6.2 ACF and PACF of ARMA(p,q)

6.2.1 ACF of ARMA(p,q)

In Section 4.6 we have derived the ACF for ARMA(1,1) process. We have used the linear process representation and the fact that

$$\gamma(\tau) = \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+\tau}.$$

We have calculated the coefficients ψ_j from the relation

$$\psi(B) = \frac{\theta(B)}{\phi(B)},$$

which (as in the above example) gives the values

$$\psi_j = \phi_1^{j-1}(\theta_1 + \phi_1).$$

This allows us to calculate the ACF of the process

$$\rho(\tau) = \frac{\gamma(\tau)}{\gamma(0)}.$$

Another way of finding the coefficients ψ is using the homogeneous difference equations. However, we may obtain such equation directly in terms of $\gamma(\tau)$ or $\rho(\tau)$.

For ARMA(1,1)

$$X_t - \phi X_{t-1} = Z_t + \theta Z_{t-1}$$

we can write

$$\begin{aligned}
\gamma(\tau) &= \mathrm{cov}(X_{t+\tau}, X_t) \\
&= \mathrm{E}(X_{t+\tau} X_t) \\
&= \mathrm{E}[(\phi X_{t+\tau-1} + Z_{t+\tau} + \theta Z_{t+\tau-1}) X_t] \\
&= \mathrm{E}[\phi X_{t+\tau-1} X_t + Z_{t+\tau} X_t + \theta Z_{t+\tau-1} X_t] \\
&= \phi\, \mathrm{E}[X_{t+\tau-1} X_t] + \mathrm{E}[Z_{t+\tau} X_t] + \theta\, \mathrm{E}[Z_{t+\tau-1} X_t]
\end{aligned}$$

Here we consider a causal ARMA(1,1) process, hence

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}.$$

This gives

$$\begin{aligned}
\mathrm{E}[Z_{t+\tau}X_t] &= \mathrm{E}[Z_{t+\tau}\sum_{j=0}^{\infty}\psi_j Z_{t-j}] \\
&= \sum_{j=0}^{\infty}\psi_j \,\mathrm{E}[Z_{t+\tau}Z_{t-j}] \\
&= \begin{cases} \psi_0\sigma^2 & \text{for } \tau = 0, \\ 0 & \text{for } \tau \geq 1. \end{cases}
\end{aligned}$$

Also,

$$\begin{aligned}
\mathrm{E}[Z_{t+\tau-1}X_t] &= \mathrm{E}[Z_{t+\tau-1}\sum_{j=0}^{\infty}\psi_j Z_{t-j}] \\
&= \sum_{j=0}^{\infty}\psi_j \,\mathrm{E}[Z_{t+\tau-1}Z_{t-j}] \\
&= \begin{cases} \psi_1\sigma^2 & \text{for } \tau = 0, \\ \psi_0\sigma^2 & \text{for } \tau = 1 \\ 0 & \text{for } \tau \geq 2. \end{cases}
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\psi_0 &= 1 \\
\psi_1 &= \phi + \theta.
\end{aligned}$$

Putting all these together we obtain

$$\begin{aligned}
\gamma(\tau) &= \phi\,\mathrm{E}[X_{t+\tau-1}X_t] + \mathrm{E}[Z_{t+\tau}X_t] + \theta\,\mathrm{E}[Z_{t+\tau-1}X_t] \\
&= \begin{cases} \phi\gamma(1) + \sigma^2(1+\phi\theta+\theta^2) & \text{for } \tau = 0, \\ \phi\gamma(0) + \sigma^2\theta & \text{for } \tau = 1, \\ \phi\gamma(\tau-1) & \text{for } \tau \geq 2. \end{cases}
\end{aligned}$$

The ACVF is in fact given here in the form of a homogeneous difference equation of order 1 with initial conditions specifying $\gamma(0)$ and $\gamma(1)$. Namely, we have

$$\gamma(\tau) - \phi\gamma(\tau-1) = 0 \tag{6.12}$$

and the initial conditions are

$$\begin{cases} \gamma(0) = \phi\gamma(1) + \sigma^2(1+\phi\theta+\theta^2) \\ \gamma(1) = \phi\gamma(0) + \sigma^2\theta \end{cases} \tag{6.13}$$

