4.5 Autoregressive Processes AR(p)

The idea behind the autoregressive models is to explain the present value of the series, X_t, by a function of p past values, $X_{t-1}, X_{t-2}, \ldots, X_{t-p}$.

Definition 4.7. *An **autoregressive process of order p** is written as*

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \ldots + \phi_p X_{t-p} + Z_t, \tag{4.20}$$

where $\{Z_t\}$ is white noise, i.e., $\{Z_t\} \sim WN(0, \sigma^2)$, and Z_t is uncorrelated with X_s for each $s < t$.

Remark 4.12. We assume (for simplicity of notation) that the mean of X_t is zero. If the mean is $\mathrm{E}\, X_t = \mu \neq 0$, then we replace X_t by $X_t - \mu$ to obtain

$$X_t - \mu = \phi_1(X_{t-1} - \mu) + \phi_2(X_{t-2} - \mu) + \ldots + \phi_p(X_{t-p} - \mu) + Z_t,$$

what can be written as

$$X_t = \alpha + \phi_1 X_{t-1} + \phi_2 X_{t-2} + \ldots + \phi_p X_{t-p} + Z_t,$$

where

$$\alpha = \mu(1 - \phi_1 - \ldots - \phi_p).$$

Other ways of writing AR(p) model use:

Vector notation: Denote

$$\boldsymbol{\phi} = (\phi_1, \phi_2, \ldots, \phi_p)^{\mathrm{T}},$$
$$\boldsymbol{X_{t-1}} = (X_{t-1}, X_{t-2}, \ldots, X_{t-p})^{\mathrm{T}}.$$

Then the formula (4.20) can be written as

$$X_t = \boldsymbol{\phi}^{\mathrm{T}} \boldsymbol{X}_{t-1} + Z_t.$$

Backshift operator: Namely, writing the model (4.20) in the form

$$X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} - \ldots - \phi_p X_{t-p} = Z_t,$$

and applying $BX_t = X_{t-1}$ we get

$$(1 - \phi_1 B - \phi_2 B^2 - \ldots - \phi_p B^p) X_t = Z_t$$

or, using the concise notation we write

$$\phi(B)X_t = Z_t, \tag{4.21}$$

where $\phi(B)$ denotes the **autoregressive operator**

$$\phi(B) = 1 - \phi_1 B - \phi_2 B^2 - \ldots - \phi_p B^p.$$

Then the AR(p) can be viewed as a solution to the equation (4.21), i.e.,

$$X_t = \frac{1}{\phi(B)} Z_t. \tag{4.22}$$

4.5.1 AR(1)

According to Definition 4.7 the autoregressive process of order 1 is given by

$$X_t = \phi X_{t-1} + Z_t, \tag{4.23}$$

where $Z_t \sim WN(0, \sigma^2)$ and ϕ is a constant.

Is AR(1) a stationary TS?

Corollary 4.1 says that an infinite combination of white noise variables is a stationary process. Here, due to the recursive form of the TS we can write AR(1) in such a form. Namely

$$\begin{aligned} X_t &= \phi X_{t-1} + Z_t \\ &= \phi(\phi X_{t-2} + Z_{t-1}) + Z_t \\ &= \phi^2 X_{t-2} + \phi Z_{t-1} + Z_t \\ &\vdots \\ &= \phi^k X_{t-k} + \sum_{j=0}^{k-1} \phi^j Z_{t-j}. \end{aligned}$$

This can be rewritten as

$$\phi^k X_{t-k} = X_t - \sum_{j=0}^{k-1} \phi^j Z_{t-j}.$$

What would we obtain if we have continued the backwards operation, i.e., what happens when $k \to \infty$?

