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Original article

Superstatistics: theory and applications

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Abstract. Superstatistics is a superposition of two different statistics relevant to driven nonequilibrium systems with a stationary state and intensive parameter fluctuations. It contains Tsallis statistics as a special case. After briefly summarizing some of the theoretical aspects, we describe recent applications of this concept to three different physical problems, namely a) fully developed hydrodynamic turbulence, b) pattern formation in thermal convection states, and c) the statistics of cosmic rays.

Key words: nonextensive statistical mechanics, fluctuations of temperature, generalized entropies
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1 Introduction

The formalism of nonextensive statistical mechanics [1–4] can be regarded as an embedding of ordinary statistical mechanics into a more general framework. The basic idea is that some systems to which ordinary statistical mechanics does not apply may still obey a similar formalism if the Shannon entropy is replaced by more general entropy measures. Good candidates for these more general entropy measures are the Tsallis entropies, which depend on a real parameter q and which reduce to the Shannon entropy for $q \to 1$. For general q, one obtains generalizations of the canonical distributions, represented by q-exponentials, often called Tsallis distributions. There can be a variety of reasons why ordinary statistical mechanics is not applicable to a particular system: there may be long-range interactions, metastability, driving forces that keep the system out of equilibrium, etc.

Tsallis distributions are indeed observed in a large variety of physical systems [3–9], many of which are in a driven stationary state that is far from equilibrium. An important question is why these or similar distributions are often observed in experiments. Can we give dynamical reasons for the occurrence of Tsallis statistics in suitable classes of nonequilibrium systems?

This is indeed possible. One can easily construct classes of stochastic differential equations with fluctuating parameters for which one can rigorously prove that they generate Tsallis statistics [10,11]. Indeed, for many systems, the reason that Tsallis distributions are observed can be easily related to the fact that there are spatio-temporal fluctuations of an intensive parameter (e.g. the inverse temperature, a friction constant, the amplitude of the Gaussian white noise, or the energy dissipation in turbulent flows). If these fluctuations evolve on a long time scale and are distributed according to a particular distribution, the χ^2-distribution, one ends up with Tsallis statistics in a natural way. For other distributions of the intensive parameter, one ends up with more general statistics: the so-called superstatistics [12], which contain Tsallis statistics as a special case. Generalized entropies (analogues of the Tsallis entropies) can be defined for these superstatistics as well [13,14], and generalized versions of statistical mechanics can be also constructed, at least in principle. It has been shown that the corresponding generalized entropies are stable [15,16].

In this paper we will briefly review the superstatistics concept, and then show that the corresponding stationary probability distributions are not just a theoretical construct, but are also of practical physical relevance. We will concentrate on three physically relevant examples of applications: hydrodynamic fully developed turbulence, defect motion in convection states, and the statistics of cosmic rays.

2 What is superstatistics?

2.1 The basic idea

Let us give a short introduction to the 'superstatistics' concept [12]. Consider a driven nonequilibrium system with spatio-temporal fluctuations of an intensive parameter β. This may be the inverse temperature, a chemical potential, or a function of the fluctuating energy dissipation in the flow (for the turbulence application). Locally, i.e. in spatial regions (cells) where β is approximately constant, the system is described by ordinary statistical mechanics, i.e. ordinary Boltzmann factors $e^{-\beta E}$, where E is an effective energy in each cell. In the long-term, the system is described by a spatio-temporal average over the fluctuating β. In this way one obtains a superposition of two statistics (that of β and that of $e^{-\beta E}$), hence the name 'superstatistics'. One may define an averaged Boltzmann factor $B(E)$ as

$$B(E) = \int_0^\infty f(\beta) e^{-\beta E} d\beta, \tag{2.1}$$

where $f(\beta)$ is the probability distribution of β. For so-called type-A superstatistics, one normalizes this effective Boltzmann factor and obtains the stationary long-term probability distribution

$$p(E) = \frac{1}{Z} B(E), \tag{2.2}$$

where

$$Z = \int_0^\infty B(E) dE. \tag{2.3}$$

For type-B superstatistics, one includes the β-dependent normalization constant into the averaging process. In this case

$$p(E) = \int_0^\infty f(\beta) \frac{1}{Z(\beta)} e^{-\beta E} d\beta, \tag{2.4}$$

where $Z(\beta)$ is the normalization constant of $e^{-\beta E}$ for a given β. Both approaches can be easily mapped onto each other, by defining a new probability density $\tilde{f}(\beta) = cf(\beta)/Z(\beta)$, where c is a normalization constant. It is obvious that Type-B superstatistics with f is equivalent to type-A superstatistics with $\tilde{f}$.

Why should one be interested in superstatistics? For many physical systems this concept yields a plausible reason for why non-trivial distributions $p(E)$ similar to Tsallis distributions occur. The big advantage of the superstatistics concept is that it is just based on ordinary statistical mechanics (ordinary Boltzmann factors $e^{-\beta E}$, obtained by locally maximizing the ordinary Shannon entropy) plus spatio-temporal fluctuations in β. Nonextensive behavior arises naturally in this context due to the fluctuations of β. Generalized entropies that take on a maximum for the observed $p(E)$ can be formally constructed but they relate to an *effective* thermodynamic description of a nonequilibrium system where one averages over the fluctuations. Hence, they are useful but less fundamental. Moreover, in the superstatistics approach concrete formulas for the effective entropic index q can be derived, which relate q to the relative variance of the β-fluctuations (see Sect. 2.6).

As will be proved later, in our fluctuating approach, $p(E)$ becomes a Tsallis distribution if $f(\beta)$ is chosen as a χ^2-distribution. But at the moment we keep $f(\beta)$ general. Many relevant concepts of superstatistics can be formulated for general $f(\beta)$.