Note that the equation (6.12)

$$\gamma(\tau) = \phi\gamma(\tau-1)$$

has an iterative form and we can write

$$
\begin{aligned}
&\gamma(2) = \phi\gamma(1)\\
&\gamma(3) = \phi\gamma(2) = \phi^2\gamma(1)\\
&\gamma(4) = \phi\gamma(3) = \phi^3\gamma(1)\\
&\dots\\
&\gamma(\tau) = \phi^{\tau-1}\gamma(1)
\end{aligned}
$$

The polynomial associated with the equation (6.12) is

$$1 - \phi z = 0$$

with root

$$z_0 = \frac{1}{\phi}.$$

So we can write

$$\gamma(\tau) = (z_0^{-1})^{\tau-1}\gamma(1).$$

This depends only on the root of the associated polynomial and on the initial conditions. Solving (6.13) for $\gamma(0)$ and $\gamma(1)$ we obtain

$$\gamma(0) = \sigma^2\frac{1 + 2\theta\phi + \theta^2}{1 - \phi^2}$$

and

$$\gamma(1) = \sigma^2\frac{(1 + \theta\phi)(\phi + \theta)}{1 - \phi^2}$$

This gives us

$$\gamma(\tau) = \sigma^2\frac{(1 + \theta\phi)(\phi + \theta)}{1 - \phi^2}\phi^{\tau-1}, \quad \text{for } \tau \geq 1.$$

Finally dividing by $\gamma(0)$ we get the ACF, which is the same as the one derived in Section 4.6, that is

$$\rho(\tau) = \frac{(1 + \theta\phi)(\phi + \theta)}{1 + 2\theta\phi + \theta^2}\phi^{\tau-1}, \quad \text{for } \tau \geq 1. \tag{6.14}$$

ACF for ARMA(p,q)

Assume that the model

$$\phi(B)X_t = \theta(B)Z_t$$

is causal, that is the roots of $\phi(B)$ are outside the unit circle. Then we can write

$$X_t = \psi(B)Z_t,$$

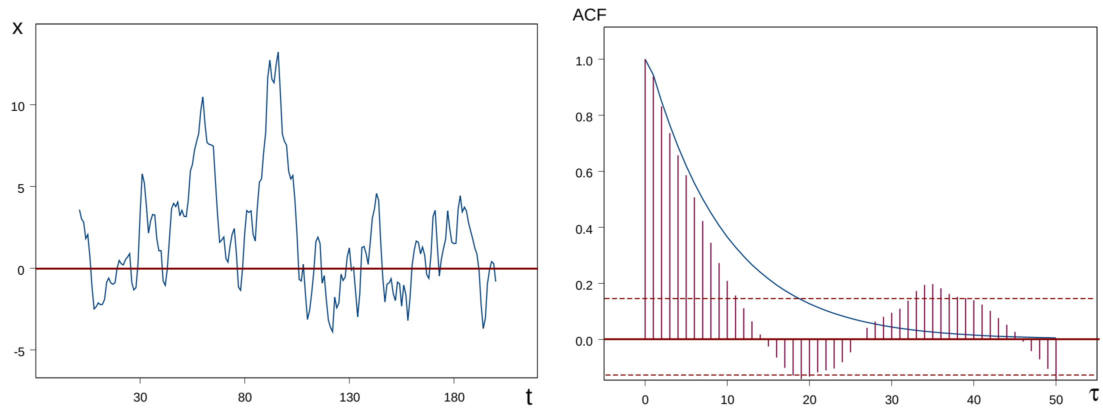

Figure 6.2: ARMA(1,1) simulated process $x_t - 0.9x_{t-1} = z_t + 0.5z_{t-1}$, sample ACF and the theoretical ACF of this process.

where

$$\psi(B) = \sum_{j=0}^{\infty} \psi_j B^j$$

and it follows immediately that $\mathrm{E}(X_t) = 0$.