Taking the expectation we obtain

$$\lim_{k\to\infty} \mathrm{E}\left(X_t - \sum_{j=0}^{k-1} \phi^j Z_{t-j}\right)^2 = \lim_{k\to\infty} \phi^{2k}\,\mathrm{E}(X_{t-k}^2) = 0$$

if $|\phi| < 1$ and the variance of X_t is bounded. Hence, we can represent AR(1) as

$$X_t = \sum_{j=0}^{\infty} \phi^j Z_{t-j}$$

in the mean square sense. This is a linear process (4.15) with

$$\psi_j = \begin{cases} \phi^j & \text{for } j \geq 0, \\ 0 & \text{for } j < 0. \end{cases}$$

This technique of iterating backwards works well for AR of order 1 but not for other orders. A more general way to convert the series into a linear process form is the method of matching coefficients.

The AR(1) model is

$$\phi(B)X_t = Z_t,$$

where $\phi(B) = 1 - \phi B$ and $|\phi| < 1$. We want to write the model as a linear process

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j} = \psi(B)Z_t,$$

where $\psi(B) = \sum_{j=0}^{\infty} \psi_j B^j$. It means we want to find the coefficients ψ_j. Substituting Z_t from the AR model into the linear process model we obtain

$$X_t = \psi(B)Z_t = \psi(B)\phi(B)X_t. \tag{4.24}$$

In full, the coefficients of both sides of the equation can be written as

$$\begin{aligned} 1 &= (1 + \psi_1 B + \psi_2 B^2 + \psi_3 B^3 + \ldots)(1 - \phi B) \\ &= 1 + \psi_1 B + \psi_2 B^2 + \psi_3 B^3 + \ldots - \phi B - \psi_1 \phi B^2 - \psi_2 \phi B^3 - \psi_3 \phi B^4 - \ldots \\ &= 1 + (\psi_1 - \phi)B + (\psi_2 - \psi_1\phi)B^2 + (\psi_3 - \psi_2\phi)B^3 + \ldots \end{aligned}$$

Now, equating coefficients of B^j on the LHS and RHS of this equation we see that all the coefficients of B^j must be zero, i.e.,

$$\begin{aligned} \psi_1 &= \phi \\ \psi_2 &= \psi_1\phi = \phi^2 \\ \psi_3 &= \psi_2\phi = \phi^3 \\ &\vdots \\ \psi_j &= \psi_{j-1}\phi = \phi^j. \end{aligned}$$

So, we obtained the linear process form of the AR(1)

$$X_t = \sum_{j=0}^{\infty} \phi^j Z_{t-j} = \sum_{j=0}^{\infty} \phi^j B^j Z_t.$$

Remark 4.13. Note, that from the equation (4.24) it follows that $\psi(B)$ is an inverse of $\phi(B)$, that is

$$\psi(B) = \frac{1}{\phi(B)}. \tag{4.25}$$

For an AR(1) we have

$$\psi(B) = \frac{1}{1 - \phi B} = 1 + \phi B + \phi^2 B^2 + \phi^3 B^3 + \dots \tag{4.26}$$

As a linear process AR(1) is stationary with mean

$$\mathrm{E}\, X_t = \sum_{j=0}^{\infty} \phi^j\, \mathrm{E}(Z_{t-j}) = 0 \tag{4.27}$$

and autocovariance function given by (4.19), that is

$$\gamma(\tau) = \sigma^2 \sum_{j=0}^{\infty} \phi^j \phi^{j+\tau} = \sigma^2 \phi^\tau \sum_{j=0}^{\infty} \phi^{2j}.$$

However, the infinite sum in this expression is the sum of a geometric progression as $|\phi| < 1$, i.e.,

$$\sum_{j=0}^{\infty} \phi^{2j} = \frac{1}{1 - \phi^2}.$$

This gives us the following form for the ACVF of AR(1).