2.2 Dynamical realization

A simple dynamical realization of superstatistics can be constructed by considering stochastic differential equations with spatio-temporally fluctuating parameters [11]. Consider a Langevin equation for a variable u,

$$\dot{u} = \gamma F(u) + \hat{\sigma} L(t), \tag{2.5}$$

where $L(t)$ is Gaussian white noise, $\gamma > 0$ is a friction constant, $\hat{\sigma}$ describes the strength of the noise, and $F(u) = -(\partial/\partial u)V(u)$ is a drift force. If γ and $\hat{\sigma}$ are constant then the stationary probability density of u is proportional to $e^{-\beta V(u)}$, where $\beta := \gamma/\hat{\sigma}^2$ can be identified with the inverse temperature of ordinary statistical mechanics. Most generally, however, we may let the parameters γ and $\hat{\sigma}$ fluctuate so that $\beta = \gamma/\hat{\sigma}^2$ has the probability density $f(\beta)$. These fluctuations are assumed to be on a long time scale so that the system can temporarily reach local equilibrium. In this case one obtains for the conditional probability $p(u|\beta)$ (i.e. the probability of u given some value of β)

$$p(u|\beta) = \frac{1}{Z(\beta)} \exp\{-\beta V(u)\}, \tag{2.6}$$

for the joint probability $p(u, \beta)$ (i.e. the probability of observing both a certain value of u and a certain value of β)

$$p(u, \beta) = p(u|\beta) f(\beta), \tag{2.7}$$

and for the marginal probability $p(u)$ (i.e. the probability of observing a certain value of u no matter what β is)

$$p(u) = \int_0^\infty p(u|\beta) f(\beta) d\beta. \tag{2.8}$$

This marginal distribution is the generalized canonical distribution of the superstatistics considered. The above formulation corresponds to type-B superstatistics.

Let us now consider a few examples of possible superstatistics, by considering different examples of distributions $f(\beta)$. Note that β lives on a positive support, so Gaussian distributions of β are unsuitable.

2.3 χ^2-superstatistics

One of the most natural choices of the probability density of β is given by the χ^2-distribution

$$f(\beta) = \frac{1}{\Gamma\left(\frac{n}{2}\right)} \left\{\frac{n}{2\beta_0}\right\}^{\frac{n}{2}} \beta^{\frac{n}{2}-1} \exp\left\{-\frac{n\beta}{2\beta_0}\right\}. \tag{2.9}$$

Note that if one adds Gaussian random variables then the sum is again a Gaussian random variable. But if one adds Gaussian random variables squared, one gets a χ^2-distributed random variable. In this sense the χ^2-distribution (also called the Γ-distribution) is a typical distribution for positive random variables that naturally arises in many circumstances. Let us denote the n independent Gaussian random variables as $X_i,\ i = 1, \ldots, n$, and assume they have average 0. Then

$$\beta := \sum_{i=1}^{n} X_i^2 \tag{2.10}$$

has the probability density (2.9). The average of the fluctuating β is given by

$$\langle \beta \rangle = n \langle X^2 \rangle = \int_0^\infty \beta f(\beta) d\beta = \beta_0 \tag{2.11}$$

and the variance by

$$\langle \beta^2 \rangle - \beta_0^2 = \frac{2}{n} \beta_0^2. \tag{2.12}$$

For linear drift forces $F(u) = -u$ one obtains for the conditional probability $p(u|\beta)$

$$p(u|\beta) = \sqrt{\frac{\beta}{2\pi}} \exp\left\{-\frac{1}{2}\beta u^2\right\}, \tag{2.13}$$

and for the marginal probability $p(u)$ a short calculation yields

$$p(u) = \int_0^\infty p(u|\beta) f(\beta) d\beta = \frac{\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \left(\frac{\beta_0}{\pi n}\right)^{\frac{1}{2}} \frac{1}{\left(1+\frac{\beta_0}{n}u^2\right)^{\frac{n}{2}+\frac{1}{2}}}. \tag{2.14}$$

Thus the stochastic differential equation (2.5) with χ^2-distributed $\beta = \gamma/\hat{\sigma}^2$ generates the generalized canonical distributions of nonextensive statistical mechanics

$$p(u) \sim \frac{1}{\left(1+\frac{1}{2}\tilde{\beta}(q-1)u^2\right)^{\frac{1}{q-1}}}, \tag{2.15}$$

provided that the following identifications are made:

$$\frac{1}{q-1} = \frac{n}{2} + \frac{1}{2} \Longleftrightarrow q = 1 + \frac{2}{n+1}, \tag{2.16}$$

$$\frac{1}{2}(q-1)\tilde{\beta} = \frac{\beta_0}{n} \Longleftrightarrow \tilde{\beta} = \frac{2}{3-q}\beta_0. \tag{2.17}$$

2.4 Log-normal superstatistics

Of particular interest for models of hydrodynamic turbulence is the log-normal distribution

$$f(\beta) = \frac{1}{\beta s\sqrt{2\pi}} \exp\left\{\frac{-(\log\frac{\beta}{m})^2}{2s^2}\right\}. \tag{2.18}$$

It yields yet another possible superstatistics (see [17–20] for related turbulence models); m and s are parameters. The average β_0 of the above log-normal distribution is given by $\beta_0 = m\sqrt{w}$ and the variance by $\sigma^2 = m^2 w(w-1)$, where $w := e^{s^2}$. One obtains for linear drift forces $F(u) = -u$ the superstatistics distribution

$$p(u) = \frac{1}{2\pi s} \int_0^\infty d\beta\, \beta^{-1/2} \exp\left\{\frac{-(\log\frac{\beta}{m})^2}{2s^2}\right\} e^{-\frac{1}{2}\beta u^2}. \tag{2.19}$$

The integral cannot be evaluated in closed form, but it can be easily numerically evaluated.