As in the example for ARMA(1,1), we can obtain a homogeneous differential equation in terms of $\gamma(\tau)$ with some initial conditions. Namely

$$\begin{aligned}
\gamma(\tau) &= \operatorname{cov}(X_{t+\tau}, X_t) \\
&= \mathrm{E}\left[\left(\sum_{j=1}^{p} \phi_j X_{t+\tau-j} + \sum_{j=0}^{q} \theta_j Z_{t+\tau-j}\right) X_t\right] \\
&= \sum_{j=1}^{p} \phi_j \,\mathrm{E}[X_{t+\tau-j} X_t] + \sum_{j=0}^{q} \theta_j \,\mathrm{E}[Z_{t+\tau-j} X_t] \\
&= \sum_{j=1}^{p} \phi_j \gamma(\tau - j) + \sigma^2 \sum_{j=\tau}^{q} \theta_j \psi_{j-\tau}
\end{aligned}$$

Here, as before, we used the linear representation of X_t, the fact that Z_{t+i} and X_t are uncorrelated for $i > 0$ and that $\psi_i = 0$ for $i < 0$.

This gives the general homogeneous difference equation for $\gamma(\tau)$,

$$\gamma(\tau) - \phi_1\gamma(\tau-1) - \ldots - \phi_p\gamma(\tau-p) = 0 \quad \text{for } \tau \geq \max(p, q+1), \tag{6.15}$$

with initial conditions

$$\gamma(\tau) - \phi_1\gamma(\tau-1) - \ldots - \phi_p\gamma(\tau-p) = \sigma^2(\theta_\tau\psi_0 + \theta_{\tau+1}\psi_1 + \ldots + \theta_q\psi_{q-\tau}) \tag{6.16}$$

for $0 \leq \tau < \max(p, q+1)$.

Example 6.4. **ACF of an AR(2) process**
Let

$$X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} = Z_t$$

be a causal AR(2) process. From (6.15) we have

$$\gamma(\tau) - \phi_1\gamma(\tau-1) - \phi_2\gamma(\tau-2) = 0 \quad \text{for } \tau \geq 2$$

with initial conditions

$$\begin{cases} \gamma(0) - \phi_1\gamma(-1) - \phi_2\gamma(-2) = \sigma^2 \\ \gamma(1) - \phi_1\gamma(0) - \phi_2\gamma(-1) = 0 \end{cases}$$

It is convenient to write these equations in terms of the autocorrelation function $\rho(\tau)$. Dividing them by $\gamma(0)$ we obtain

$$\begin{cases} \rho(\tau) - \phi_1\rho(\tau-1) - \phi_2\rho(\tau-2) = 0, \quad \text{for } \tau \geq 2 \\ \rho(0) = 1 \\ \rho(1) = \dfrac{\phi_1}{1-\phi_2} \end{cases} \tag{6.17}$$

We know that a general solution to a second order difference equation is

$$\rho(\tau) = c_1 z_1^{-\tau} + c_2 z_2^{-\tau}$$

where z_1 and z_2 are the roots of the associated polynomial

$$\phi(z) = 1 - \phi_1 z - \phi_2 z^2,$$

and c_1 and c_2 can be found from the initial conditions.

Take $\phi_1 = 0.7$ and $\phi_2 = -0.1$, that is the AR(2) process is

$$X_t - 0.7X_{t-1} + 0.1X_{t-2} = Z_t.$$

It is a causal process as the coefficients lie in the admissible parameter space. Also, the roots of the associated polynomial

$$\phi(z) = 1 - 0.7z + 0.1z^2$$

are $z_1 = 2$ and $z_2 = 5$, i.e., they are outside the unit circle. The initial conditions are

$$\begin{cases} \rho(0) = 1 \\ \rho(1) = \dfrac{0.7}{1+0.1} = \dfrac{7}{11} \end{cases}$$

