + (O_i^* - C_{i-1}^*)^2]}. \qquad (8.11)$$