2.5 F-superstatistics

Another example is superstatistics based on F-distributions. The F-distribution is given by

$$f(\beta) = \frac{\Gamma((v+w)/2)}{\Gamma(v/2)\Gamma(w/2)} \left(\frac{bv}{w}\right)^{v/2} \frac{\beta^{\frac{v}{2}-1}}{(1+\frac{vb}{w}\beta)^{(v+w)/2}}. \tag{2.20}$$

Here w and v are positive integers and $b > 0$ is a parameter. We note that we obtain a Tsallis distribution for $v = 2$. However, this is a Tsallis distribution in β-space, not in E-space.

The average of β is given by

$$\beta_0 = \frac{w}{b(w-2)} \tag{2.21}$$

and the variance by

$$\sigma^2 = \frac{2w^2(v+w-2)}{b^2 v(w-2)^2(w-4)}. \tag{2.22}$$

Again, the integral leading to the marginal distribution $p(u)$ cannot be obtained in closed form, but is easily numerically evaluated. Superstatistics based on F-distributions has been studied in more detail in [21], and possible applications in plasma physics were sketched there.

2.6 General properties of superstatistics

For small E, all superstatistics have been shown to have the same first-order corrections to the Boltzmann factor of ordinary statistical mechanics as Tsallis statistics [12]. For moderately large E, one often observes similar behaviour as for Tsallis statistics (see [20] for examples). On the other hand, the extreme tails of $p(E)$ for very large E are very different for the various superstatistics. Tsallis distributions decay with a power law for $E \to \infty$. General superstatistics can have all kinds of asymptotic decays.

For *any* superstatistics, and not only Tsallis statistics, one can generally define a parameter q by the relation

$$(q-1)\beta_0^2 = \sigma^2, \tag{2.23}$$

where σ^2 is the variance of $f(\beta)$, hence

$$q = \frac{\langle \beta^2 \rangle}{\langle \beta \rangle^2}. \tag{2.24}$$

For χ^2-superstatistics, this q coincides with the index q of the Tsallis entropies. For the log-normal distribution we obtain from (2.24) $q = w$, and for the F-distribution $q = 1 + \frac{2(v+w-2)}{v(w-4)}$. The meaning of this generally defined parameter q is that $\sqrt{q-1} = \frac{\sigma}{\beta_0}$ is just the coefficient of variation of the distribution $f(\beta)$, defined by the ratio of the standard deviation to the mean. If there are no fluctuations of β at all, we obtain $q = 1$, i.e. ordinary statistical mechanics. Our formula (2.24) relating q and the variance of the β fluctuations is valid for both type-A and type-B superstatistics, just that one has to form averages with either f and $\tilde{f}$.

One remark is in order. The relation (2.24) necessarily means that $q \geq 1$. Hence the superstatistics concept is only applicable to systems with $q \geq 1$. The range $0 < q < 1$ is also important in nonextensive statistical mechanics, but q-values smaller than 1 cannot be realized by our fluctuating approach in a straightforward way.

Another remark is that Tsallis and Souza [13,16] have recently shown that general superstatistics maximize more general classes of entropy-like functions subject to suitable constraints. This can be used as a starting point for constructing a generalized statistical mechanics for general superstatistics.

2.7 Many particles

So far our dynamical realization of a superstatistics in terms of a stochastic differential equation was written down for one test particle in one dimension. To generalize to N particles in d space dimensions, we may consider coupled systems of equations with fluctuating friction forces, as given by

$$\dot{\boldsymbol{u}}_i = -\gamma_i \boldsymbol{F}_i(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_N) + \hat{\sigma}_i \boldsymbol{L}_i(t) \qquad i = 1, \ldots, N. \tag{2.25}$$

Suppose that a potential $V(\boldsymbol{u_1}, \ldots, \boldsymbol{u_N})$ exists for this problem such that $\boldsymbol{F}_i = -\frac{\partial}{\partial \boldsymbol{u}_i} V$. We can then proceed to marginal stationary distributions in a similar way as before. One just has to specify the statistics of the β_i in the vicinity of each particle i.

If all $\beta_i = \frac{\gamma_i}{\hat{\sigma}_i^2}$ are given by the same fluctuating random variable $\beta_i = \beta$, the integration yields marginal distributions of the form

$$p(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_N) = \int_0^\infty \frac{f(\beta)}{Z(\beta)} e^{-\beta V(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_N)} d\beta \tag{2.26}$$

(type-B superstatistics). The result depends on both $f(\beta)$ and $Z(\beta)$.