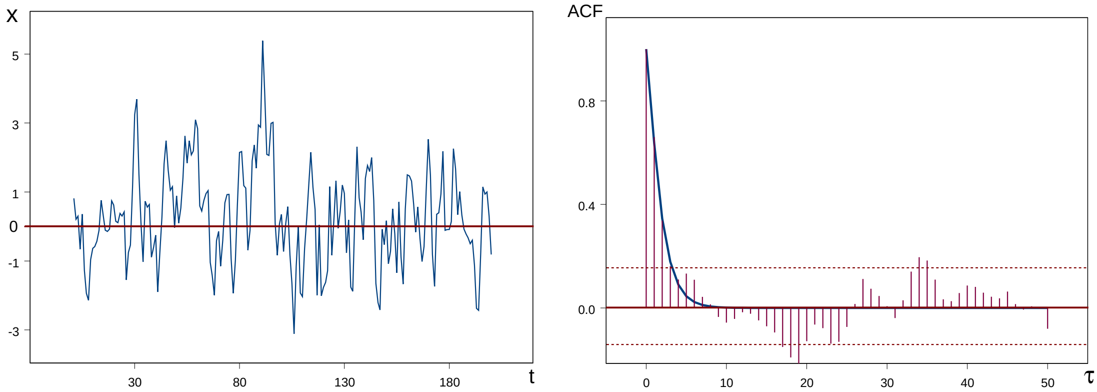

Figure 6.3: AR(2) simulated process $x_t - 0.7x_{t-1} + 0.1x_{t-2} = z_t$, sample ACF and the theoretical ACF of this process.

They give the set of equations for c_1 and c_2, namely

$$\begin{cases} c_1 + c_2 = 1 \\ \frac{1}{2}c_1 + \frac{1}{5}c_2 = \frac{7}{11} \end{cases}$$

These give

$$c_1 = \frac{16}{11}, \quad c_2 = -\frac{5}{11}$$

and finally we obtain the ACF for this AR(2) process

$$\rho(\tau) = \frac{16}{11}2^{-\tau} - \frac{5}{11}5^{-\tau} = \frac{2^{4-\tau} - 5^{1-\tau}}{11}.$$

Simulated AR(2) process, its sample ACF and the theoretical ACF are shown in Figure 6.3. As we can see, the theoretical ACF decreases quickly towards zero, but it never attains zero, we say it tails off.

6.2.2 PACF of ARMA(p,q)

We have seen earlier that the autocorrelation function of MA(q) models is zero for all lags greater than q as these are q-correlated processes. Hence, the ACF is a good indication of the order of the process. However AR(p) and ARMA(p,q) processes are "fully" correlated, their ACF tails off and never becomes zero, though it may be very close to zero. In such cases it is difficult to identify the process on the ACF basis only.

In this section we will consider another correlation function, which together with the ACF will help to identify the models. The function is called **Partial Autocorrelation Function (PACF)**. Before introducing a formal definition of PACF we motivate the idea for AR(1). Let

$$X_t = \phi X_{t-1} + Z_t$$

be a causal AR(1) process. Then

$$\begin{aligned}
\gamma(2) &= \operatorname{cov}(X_t, X_{t-2}) \\
&= \operatorname{cov}(\phi X_{t-1} + Z_t, X_{t-2}) \\
&= \operatorname{cov}(\phi^2 X_{t-2} + \phi Z_{t-1} + Z_t, X_{t-2}) \\
&= \mathrm{E}[(\phi^2 X_{t-2} + \phi Z_{t-1} + Z_t) X_{t-2}] \\
&= \phi^2 \gamma(0).
\end{aligned}$$

The autocorrelation is not zero because X_t depends on X_{t-2} through X_{t-1}. Due to the iterative kind of AR models there is a chain of dependence. We can break this dependence removing the influence of X_{t-1} from both X_t and X_{t-2} to obtain

$$X_t - \phi X_{t-1} \quad \text{and } X_{t-2} - \phi X_{t-1}$$

for which the covariance is zero, i.e.,

$$\operatorname{cov}(X_t - \phi X_{t-1}, X_{t-2} - \phi X_{t-1}) = \operatorname{cov}(Z_t, X_{t-2} - \phi X_{t-1}) = 0.$$

Similarly, we obtain zero covariance for X_t and X_{t-3} after breaking the chain of dependence, i.e. removing the dependence of the two variables on X_{t-1} and X_{t-2}, i.e. for $X_t - f(X_{t-1}, X_{t-2})$ and $X_{t-3} - f(X_{t-1}, X_{t-2})$ for some function f. Continuing this we would obtain zero covariances for variables $X_t - f(X_{t-1}, X_{t-2}, \ldots, X_{t-\tau+1})$ and $X_{t-\tau} - f(X_{t-1}, X_{t-2}, \ldots, X_{t-\tau+1})$. Then the only nonzero covariance is for X_t and X_{t-1} (nothing in between to break the chain of dependence). These covariances with an appropriate function f divided by the variance of the process are the partial autocorrelations. Hence, for a causal AR(1) process we would have the PACF at lag 1 equal to $\rho(1)$ and at lags > 1 equal to 0. This, together with the tailing off shape of the ACF identifies the process.

