Queen Mary
UNIVERSITY OF LONDON

B.Sc. EXAMINATION

MTH6120 Further Topics in Mathematical Finance

May 2009

Duration: 2 hours.

You may attempt as many questions as you wish and all questions carry equal marks. Except for the award of a bare pass, only the best 4 questions will be counted.

Calculators are not permitted in this examination.

1.

(a) [3 marks] Explain what is an arbitrage opportunity.

(b) [5 marks] State the Arbitrage Theorem.

(c) [5 marks] A betting shop quotes the odds for the three possible outcomes of a football game (win, draw, loose) to be 1:1, 2:1, and 4:1. Is there an arbitrage opportunity? Give reasons.

(d) [3 marks] Is the principle of no arbitrage opportunities sufficient to uniquely price a call option? Give examples.

(e) [9 marks] Consider a 3-state model of share price evolution where the share price $S(0)$ can evolve to three different values s_1, s_2, s_3, where $s_1 > s_2 = S(0) > s_3$. The interest rate is $r = 0$ and the strike price K satisfies $s_1 > K > s_2$. Assuming there are no arbitrage opportunities, determine the allowed range of option prices.

2.

(a) [3 marks] Briefly explain what is i) a European call option ii) a European put option. What is the difference between European and American options?

(b) [5 marks] What are exotic options? Give a few examples.

(c) [8 marks] Consider a European call option whose return at expiration time is capped by the amount $B > 0$, i.e. the payoff is given by $min((S(t)-K)^+, B)$. Assuming that the Black-Scholes formula is valid for plain (uncapped) options, derive the no-arbitrage cost of the capped option.

(d) [5 marks] State the main result of the Black-Scholes model, the Black-Scholes option pricing formula. Explain the meaning of the parameters s, t, K, σ, r and ω.

(e) [4 marks] Explain how the Black-Scholes option pricing formula is modified if i) a dividend is paid *continuously* at a rate equal to a fixed fraction f of the price of the share ii) a dividend is paid *at a fixed time* t_D at a certain fraction f of the share price.

3.

(a) [3 marks] Explain what is a utility function.

(b) [6 marks] Are the following utility functions strictly concave?

i) $u(x) = \log x$

ii) $u(x) = -(1 - e^{-bx})^2 \quad (b \neq 0)$

iii) $u(x) = x^x \quad (x > 0)$

Give reasons.

(c) [5 marks] Write down the utility function of a risk indifferent investor. Is this function i) concave ii) strictly concave iii) convex iv) strictly convex? Answer with yes or no.

(d) [8 marks] An investor with capital x can invest any amount αx, where $0 \leq \alpha \leq 1$. If the amount αx is invested she may either receive $5\alpha x$ with probability p or 0 with probability $1-p$. No interest is paid. The investor's utility function is $u(x) = \sqrt{x}$. Given p, how much money should she invest?

(e) [3 marks] For the above question 3(d), is there a critical value p_{crit} of the probability p below which no money will be invested? If yes, what is p_{crit}?

4.

(a) [3 marks] What is a portfolio?

(b) [3 marks] Briefly explain the portfolio selection problem and how it is solved using utility functions.

(c) [7 marks] Five different share prices evolve independently. The rates of returns have the same mean values but different variances $v_1^2 = \frac{1}{7}$, $v_2^2 = \frac{1}{5}$, $v_3^2 = \frac{1}{4}$, $v_4^2 = \frac{1}{3}$, $v_5^2 = 1$. You have *£10000* to invest. Construct the optimal portfolio corresponding to maximum expected utility.

(d) [4 marks] Define what is a share price index $I(t)$. Express the strike price K of an option on the share price index in terms of the individual strike prices K_j.

(e) [8 marks] Let G be the payoff of an index call option at time t, and G_i be the payoff of a call option on the individual share i. Let w_i be the weight of share i in the share price index. Show that $G \leq \sum_{i=1}^{n} w_i G_i$.

5.

(a) [3 marks] Define the rate of return R_i of a stock labelled i.

(b) [5 marks] Let a data set of 20 daily closing prices of stock 1 and stock 2 be given. Explain how you would estimate from this data set i) the average of the rate of return of each stock ii) $cov(R_1, R_2)$.

(c) [5 marks] Briefly explain the Capital Assets Pricing Model (CAPM) with a given interest rate r. What does it mean if share i has a large beta-value β_i?

(d) [4 marks] Derive a formula for the parameter β_i of the CAPM in terms of the mean rate of returns r_i and r_m and the interest rate r.

(e) [8 marks] Starting from the equations of CAPM, derive a relation between $cov(R_i, R_m)$ and $var(R_m)$, where R_m is the return of the market index.