Let us consider an example. We consider a partition function $Z(\beta)$ whose β-dependence is of the form

$$Z(\beta) = \int d\boldsymbol{u}_1 \cdots d\boldsymbol{u}_N e^{-\beta V} \sim \beta^x e^{-\beta y}, \tag{2.27}$$

where x and y are some suitable numbers. We consider χ^2-superstatistics for this problem. The marginal distributions are obtained as

$$p(\boldsymbol{u_1}, \ldots, \boldsymbol{u_N}) \sim \frac{1}{(1 + \tilde{\beta}(q-1)V(\boldsymbol{u_1}, \ldots, \boldsymbol{u}_N))^{\frac{1}{q-1}}}, \tag{2.28}$$

where

$$q = 1 + \frac{2}{n - 2x} \tag{2.29}$$

and

$$\tilde{\beta} = \frac{\beta_0}{1 + (q-1)(x - \beta_0 y)}. \tag{2.30}$$

Is the assumption of a single fluctuating β realistic for many particles? No. In many physical applications, the various particles will be dilute and only weakly interacting. Hence, in this case β is expected to fluctuate spatially in such a way that the local inverse temperature β_i surrounding one particle i is almost independent of the local β_j surrounding another particle j. Moreover, the potential is approximately just a sum of single-particle potentials $V(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_N) = \sum_{i=1}^N V_s(\boldsymbol{u}_i)$. In this case integration over all β_i leads to marginal densities of the form

$$p(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_N) = \int_0^\infty d\beta_1 \cdots \int_0^\infty d\beta_N \prod_{i=1}^N \frac{f(\beta_i)}{Z_s(\beta_i)} e^{-\beta_i V_s(\boldsymbol{u}_i)}, \tag{2.31}$$

and for χ^2-superstatistics one obtains

$$p(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_N) \sim \prod_{i=1}^N \frac{1}{(1 + \tilde{\beta}(q-1)V_s(\boldsymbol{u}_i))^{\frac{1}{q-1}}}. \tag{2.32}$$

This means that the N-particle nonextensive system reduces to products of 1-particle nonextensive systems.

We notice that for many particles there is no unique answer as to what the physically relevant generalized canonical distributions are – it depends on the spatio-temporal statistics of the β_i. Systems of very high particle density may be better described by (2.26), dilute systems better by (2.31). Our physical applications in Sect. 3 mainly relate to the form (2.31).

2.8 Further generalizations

We just briefly mention some further generalizations, without working them out in much detail.

- We may regard the temperature T rather than $\beta = T^{-1}$ as the fundamental variable, and can then take for T a χ^2-distribution, log-normal distribution, F-distribution, etc. This case is easily reduced to the framework studied so far, since any chosen probability distribution f_T of T implies a corresponding probability distribution f_β of β given by

$$f_\beta(\beta) = f_T(T)|\frac{dT}{d\beta}| = f_T(\beta^{-1})\frac{1}{\beta^2}. \tag{2.33}$$

- For each particle there may be several different intensive parameters that fluctuate, for example, not only the inverse temperature β but also the chemical potential μ may fluctuate in a spatio-temporal way. In order to proceed to the marginal distribution, one then simply has to do two integrals (over β and μ) rather than one.
- Rather than starting from ordinary Boltzmann factors $e^{-\beta E}$ and averaging these over a fluctuating β, we could also construct a superstatistics for which the statistics in the single cells is already given by Tsallis

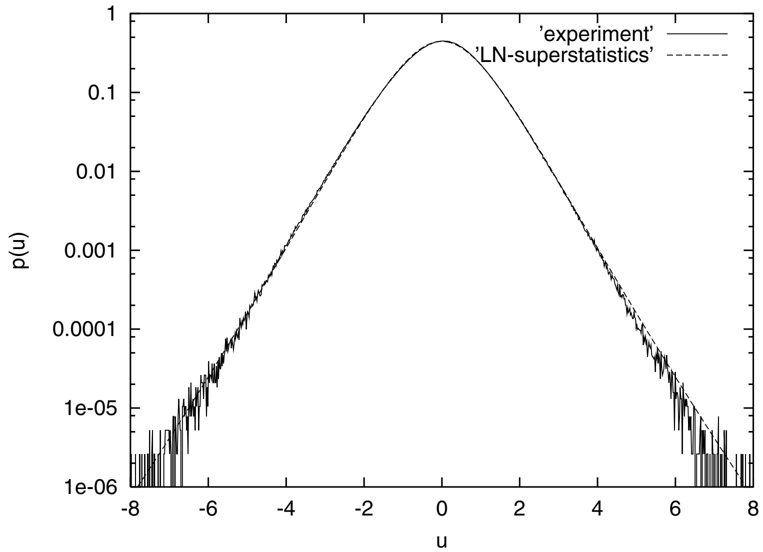

Fig. 3.1. Histogram of velocity differences u measured in Swinney's experiment and the prediction of the log-normal superstatistics equation (2.19) with $s^2 = 0.28$

statistics, i.e. one would be β-averaging q-exponentials $e_q^{-\beta E}$ in this case. This is related to the case of two fluctuating parameters discussed above: we may think of cases in which the integration over the fluctuating μ yields q-exponentials, which then still have to be averaged over β.

3 Applications

3.1 Fully developed hydrodynamic turbulence

In the turbulence application, u stands for a local velocity *difference* in the turbulent flow. On a very small time scale, this velocity difference is essentially the acceleration. The basic idea is that turbulent velocity differences locally relax with a certain damping constant γ and are at the same time driven by rapidly fluctuating chaotic force differences, which are modelled to a good approximation by Gaussian white noise. One knows that the energy dissipation fluctuates in space and time in turbulent flows. Moreover, it has been known since the early papers by Kolmogorov in 1962 that the probability density of the energy dissipation is approximately log-normal in a turbulent flow. Hence, if β is a simple power-law function of the energy dissipation, a log-normally distributed β is implied. We thus end up in a natural way with log-normal superstatistics.

Figure 3.1 shows an experimentally measured $p(u)$ of velocity differences u at the scale $r = 92.5\eta$ in a turbulent Taylor-Couette flow obtained by Swinney et al.[5]. η denotes the Kolmogorov length scale. The data have been re-scaled to a variance of 1. The dashed line is the prediction of log-normal superstatistics. There is excellent agreement between the measured density and the log-normal superstatistics given by (2.19). The fitting parameter for this example was $s^2 = 0.28$. This is the only fitting parameter; $m = \sqrt{w}$ is fixed by the condition of the variance of 1 of $p(u)$.

