

# Superstatistical Brownian motion

Christian Beck

*School of Mathematical Sciences, Queen Mary,  
University of London, Mile End Road, London E1 4NS, UK*

(Dated: August 10, 2005)

## Abstract

As a main example for the superstatistics approach, we study a Brownian particle moving in a  $d$ -dimensional inhomogeneous environment with macroscopic temperature fluctuations. We discuss the average occupation time of the particle in spatial cells with a given temperature. The Fokker-Planck equation for this problem becomes a stochastic partial differential equation. We illustrate our results using experimentally measured time series from hydrodynamic turbulence.

## I. INTRODUCTION

Nonextensive statistical mechanics [1, 2, 3, 4], originally developed as an equilibrium formalism, has indeed many interesting applications for driven nonequilibrium systems of sufficient complexity. Consider e.g. a Brownian particle moving through a changing environment. We may assume that the environment is inhomogeneous and a suitable parameter describing the state of this environment, e.g. the inverse temperature  $\beta$ , fluctuates on a relatively large spatio-temporal scale. This means that there is a fast dynamics given by the velocity of the Brownian particle and a slow one given by temperature changes of the environment. The two effects produce a superposition of two statistics, or in a short, a superstatistics [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23]. The stationary probability distributions for an ordinary Brownian particle in a constant environment are certainly Gaussian distributions, but for a fluctuating environment one obtains non-Gaussian behaviour with fat tails. These tails can decay e.g. with a power law, or as a stretched exponential, or in an even more complicated way [8]. Which type of tails are produced depends on the probability distribution  $f(\beta)$  of the parameter  $\beta$ . For the above simple example of a generalized Brownian particle,  $\beta$  is the fluctuating inverse temperature of the environment, but in general  $\beta$  can also be an effective friction constant, a changing mass parameter, a changing amplitude of Gaussian white noise, the fluctuating energy dissipation in turbulent flows, or simply a local variance parameter extracted from a signal. Recent applications of the superstatistics concept include a variety of physical systems, such as Lagrangian [24, 25, 26, 27] and Eulerian turbulence [28, 29, 30], defect turbulence [31], atmospheric turbulence [32, 33], cosmic ray statistics [34], solar flares [35], solar wind statistics [36], networks [37, 38], random matrix theory [39], and mathematical finance [40, 41, 42].

If  $\beta$  is distributed according to a particular probability distribution, the  $\chi^2$ -distribution, then the corresponding marginal stationary velocity distributions of the Brownian particle obtained by integrating over all  $\beta$  are given by the generalized canonical distributions of nonextensive statistical mechanics [1, 2, 3, 4]. For other distributions of the intensive parameter  $\beta$ , one ends up with more general statistics, which contain Tsallis statistics as a special case.

In this paper we first briefly review the superstatistics concept. We then look at typical occupation times of the superstatistical Brownian particle in cells of constant temperature,

distinguishing between temporal and spatial temperature fluctuations. In section 4 we emphasize a certain analogy between the step from statistics to superstatistics and the step from 1st quantization to 2nd quantization: In both cases the rate equations for probability densities are upgraded from deterministic partial differential equations to stochastic ones. Finally, in the last section we compare with experimental data in turbulent flows, where superstatistical models are a very useful tool to effectively describe the dynamics.

## II. VARIOUS TYPES OF SUPERSTATISTICS

It is well known that for equilibrium systems of ordinary statistical mechanics the probability to observe a state with energy  $E$  is given by

$$p(E) = \frac{1}{Z(\beta)} \rho(E) e^{-\beta E}. \quad (1)$$

This formula just describes the ordinary canonical ensemble.  $e^{-\beta E}$  is the Boltzmann factor,  $\rho(E)$  is the density of states and  $Z(\beta)$  is the normalization constant of  $\rho(E)e^{-\beta E}$ . For superstatistical systems, one generalizes this approach by assuming that  $\beta$  is a random variable as well. Indeed, a driven nonequilibrium system is often inhomogeneous and consist of many spatial cells with different values of  $\beta$  in each cell. The cell size is effectively determined by the correlation length of the continuously varying  $\beta$ -field. If we assume that each cell reaches local equilibrium very fast, i.e. the associated relaxation time is short, then in the long-term run the stationary probability distributions arise as a superposition of Boltzmann factors  $e^{-\beta E}$  weighted with the probability density  $f(\beta)$  to observe some value  $\beta$  in a randomly chosen cell:

$$p(E) = \int_0^\infty f(\beta) \frac{1}{Z(\beta)} \rho(E) e^{-\beta E} d\beta \quad (2)$$