$$\gamma(\tau) = \frac{\sigma^2 \phi^\tau}{1 - \phi^2}. \tag{4.28}$$

Then the variance of AR(1) is

$$\gamma(0) = \frac{\sigma^2}{1 - \phi^2}.$$

Hence, the autocorrelation function of AR(1) is

$$\rho(\tau) = \frac{\gamma(\tau)}{\gamma(0)} = \phi^\tau. \tag{4.29}$$

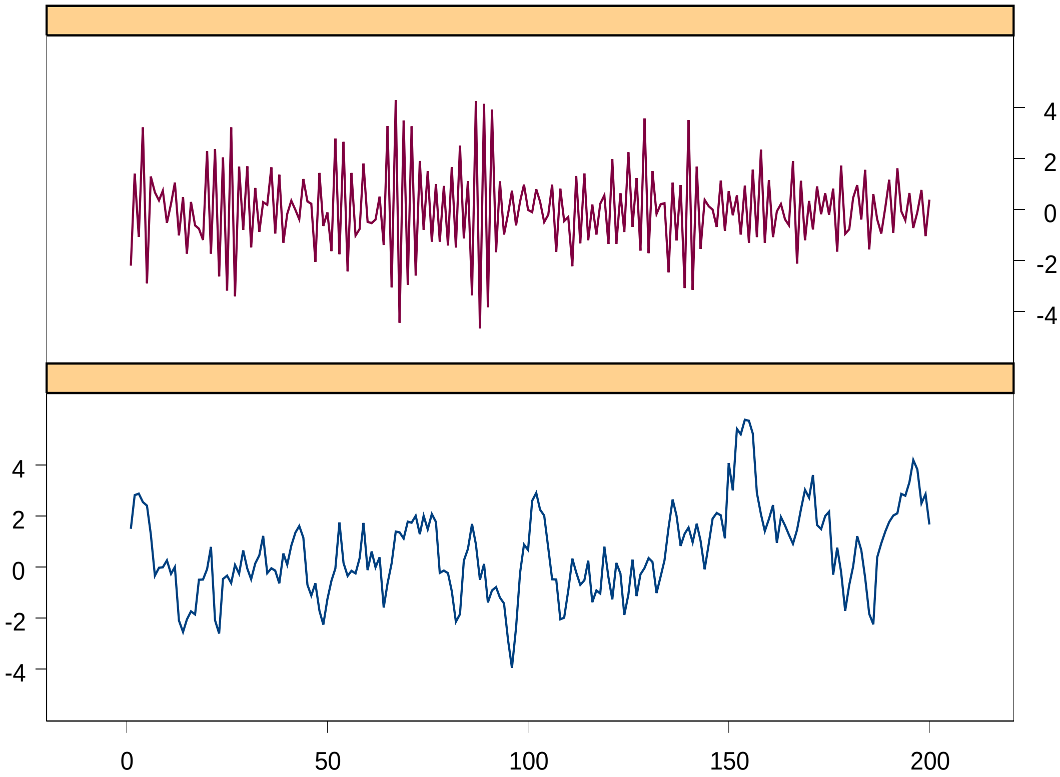


Figure 4.7: Simulated AR(1) processes for $\phi = -0.9$ (top) and for $\phi = 0.9$ (bottom).