Figure 3.2 shows the measured probability density of the acceleration of a Lagrangian test particle in a turbulent flow obtained in the experiment of Bodenschatz et al. [22,23]. Log-normal superstatistics with $s^2 = 3.0$ yields a good fit. Since Bodenschatz's data reach rather large accelerations a (in units of the standard deviation), the measured tails of the distributions allow for a sensitive distinction between various superstatistics. Though χ^2-superstatistics (= Tsallis statistics) also yields a reasonably good fit of the data for moderately large accelerations, for the extreme tails, log-normal superstatistics seems to do a better job. The main difference between χ^2-superstatistics and log-normal superstatistics is the fact that $p(a)$ decays with a power law for the former, whereas it decays with a more complicated logarithmic law for the latter.

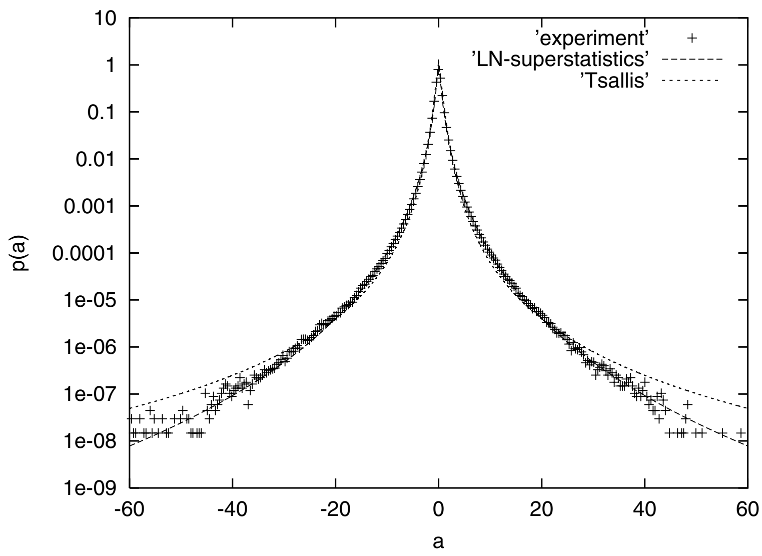

Fig. 3.2. Histogram of accelerations a measured in Bodenschatz's experiment, the prediction of the log-normal superstatistics equation (2.19) with $s^2 = 3.0$, and the prediction of Tsallis statistics with $q = 1.5$

3.2 Chaotic defect motion in inclined layer convection

Let us now consider another physically relevant example, so-called 'defect turbulence'. Defect turbulence shares with ordinary turbulence only its name. Otherwise it is very different. It is a phenomenon related to convection and has nothing to do with fully developed hydrodynamic turbulence. Consider a Raleigh-Benard convection experiment: a liquid is heated from below and cooled from above. For large enough temperature differences, interesting convection patterns start to evolve. An inclined-layer convection experiment [7,24] is a kind of Raleigh-Benard experiment in which the apparatus is tilted at an angle (say 30 degrees). Moreover, the liquid is confined between two very narrow plates. For large enough temperature differences, the convection rolls evolve chaotically. Of particular interest are the defects in this pattern, i.e. points where two convection rolls merge into one (see Fig. 3.3). These defects behave very much like particles. They have well-defined positions and velocities, they are created and annihilated (in pairs of course), and one can even formally attribute a 'charge' to them: there are positive and negative defects. For more details, see [7,24] and references therein.

The probability density of defect velocities has been quite precisely measured. As shown in Fig. 3.4, it quite precisely coincides with a Tsallis distribution with $q \approx 1.46$. Other dynamical properties of the defects also coincide quite well with typical predictions of nonextensive models [7]. In total, these defects seem to behave very much like an ideal gas of nonextensive statistical mechanics with entropic index $q \approx 1.5$. Defect turbulence thus may serve as an example for which generalized versions of statistical mechanics are not only mathematically beautiful but physically useful. Note that this is an application to a physical system far from equilibrium, for which ordinary statistical mechanics has little to say.

Our dynamical realization in terms of the generalized Langevin equation (2.5) with $F(u) = -u$ and fluctuating effective friction γ makes sense as a very simple model for the defect velocity u. While ordinary Brownian particles have constant damping due to Stokes' law, $\gamma = \frac{6\pi\nu\rho a}{m}$, where ν is the kinematic viscosity of the liquid, ρ is its density, m is the mass of the particle, and a is the radius of the particles, defects are no ordinary particles: they have neither a well-defined mass m nor a well-defined radius a, and thus one expects that there is an ensemble of damping constants that depend on the topology of the defect and its fluctuating environment. In particular, the fastest velocities result from circumstances in which the defect is moving in a local environment with only a very small effective damping γ acting. The driving forces $L(t)$ are hardly damped during such a time interval, and lead to very large velocities for a limited amount of time, until another region with another γ is reached.

The experimentally observed value $q \approx 1.46...1.50$ for the defect statistics means, according to (2.16), that there are effectively about three independent degrees of freedom that contribute to the fluctuating local defect environment. We do not know from where these three effective degrees of freedom come, but a very simple picture would be that the fluctuating environment of the defect is mainly characterized by the states of the three convection rolls that merge when forming a defect.