The simplest example is a Brownian particle of mass  $m$  moving through a changing environment in  $d$  dimensions. For its velocity  $\vec{v}$  one has the local Langevin equation

$$\dot{\vec{v}} = -\gamma \vec{v} + \sigma \vec{L}(t) \quad (3)$$

( $\vec{L}(t)$ :  $d$ -dimensional Gaussian white noise,  $\gamma$ : friction constant,  $\sigma$ : strength of noise) which becomes superstatistical because for a fluctuating environment the parameter  $\beta := \frac{2}{m} \frac{\gamma}{\sigma^2}$  becomes a random variable as well: it varies from cell to cell on the large spatio-temporal

scale  $T$ . Of course, for this example  $E = \frac{1}{2}m\bar{v}^2$ . During the time scale  $T$  the local stationary distribution in each cell is Gaussian,

$$p(\vec{v}|\beta) = \left(\frac{\beta}{2\pi}\right)^{d/2} e^{-\frac{1}{2}\beta m\vec{v}^2}. \quad (4)$$

But the marginal distribution describing the long-time behavior of the particle for  $t \gg T$ ,

$$p(\vec{v}) = \int_0^\infty f(\beta)p(\vec{v}|\beta)d\beta \quad (5)$$

exhibits fat tails. The large- $|v|$  tails of the distribution (5) depend on the behaviour of  $f(\beta)$  for  $\beta \rightarrow 0$  [8]. Many different superstatistical models corresponding to different  $f(\beta)$  are possible: The function  $f$  is determined by the environmental dynamics of the nonequilibrium system under consideration.

Consider, for example, a simple case where there are  $n$  independent Gaussian random variables  $X_1, \dots, X_n$  underlying the dynamics of  $\beta$  in an additive way.  $\beta$  needs to be positive and a positive  $\beta$  is obtained by squaring these Gaussian random variables. The probability distribution of a random variable that is the sum of squared Gaussian random variables  $\beta = \sum_{i=1}^n X_i^2$  is well known in statistics: It is the  $\chi^2$ -distribution of degree  $n$ , i.e. the probability density  $f(\beta)$  is given by

$$f(\beta) = \frac{1}{\Gamma(\frac{n}{2})} \left(\frac{n}{2\beta_0}\right)^{n/2} \beta^{n/2-1} e^{-\frac{n\beta}{2\beta_0}}, \quad (6)$$

where  $\beta_0$  is the average of  $\beta$ .

One can now simply do the integration in eq. (5) using eq. (6) [12, 13]. The result is the generalized canonical distribution of nonextensive statistical mechanics, i.e. a  $q$ -exponential of the form

$$p(\vec{v}) \sim \frac{1}{(1 + \tilde{\beta}(q-1)\frac{1}{2}m\vec{v}^2)^{\frac{1}{q-1}}} \quad (7)$$

with

$$q = 1 + \frac{2}{n+d} \quad (8)$$

and

$$\tilde{\beta} = \frac{2\beta_0}{2 - (q-1)d}. \quad (9)$$

Thus these types of generalized Boltzmann factors occur as stationary states for nonequilibrium systems with suitable fluctuations of some intensive parameter  $\beta$ .

Instead of  $\beta$  being a sum of many contributions, for other systems the random variable  $\beta$  may be generated by multiplicative random processes. In this case one typically ends up with a log-normally distributed  $\beta$  [30], i.e. the probability density is given by

$$f(\beta) = \frac{1}{\sqrt{2\pi s}\beta} \exp \left\{ -\frac{(\ln \frac{\beta}{\mu})^2}{2s^2} \right\}, \quad (10)$$

where  $\mu$  and  $s$  are parameters. Both  $\chi^2$ -superstatistics (= Tsallis statistics) and lognormal superstatistics are relevant for turbulent systems. The observation is that the former one particularly well describes atmospheric turbulence data [32, 33], whereas the latter one gives good results for laboratory turbulence (e.g. Taylor-Couette flow) [28, 29, 30]. Note that the Reynolds number is controlled for laboratory turbulence, whereas it fluctuates for the atmospheric case.