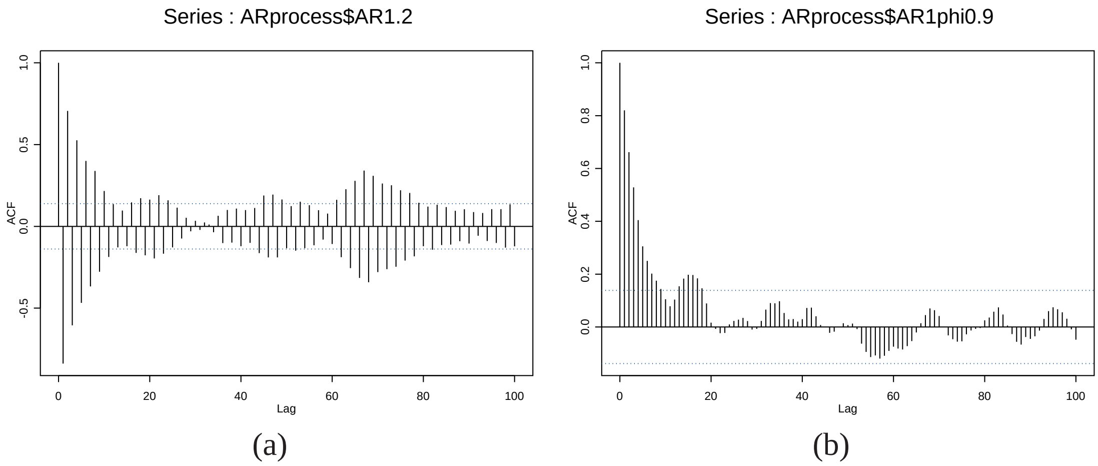


Figure 4.8: Sample ACF for AR(1): (a) $x_t = -0.9x_{t-1} + z_t$ and (b) $x_t = 0.9x_{t-1} + z_t$.

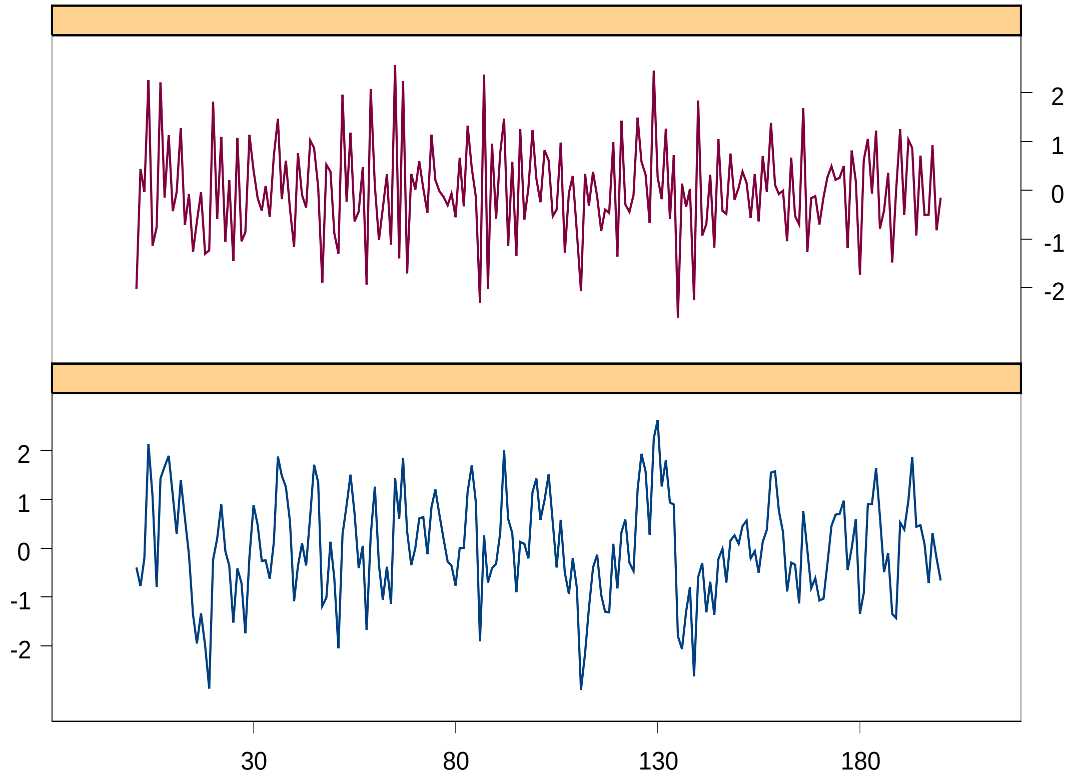


Figure 4.9: Simulated AR(1) processes for $\phi = -0.5$ (top) and for $\phi = 0.5$ (bottom).