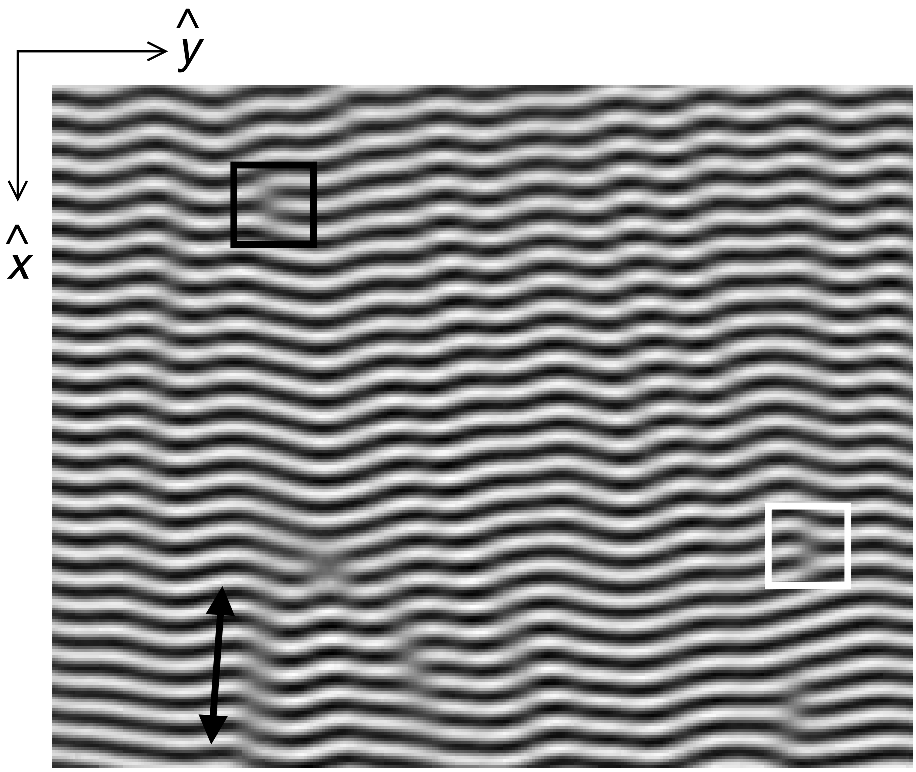

Fig. 3.3. A positive defect (black box) and a negative defect (white box), as seen in the experiment of Daniels et al. [24]

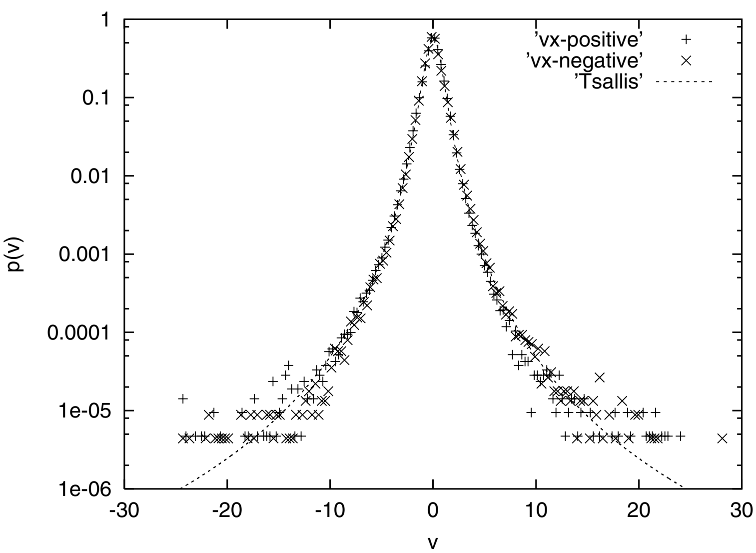

Fig. 3.4. Measured distribution of positive and negative defect velocities v_x and a comparison with a Tsallis distribution with $q = 1.46$. The dimensionless temperature difference is $\frac{\Delta T}{T_c} - 1 = 0.17$. All distributions have been re-scaled to a variance of 1

3.3 Statistics of cosmic rays

Our third example is from high-energy physics. We will proceed to extremely high temperatures, at which (similarly to defect turbulence) particles are created and annihilated in pairs. Nonextensive statistical mechanics has been shown to work well for reproducing experimentally measured cross-sections in e^+e^- collider experiments [8,25]. Here we apply it to high-energy collision processes induced by astrophysical sources, which lead to the creation of cosmic ray particles [26] that are ultimately observed on Earth. The idea to apply nonextensive

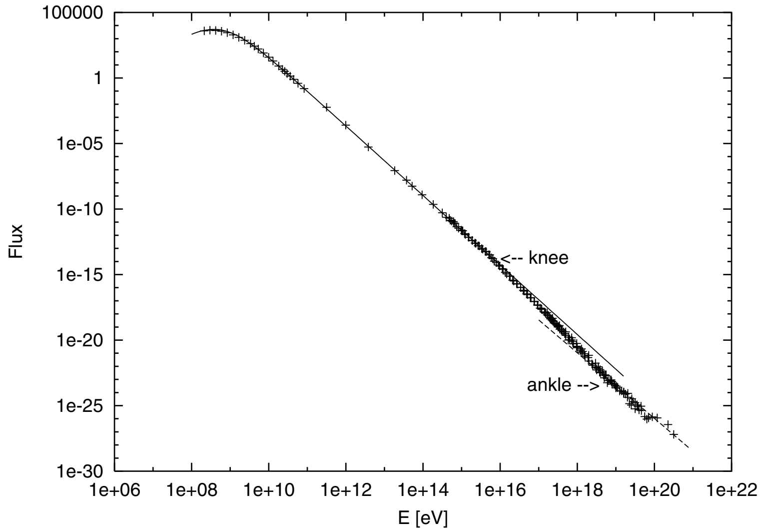

Fig. 3.5. Measured distribution of primary cosmic ray particles with a given energy E and a comparison with the distribution (3.1) with $q = 1.215$ and $\tilde{\beta}^{-1} = 107$ MeV (solid line)

statistics to the measured cosmic ray spectrum was first presented in [9]. Some related work can also be found in [27,28].