### III. OCCUPATION TIMES IN CELLS OF CONSTANT LOCAL TEMPERATURE

Let us now discuss some subtleties of the superstatistical modeling approach. An important question is what really causes the fluctuations of temperature around the Brownian particle. Clearly these are produced by the fact that we consider a driven nonequilibrium system in a stationary state which has some external energy input and also some energy dissipation (like the sun acting on the earth's atmosphere and producing spatio-temporal temperature fluctuations in form of the weather). There are, however, different possibilities: Either there could be explicit temporal temperature changes around the Brownian particle, or the particle could move through spatial cells of size  $L^d$  which all have different temperatures but no explicit time dependence. The latter case is more a spatial than a temporal effect, and corresponds to a frozen random pattern.

To treat the second case properly, we have to take into account the average occupation time of the Brownian particle in a cell of size  $L^d$  where the temperature is approximately constant. Locally, in each cell the velocity of our particle is described by the Ornstein-Uhlenbeck process and its position by the Ornstein-Uhlenbeck position process [43]. The latter process is a Gaussian diffusion process. For each position component  $x_i(t)$ ,  $i = 1, \dots, d$  of the Brownian particle one has for large times  $t$

$$\langle x_i^2(t) \rangle = 2Dt, \quad (11)$$

where the diffusion constant  $D$  is given by the Einstein relation

$$D = \frac{1}{m\gamma\beta}. \quad (12)$$

Of course, the larger the temperature  $\beta^{-1}$  in a given cell, the faster the particle will diffuse, hence it will occupy a given cell of size  $L^d$  for a shorter time on average if the temperature is higher in that cell. In fact, the average time  $\bar{t}$  the particle spends in a cell of temperature  $\beta^{-1}$  is given by

$$L^2 = 2D\bar{t} \iff \bar{t} = \frac{L^2}{2D} = \frac{1}{2}L^2m\gamma\beta. \quad (13)$$

Assuming that on average all spatial cells with a given local temperature have the same typical size  $L^d$ , this means the average occupation time  $\bar{t}$  in a given cell is proportional to  $\beta$ . The probability to find the particle in a given cell is proportional to the occupation time, hence this means we have to adjust the probability density  $f_{space}(\beta)$  describing the spatial distribution of  $\beta$  in the various cells by a factor  $\beta$  if we proceed to a temporal description  $f_{temp}(\beta)$ , i.e. if we take into account how much time the particle spends in each cell:

$$f_{temp}(\beta) \sim \bar{t}f_{space}(\beta) \sim \beta f_{space}(\beta) \quad (14)$$

In general, one has for a given effective energy  $E$  the local Boltzmann-Gibbs distributions

$$p_{loc}(E|\beta) = \beta e^{-\beta E} \quad (15)$$

in each cell (assuming for simplicity a constant density of states). These distributions are properly normalized,

$$\int_0^\infty p_{loc}(E|\beta)dE = \int_0^\infty \beta e^{-\beta E}dE = 1. \quad (16)$$

Taking temporal averages over all possible  $\beta$ , one obtains the marginal distributions

$$p(E) = \int_0^\infty \beta f_{temp}(\beta)e^{-\beta E}d\beta. \quad (17)$$

Our previous consideration shows that proceeding to a spatial probability density  $f_{space}(\beta)$  describing the distribution of  $\beta$  in the various spatial cells requires an additional weighting by a factor  $\beta$ , since a typical particle will spend more time in cells with small temperature.

We thus arrive at

$$p(E) \sim \int_0^\infty \beta^2 f_{space}(\beta)e^{-\beta E}d\beta. \quad (18)$$

Our consideration shows that care has to be taken in the interpretation of the probability density  $f(\beta)$ , namely whether it describes the distribution of  $\beta$  in spatial cells or in temporal time slices. In general, one must also take into account the density of state and the fact that for Brownian particles probability densities are usually regarded as a function of the velocity  $\vec{v}$  rather than the energy  $E = \frac{1}{2}m\vec{v}^2$ . This corresponds to a simple transformation of random variables.