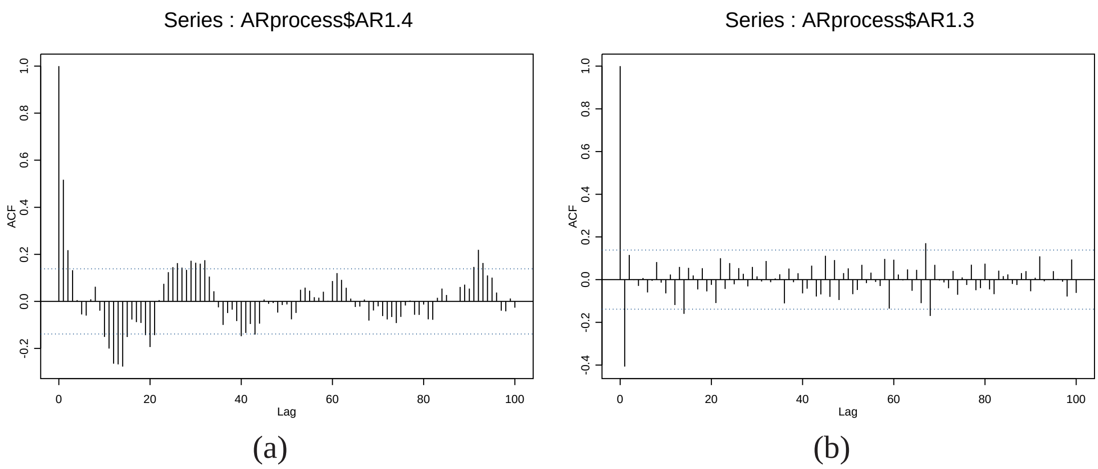


Figure 4.10: Sample ACF for AR(1): (a) $x_t = -0.5x_{t-1} + z_t$ and (b) $x_t = 0.5x_{t-1} + z_t$.

Figures 4.7, 4.9 and 4.8, 4.10 show simulated AR(1) processes for four different values of the coefficient ϕ (equal to -0.9, 0.9, -0.5 and 0.5) and the respective sample ACF functions.

Looking at these graphs we can see that for positive coefficient ϕ we obtain more smooth TS than for the negative one. Also, the ACFs are very different. We see that if ϕ is negative the neighboring observations are negatively correlated, but those two time points apart are positively correlated. In fact, if ϕ is negative the neighboring TS values have typically opposite signs. This is more evident if ϕ is close to -1.

4.5.2 Random Walk

This is a TS where at each point of time the series moves randomly away from its current position. The model can then be written as

$$X_t = X_{t-1} + Z_t, \tag{4.30}$$

where Z_t is a white noise variable with zero mean and constant variance σ^2. The model has the same form as AR(1) process, but since $\phi = 1$, it is not stationary. Such process is called Random Walk.

Repeatedly substituting for past values gives

$$\begin{aligned} X_t &= X_{t-1} + Z_t \\ &= X_{t-2} + Z_{t-1} + Z_t \\ &= X_{t-3} + Z_{t-2} + Z_{t-1} + Z_t \\ &= \dots \\ &= X_0 + \sum_{j=0}^{t-1} Z_{t-j}. \end{aligned}$$

If the initial value, X_0, is fixed, then the mean value of X_t is equal to X_0, that is

$$\operatorname{E} X_t = \operatorname{E}\left[X_0 + \sum_{j=0}^{t-1} Z_{t-j}\right] = X_0.$$

So, the mean is constant, but as we see below, the variance and covariance depend on time, not just on lag. The white noise variables Z_t are uncorrelated, hence we