Experimental data of the measured cosmic ray spectrum [29] are shown in Fig. 3.5. Also shown is a curve that corresponds to a prediction of nonextensive statistical mechanics. Up to energies of 10^{16} eV, the measured flux rate of cosmic ray particles with a given energy is well fitted by a generalized canonical distribution of the form

$$p(E) = C \frac{E^2}{(1 + \tilde{\beta}(q-1)E)^{1/(q-1)}}. \tag{3.1}$$

This is an F-distribution in E. E is the energy of the particles,

$$E = \sqrt{c^2 p_x^2 + c^2 p_y^2 + c^2 p_z^2 + m^2 c^4}, \tag{3.2}$$

$\tilde{\beta} = \tilde{T}^{-1}$ is an effective inverse temperature variable, and C is a constant representing the total flux rate. For relativistic particles the rest mass m can be neglected and one has $E \approx c|\boldsymbol{p}|$. The distribution (3.1) is a q-generalized relativistic Maxwell-Boltzmann distribution in the formalism of nonextensive statistical mechanics. The Tsallis distribution is multiplied by E^2, taking into account the available phase space volume. As seen in Fig. 3.5, the cosmic ray spectrum is very well fitted by the distribution (3.1) if the entropic index is chosen as $q = 1.215$ and the effective temperature parameter is given by $\tilde{T} = \tilde{\beta}^{-1} = 107$ MeV.

The above effective temperature is of the same order of magnitude as the so-called Hagedorn temperature T_H [30], an effective temperature well known from collider experiments. The Hagedorn temperature is much smaller than the center-of-mass energy E_{CMS} of a typical collision process and represents a kind of 'boiling temperature' of nuclear matter at the confinement phase transition. It is a kind of maximum temperature that can be reached in a collision experiment. Even the largest E_{CMS} cannot produce a larger average temperature than T_H, due to the fact that the number of possible particle states grows exponentially. The Hagedorn theory of scattering processes is known to work well for energies $E_{CMS} < 10$ GeV, whereas for larger energies there is experimental evidence from various collision experiments that power-law behaviour of differential cross-sections sets in. This power-law is not contained in the original Hagedorn theory but can be formally incorporated if one considers a nonextensive extension of the Hagedorn theory [25].

Let us now work out the assumption that the nonextensive behaviour of the measured cosmic ray spectrum is due to fluctuations of the temperature. Assume that locally, in the creation process of some cosmic ray particle, some value of the fluctuating inverse temperature β is given. We then expect the momentum of a randomly picked particle in this region to be distributed according to the relativistic Maxwell-Boltzmann distribution

$$p(E|\beta) = \frac{1}{Z(\beta)} E^2 e^{-\beta E}. \tag{3.3}$$

Here $p(E|\beta)$ denotes the conditional probability of E given some value of β. We neglect the rest mass m so that $E = c|\boldsymbol{p}|$. The normalization constant is given by

$$Z(\beta) = \int_0^\infty E^2 e^{-\beta E} dE = \frac{2}{\beta^3}. \tag{3.4}$$

Now assume that β is χ^2-distributed. The observed cosmic ray distribution on Earth does not contain any information about the local temperature at which the various particles were produced. Hence we have to average over all possible fluctuating temperatures, obtaining the measured energy spectrum as the marginal distribution

$$p(E) = \int_0^\infty p(E|\beta) f(\beta) d\beta. \tag{3.5}$$

The integral (3.5), with $f(\beta)$ given by (2.9) and $p(E|\beta)$ given by (3.3), is easily evaluated and one obtains (3.1) with

$$q = 1 + \frac{2}{n+6} \tag{3.6}$$

and

$$\tilde{\beta} = \frac{\beta_0}{4 - 3q}. \tag{3.7}$$

This just corresponds to (2.29) and (2.30) for $x = -3$ and $y = 0$.

The variables X_i in (2.10) describe the independent degrees of freedom contributing to the fluctuating temperature. At very large center-of-mass energies, due to the uncertainty relation $r \sim 1/E_{CMS}$, the probed volume r^3 is very small, and all relevant degrees of freedom in this small volume are basically represented by the three spatial dimensions into which heat can flow. We may physically interpret X_i^2 as the heat loss in the spatial i-direction, where $i = x, y, z$, during the collision process that generates the cosmic ray particle. The more heat is lost, the smaller the local T, i.e. the larger the local β given by (2.10). The three spatial degrees of freedom yield $n = 3$ or, according to (3.6),

$$q = \frac{11}{9} = 1.222. \tag{3.8}$$

For cosmic rays, E_{CMS} is very large. Hence we expect a q-value that is close to this asymptotic value. The fit in Fig. 3.5 in fact uses $q = 1.215$, which agrees with the predicted value in (3.8) to about three digits. It also coincides well with the fitting value of $q = 1.225$ used by Tsallis et al. [9] using multi-parameter generalizations of nonextensive canonical distributions.