#### IV. STOCHASTIC FOKKER-PLANCK EQUATION

Let us once again emphasize that for a superstatistical Brownian particle one has a superposition of two stochastic processes on different time scales: A fast stochastic process given by a local Ornstein-Uhlenbeck process and a slow stochastic process given by  $\beta(t)$ . The  $\beta$ -process can a priori be anything. We assume that the parameters  $\sigma$  and  $\gamma$  in eq. (3) are constant for a sufficiently long time scale  $T$ , and then change to new values, either by an explicit time dependence, or by a change of the environment through which the Brownian particle moves.

For simplicity, let us consider the 1-dimensional case. Locally, in a given cell with constant temperature the probability density  $P(v, t)$  obeys the Fokker-Planck equation [43]

$$\frac{\partial P}{\partial t} = \gamma \frac{\partial(vP)}{\partial v} + \frac{1}{2}\sigma^2 \frac{\partial^2 P}{\partial v^2} \quad (19)$$

with the local stationary solution

$$P(v|\beta) = \sqrt{\frac{m\beta}{2\pi}} \exp\left\{-\frac{1}{2}\beta mv^2\right\}. \quad (20)$$

In the superstatistical approach, one regards  $\gamma(t)$  and  $\sigma(t)$  as stochastic processes as well, which evolve on a much larger typical time scale  $T$  as compared to  $v(t)$ . This means that for superstatistical systems the above equation is not an ordinary Fokker-Planck equation: It is a Fokker-Planck equation whose coefficients  $\gamma(t)$  and  $\sigma(t)$  are stochastic processes, with all their properties to be specified by a characteristic functional. Hence, in the superstatistical approach the equation that determines the evolution of probabilities becomes a stochastic partial differential equation—rather than a deterministic one. At the end, one has to perform an average over all realizations of  $\gamma(t)$  and  $\sigma(t)$ . Needless to say that one can generalize all this to superstatistical versions of higher-dimensional and nonlinear Fokker-Planck equations as well.

We can see a certain analogy to the process of second quantization in quantum field theories. Consider, for example, a Klein-Gordon (or Schrödinger) field. The Klein-Gordon (or Schrödinger) equation governs the time evolution of a wave function, and the wave function squared essentially gives the quantum mechanical probability density in a first-quantized system. Thus the Klein-Gordon (or Schrödinger) equation corresponds to a deterministic evolution equation for probability densities. In our case, this role is taken over by the local (deterministic) Fokker-Planck equation. In order to proceed to second quantization in quantum mechanics a possible method is to add noise to the classical field equation (e.g. the Klein-Gordon equation), obtaining a second-quantized theory via the stochastic quantization procedure [44, 45]. By second quantization, we thus construct a stochastic partial differential equation, a Klein-Gordon equation with noise. In a similar way, for superstatistical systems we proceed from a deterministic Fokker-Planck equation to a stochastic one, by replacing the constant parameters  $\gamma$  or  $\sigma$  by suitable stochastic processes. Thus proceeding to a superstatistical description is analogous to proceeding to a second quantized description in quantum mechanics.

To completely solve the superstatistical system dynamics one has to find all solutions of the stochastic Fokker-Planck equation (which are stochastic processes) and then form expectations. In full generality, this is quite a heavy task, though some progress has been made for very simple dichotomous models [16]. The solution of the stochastic Fokker-Planck equation depends on what type of stochastic processes are chosen for  $\gamma(t)$  and  $\sigma(t)$ . As shown in [20], one has for example a power-law decay of the marginal velocity correlation function if  $\gamma$  fluctuates and  $\sigma$  is constant, and an exponential decay if  $\gamma$  is constant and  $\sigma$  fluctuates, choosing the same  $\chi^2$ -distribution of  $\beta = \frac{2}{m} \frac{\gamma}{\sigma^2}$  in both cases and assuming  $T \gg \tau$ ,  $\tau$  being the local relaxation time in each cell. To construct the best superstatistical model for a given physical application, one has to carefully compare with the experimental data, looking at both, stationary densities and correlation functions.