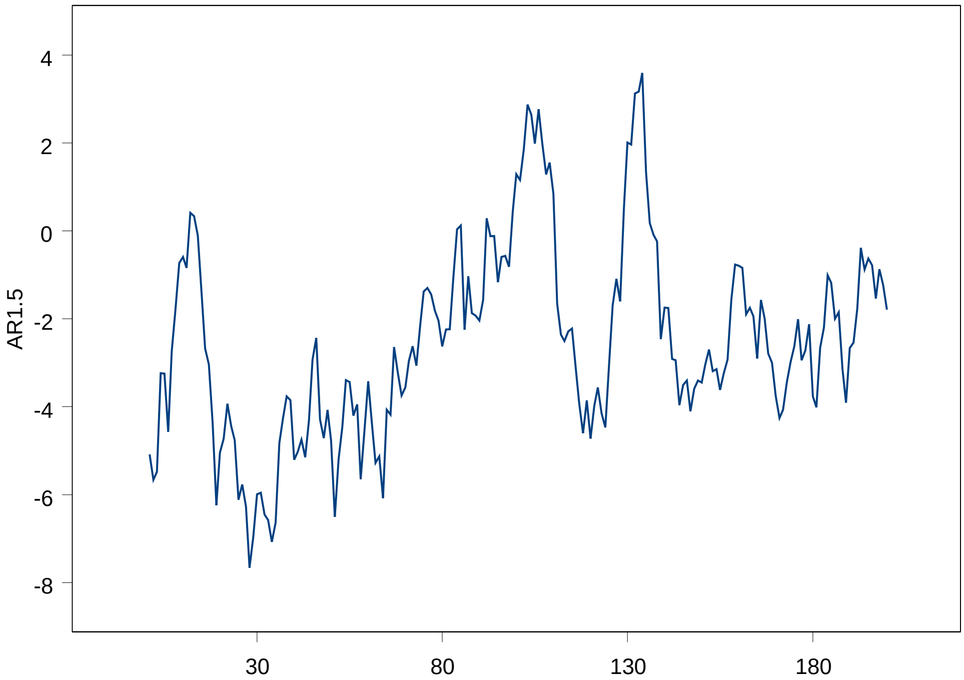


Figure 4.11: Simulated Random Walk $x_t = x_{t-1} + z_t$.

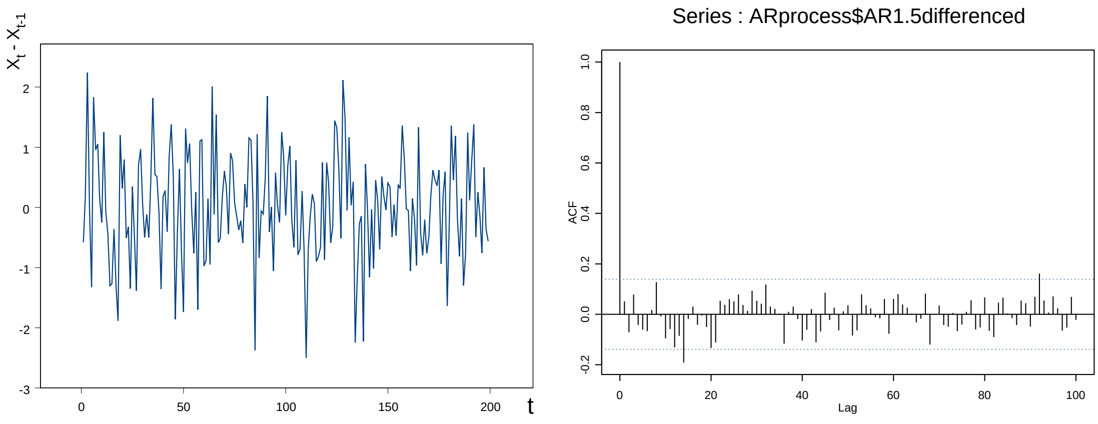
obtain

$$\begin{aligned}\operatorname{var}(X_t) &= \operatorname{var}\left(X_0 + \sum_{j=0}^{t-1} Z_{t-j}\right) \\ &= \operatorname{var}\left(\sum_{j=0}^{t-1} Z_{t-j}\right) \\ &= \sum_{j=0}^{t-1} \operatorname{var}(Z_{t-j}) = t\sigma^2\end{aligned}$$

and

$$\begin{aligned}\operatorname{cov}(X_t, X_{t-\tau}) &= \operatorname{cov}\left(\sum_{j=0}^{t-1} Z_{t-j}, \sum_{k=0}^{t-\tau-1} Z_{t-\tau-k}\right) \\ &= \operatorname{E}\left[\left(\sum_{j=0}^{t-1} Z_{t-j}\right)\left(\sum_{k=0}^{t-\tau-1} Z_{t-\tau-k}\right)\right] \\ &= |t-\tau|\sigma^2.\end{aligned}$$

A simulated series of this form is shown in Figure 4.11.