For smaller center-of-mass energies, the volume r^3 probed will be bigger and more effective degrees of freedom within this bigger interaction region can contribute to the fluctuating temperature. Hence we expect that, for decreasing E_{CMS}, n will increase and q will become smaller than $11/9$. Experimental data measured in e^+e^- annihilation experiments are in good agreement with the following parametrization [25]:

$$q(E_{CMS}) = \frac{11 - e^{-E_{CMS}/E_0}}{9 + e^{-E_{CMS}/E_0}}, \tag{3.9}$$

with $E_0 \approx 45.6$ GeV. Solving for E_{CMS} we get

$$E_{CMS} = -E_0 \log \frac{11 - 9q}{1 + q}, \tag{3.10}$$

and putting in the fitted value $q = 1.215$ of the cosmic ray spectrum, we can give an estimate of the average center-of-mass energy of the ensemble of astrophysical accelerators:

$$E_{CMS} \approx 161 \text{ GeV}. \tag{3.11}$$

It should be clear that this is a very rough estimate only. Moreover, it is an average over the unknown ensemble of all accelerating astrophysical sources. Some astrophysical objects can definitely accelerate to larger energies.

The 'knee' and 'ankle' in the cosmic ray spectrum, which occur at $E \approx 10^{16}$ eV and $E \approx 10^{19}$ eV, respectively, are produced by other effects that can be easily embedded into this nonextensive model (see [28] for details). The advantage of our superstatistics model is that a concrete prediction for q can be given.

References

1. Tsallis, C.: Possible generalization of Boltzmann-Gibbs statistics. J. Stat. Phys. **52**, 479–487 (1988)
2. Tsallis, C., Mendes, R.S., Plastino, A.R.: The role of constraints within generalized nonextensive statistics. Physica A **261**, 534–554 (1998)
3. Tsallis, C.: Nonextensive statistics: theoretical, experimental and computational evidence and connections. Braz. J. Phys. **29**, 1–35 (1999)
4. Abe, S., Okamoto, Y. (eds): *Nonextensive Statistical Mechanics and Its Applications*, Springer Berlin (2001)
5. Beck, C., Lewis, G.S., Swinney, H.L.: Measuring nonextensitivity parameters in a turbulent Couette-Taylor flow. Phys. Rev. E **63**, 035303(R) (2001)
6. Beck, C.: On the small-scale statistics of Lagrangian turbulence. Phys. Lett. A **287**, 240–244 (2001)
7. Daniels, K.E., Beck, C., Bodenschatz, E.: Defect turbulence and generalized statistical mechanics. cond-mat/0302623, Physica D (to appear) (2003)
8. Bediaga, I., Curado, E.M.F., de Miranda, J.M. A thermodynamic equilibrium approach in $e^+e^- \rightarrow$ hadrons. Physica A **286**, 156–163 (2000)
9. Tsallis, C., Anjos, J.C., Borges, E.P.: Fluxes of cosmic rays: a delicately balanced stationary state. Phys. Lett. **310**, 372–376 (2003)
10. Wilk, G., Wlodarczyk, Z.: Interpretation of the nonextensivity parameter q in some applications of Tsallis statistics and Levy distributions. Phys. Rev. Lett. **84**, 2770–2773 (2000)
11. Beck, C.: Dynamical foundations of nonextensive statistical mechanics. Phys. Rev. Lett. **87**, 180601 (2001)
12. Beck, C., Cohen, E.G.D.: Superstatistics. Physica A **322**, 267–275 (2003)
13. Tsallis, C., Souza, A.M.C.: Constructing a statistical mechanics for Beck-Cohen superstatistics. Phys. Rev. E **67**, 026106 (2003)
14. Abe, S.: Generalized entropy optimized by an arbitrary distribution. cond-mat/0211437 (2002)
15. Abe, S.: Stability of Tsallis entropy and instabilities of Renyi and normalized Tsallis entropies: a basis for q-exponential distributions. Phys. Rev. E **66**, 046134 (2002)
16. Tsallis, C., Souza, A.M.C.: Stability of the entropy for superstatistics. cond-mat/0301304 (2003)
17. Reynolds, A.M.: On the application of nonextensive statistics to Lagrangian turbulence. Phys. Fluids **15**, L1–L4 (2003)
18. Beck, C.: Lagrangian acceleration statistics in turbulent flows. cond-mat/0212566 Europhys. Lett. (to appear) (2003)
19. Castaing, B., Gagne, Y., Hopfinger, E.J.: Velocity probability density functions of high Reynolds-number turbulence. Physica D **46**, 177–200 (1990)
20. Beck, C. Superstatistics in hydrodynamic turbulence. physics/0303061 (2003) Physica D (to appear)
21. Sattin, F., Salasnich, L.: Multiparameter generalization of nonextensive statistical mechanics. Phys. Rev. E **65**, 035106(R) (2002)
22. La Porta, A., Voth, G.A., Crawford, A.M., Alexander, J., Bodenschatz, E.: Fluid particle accelerations in fully developed turbulence. Nature **409**, 1017 (2001)
23. Voth, G.A., et al: Measurement of particle accelerations in fully developed turbulence. J. Fluid Mech. **469**, 121–160 (2002)
24. Daniels, K.E., Bodenschatz, E.: Statistics of defect motion in spatio-temporal chaos in inclined layer convection. Chaos **13**, 55–63 (2003)
25. Beck, C.: Non-extensive statistical mechanics and particle spectra in elementary interactions. Physica A **286**, 164–180 (2000)
26. Gaisser, T.K., Stanev, T.: Cosmic rays. Phys. Rev. D **66**, 010001-1 (2002) (http://pdg.lbl.gov)
27. Kaniadakis, G.: Statistical mechanics in the context of special relativity. Phys. Rev. E **66**, 056125 (2002)
28. Beck, C.: Generalized statistical mechanics of cosmic rays. cond-mat/0301354 (2003) Physica A (to appear)
29. http://astroparticle.uchicago.edu/announce.htm (data compilation by Swordy, S.P.)
30. Hagedorn, R.: Nuovo Cimento **3**, 147 (1965)