## V. COMPARISON WITH TURBULENT VELOCITY FLUCTUATIONS

A major application of superstatistics is in turbulence modeling [12, 24, 25, 26, 27, 28, 29, 30]. Here essentially the local velocity  $v$  of the superstatistical Brownian particle model corresponds to a local longitudinal velocity difference  $u$  in the turbulent flow, and  $\beta$  is a

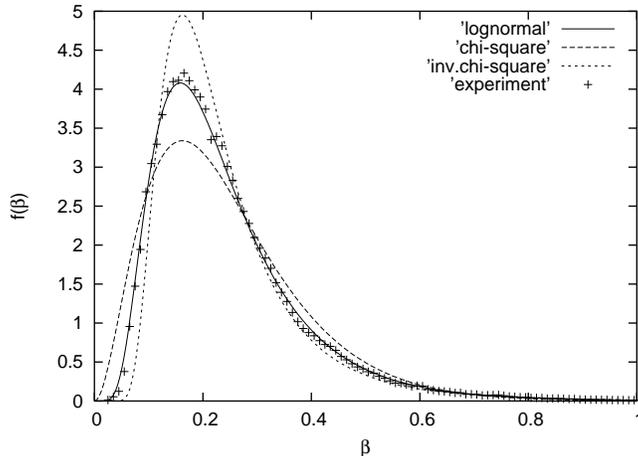


FIG. 1: Probability density  $f_{temp}(\beta)$  as extracted from an experimentally measured time series in a turbulent Taylor-Couette flow [30]. Also shown is a lognormal,  $\chi^2$ , and inverse  $\chi^2$  distribution with the same mean and variance as the experimental data.

local variance parameter connected with fluctuations in energy dissipation. As said before, to construct a suitable superstatistical turbulence model one has to compare carefully with various experimentally accessible quantities. For the example of a measured velocity time series in a turbulent Taylor-Couette flow, one basically observes the following [30]: The probability density  $f_{temp}(\beta)$  is essentially a lognormal distribution (Fig. 1). Correlation functions of the measured velocity signal  $v(t)$  decay more or less exponentially, but those of temporal velocity differences  $u(t) = v(t + \delta) - v(t)$  on a given time scale  $\delta$  decay asymptotically with a power law (Fig. 2). The marginal distribution  $p(u)$  exhibits fat tails whose flatness decreases with increasing  $\delta$ . On all scales the measured histograms  $p(u)$  are well described by lognormal superstatistics as given by eq. (5) and (10), replacing  $\vec{v} \rightarrow u$ . One observes a clear time-scale separation between the process  $u(t)$  and the corresponding process  $\beta(t)$ :  $\beta(t)$  evolves on a much larger typical time scale. This time scale separation increases with increasing Reynolds number, as well as with  $\delta$  [30]. The process  $\beta(t)$  is strongly correlated, its correlation function decays with a power law.

---

[1] C. Tsallis, J. Stat. Phys. **52**, 479 (1988)

[2] C. Tsallis, R.S. Mendes and A.R. Plastino, Physica **261A**, 534 (1998)

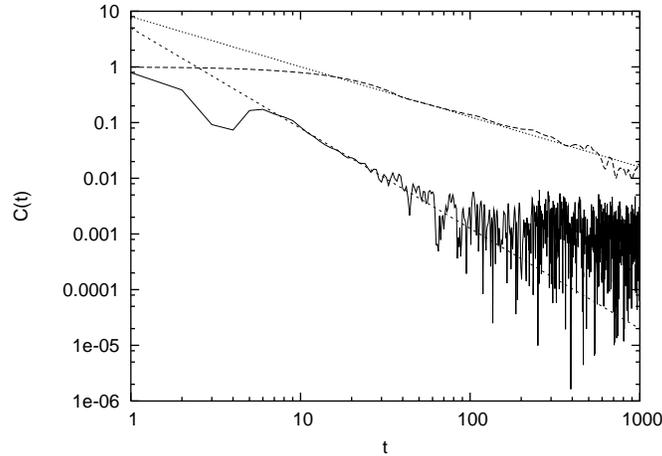


FIG. 2: Correlation functions  $C_\beta(t)$  (top) and  $|C_u(t)|$  (bottom) as measured in a turbulent Taylor-Couette flow [30] on the smallest scales. The straight lines show power laws with exponents -0.9 and -1.8.