(a) (b)

Figure 4.12: (a) Differenced Random Walk ∇x_t and (b) its sample ACF.

As we can see the random walk meanders away from its starting value in no particular direction. It does not exhibit any clear trend, but at the same time is not stationary.

However, the first difference of random walk is stationary as it is just white noise, namely

$$\nabla X_t = X_t - X_{t-1} = Z_t.$$

The differenced random walk and its sample ACF are shown in Figure 4.12.

4.5.3 Explosive AR(1) Model and Causality

As we have seen in the previous section, random walk, which is AR(1) with $\phi = 1$ is not a stationary process. So, there is a question if a stationary AR(1) process with $|\phi| > 1$ exists? Also, what are the properties of AR(1) models for $\phi > 1$?

Clearly, the sum $\sum_{j=0}^{k-1} \phi^j Z_{t-j}$ will not converge in mean square sense as $k \to \infty$ and we will not get a linear process representation of the AR(1). However, if $|\phi| > 1$ then $\frac{1}{|\phi|} < 1$ and we can express a past value of the TS in terms of a future value rewriting

$$X_{t+1} = \phi X_t + Z_{t+1}$$

as

$$X_t = \phi^{-1} X_{t+1} - \phi^{-1} Z_{t+1}.$$

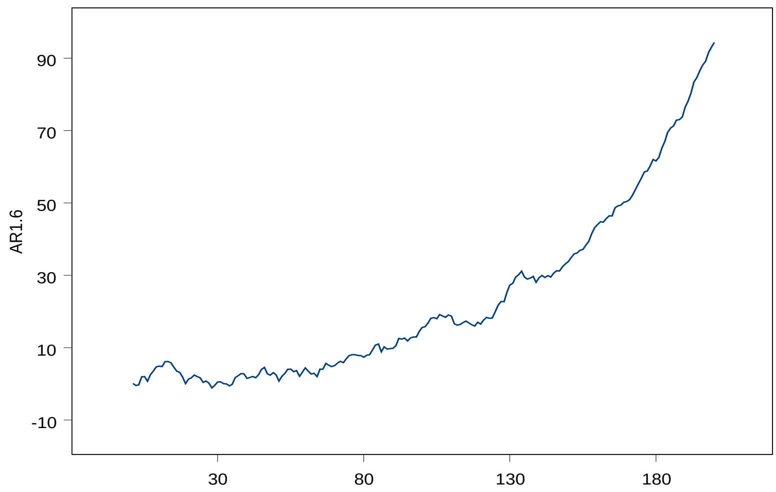


Figure 4.13: Simulated Explosive AR(1): $x_t = 1.02x_{t-1} + z_t$.

Then, substituting for X_{t+j} several times we obtain

$$\begin{aligned} X_t &= \phi^{-1}X_{t+1} - \phi^{-1}Z_{t+1} \\ &= \phi^{-1}(\phi^{-1}X_{t+2} - \phi^{-1}Z_{t+2}) - \phi^{-1}Z_{t+1} \\ &= \phi^{-2}X_{t+2} - \phi^{-2}Z_{t+2} - \phi^{-1}Z_{t+1} \\ &= \dots \\ &= \phi^{-k}X_{t+k} - \sum_{j=1}^{k-1} \phi^{-j}Z_{t+j} \end{aligned}$$

Since $|\phi^{-1}| < 1$ we obtain

$$X_t = -\sum_{j=1}^{\infty} \phi^{-j}Z_{t+j},$$

which is a future dependent stationary TS. This however, does not have any practical meaning because it requires knowledge of future values to predict the future.

When a process does not depend on the future, such as AR(1) when $|\phi| < 1$, we say that it is **causal**.

Figure 4.13 shows a simulated series $x_t = 1.02x_{t-1} + z_t$. As we can see the values of the time series quickly become large in magnitude, even for ϕ just slightly above 1. Such process is called **explosive**. This is not a causal TS.