- [3] C. Tsallis, *Braz. J. Phys.*, **29**: 1 (1999)
- [4] S. Abe, Y. Okamoto (eds.), *Nonextensive Statistical Mechanics and Its Applications*, Springer, Berlin (2001)
- [5] C. Beck and E.G.D. Cohen, *Physica* **322A**, 267 (2003)
- [6] E.G.D. Cohen, *Einstein und Boltzmann — Dynamics and Statistics*, Boltzmann award lecture at Statphys 22, Bangalore, Pramana (2005)
- [7] C. Beck, *Cont. Mech. Thermodyn.* **16**, 293 (2004)
- [8] H. Touchette and C. Beck, *Phys. Rev. E* **71**, 016131 (2005)
- [9] T. Yamano, cond-mat/0502643
- [10] C. Vignat, A. Plastino and A.R. Plastino, cond-mat/0505580
- [11] V.V. Ryazanov, cond-mat/0404357
- [12] C. Beck, *Phys. Rev. Lett.* **87**, 180601 (2001)
- [13] G. Wilk and Z. Wlodarczyk, *Phys. Rev. Lett.* **84**, 2770 (2000)
- [14] C. Tsallis and A.M.C. Souza, *Phys. Rev.* **67E**, 026106 (2003)
- [15] C. Tsallis and A.M.C. Souza, *Phys. Lett.* **319A**, 273 (2003)
- [16] J. Luczka and B. Zaborek, *Acta Phys. Polon. B* **35**, 2151 (2004)
- [17] V. Garcia-Morales and J. Pellicer, math-ph/0304013
- [18] F. Sattin and L. Salasnich, *Phys. Rev.* **65E**, 035106(R) (2002)

- [19] F. Sattin, *Physica* **338A**, 437 (2004)
- [20] C. Beck, cond-mat/0502306
- [21] P.-H. Chavanis, cond-mat/0409511, to appear in *Physica A* (2005)
- [22] J. Luczka, P. Talkner and P. Hänggi, *Physica* **278A**, 18 (2000)
- [23] P. Allegrini, F. Barbi, P. Grigolini and P. Paradisi, cond-mat/0503335
- [24] C. Beck, *Europhys. Lett.* **64**, 151 (2003)
- [25] A. Reynolds, *Phys. Rev. Lett.* **91**, 084503 (2003)
- [26] N. Mordant, A.M. Crawford and E. Bodenschatz, *Physica* **193D**, 245 (2004)
- [27] A.R. Aringazin and M.I. Mazhitov, cond-mat/0408018
- [28] C. Beck, *Physica* **193D**, 195 (2004)
- [29] S.Jung and H.L. Swinney, cond-mat/0502301, to appear in *Phys. Rev. E*.
- [30] C.Beck, E.G.D. Cohen and H.L. Swinney, cond-mat/0507411
- [31] K. E. Daniels, C. Beck and E. Bodenschatz, *Physica* **193D**, 208 (2004)
- [32] S. Rizzo and A. Rapisarda, in *Proceedings of the 8th Experimental Chaos Conference, Florence*, AIP Conf. Proc. **742**, 176 (2004) (cond-mat/0406684)
- [33] S. Rizzo and A. Rapisarda, in *Proceeding of the International Workshop on ‘Complexity, Metastability and Nonextensivity’*, eds. C.Beck, G. Benedek, A. Rapisarda and C. Tsallis, World Scientific, in press (cond-mat/0502305)
- [34] C. Beck, *Physica* **331A**, 173 (2004)
- [35] M. Baiesi, M. Paczuski and A.L. Stella, cond-mat/0411342
- [36] L.F. Burlaga and A.F. Vinas, *J. Geophys. Research* **110** (2005) (in press)
- [37] S. Abe and S. Thurner, cond-mat/0501429, to appear in *Phys. Rev. E* (2005)
- [38] H. Hasegawa, cond-mat/0506301
- [39] A.Y. Abul-Magd, cond-mat/0507034, to appear in *Physica A* (2005)
- [40] J.-P. Bouchard and M. Potters, *Theory of Financial Risk and Derivative Pricing*, Cambridge University Press, Cambridge (2003)
- [41] M. Ausloos and K. Ivanova, *Phys. Rev.* **68E**, 046122 (2003)
- [42] Y. Ohtaki and H.H. Hasegawa, cond-mat/0312568
- [43] N.G. van Kampen, *Stochastic Processes in Physics and Chemistry*, North Holland, Amsterdam (1981)
- [44] G. Parisi and Y. Wu, *Sci Sin.* **24**, 483 (1981)

- [45] P.H. Damgaard and H. Hüffel (eds.), *Stochastic Quantization*, World Scientific, Singapore (1988)