Definition 6.2. *The Partial Autocorrelation Function (PACF) of a zero-mean stationary TS $\{X_t\}_{t=0,1,\dots}$ is defined as*

$$\begin{aligned}\phi_{11} &= \operatorname{corr}(X_1, X_0) = \rho(1)\\ \phi_{\tau\tau} &= \operatorname{corr}(X_\tau - f_{(\tau-1)}, X_0 - f_{(\tau-1)}), \quad \tau \geq 2,\end{aligned} \tag{6.18}$$

where

$$f_{(\tau-1)} = f(X_{\tau-1}, \dots, X_1)$$

minimizes the mean square linear prediction error

$$\mathrm{E}(X_\tau - f_{(\tau-1)})^2.$$

Remark 6.4. The subscript at the f function denotes the number of variables the function depends on.

Remark 6.5. By stationarity, $\phi_{\tau\tau}$ is the correlation between variables X_t and $X_{t-\tau}$ with the linear effect

$$f(X_{t-1}, \dots, X_{t-\tau+1}) = \beta_1 X_{t-1} + \dots + \beta_{\tau-1} X_{t-\tau+1}$$

on each variable removed.

Example 6.5. **The PACF of AR(1)**

Consider a process

$$X_t = \phi X_{t-1} + Z_t, \quad Z_t \sim WN(0, \sigma^2),$$

where $|\phi| < 1$, i.e., a causal AR(1). Then by definition 6.2

$$\phi_{11} = \rho(1) = \phi.$$

To calculate ϕ_{22} we need to find the function $f_{(1)}$ which is of the form

$$f_{(1)} = \beta X_1.$$

We choose β to minimize

$$\begin{aligned}\mathrm{E}(X_2 - \beta X_1)^2 &= \mathrm{E}(X_2^2 - 2\beta X_1 X_2 + \beta^2 X_1^2)\\ &= \gamma(0) - 2\beta\gamma(1) + \beta^2\gamma(0)\end{aligned}$$

which is a polynomial in β. Taking the derivative with respect to β and setting it equal to zero, we obtain

$$-2\gamma(1) + 2\gamma(0)\beta = 0.$$

Hence

$$\beta = \frac{\gamma(1)}{\gamma(0)} = \rho(1) = \phi$$

and

$$f_{(1)} = \phi X_1.$$

Then

$$\phi_{22} = \mathrm{corr}(X_2 - \phi X_1, X_0 - \phi X_1) = \mathrm{corr}(Z_2, X_0 - \phi X_1) = 0$$

as by causality X_0, X_1 do not depend on Z_2. Similarly we would obtain $\phi_{33} = 0$. In fact

$$\phi_{\tau\tau} = 0 \quad \text{for } \tau > 1.$$

The PACF of AR(p)

Let

$$X_t - \phi_1 X_{t-1} - \ldots - \phi_p X_{t-p} = Z_t, \quad Z_t \sim WM(0, \sigma^2)$$

be a causal AR(p) process, i.e., we assume that the roots of $\phi(z)$ are outside the unit circle. When $\tau > p$ the linear combination minimizing the mean square linear prediction error is

$$f_{(p)} = \sum_{j=1}^{p} \phi_j X_{\tau-j}.$$

We will discuss this result later. Now we will use it to obtain the PACF for $\tau > p$, namely

$$\begin{aligned} \phi_{\tau\tau} &= \mathrm{corr}(X_\tau - f_{(p)}, X_0 - f_{(p)}) \\ &= \mathrm{corr}(Z_\tau, X_0 - f_{(p)}) = 0 \end{aligned}$$

as by causality $X_{\tau-j}$, do not depend on the future noise value Z_τ.

When $\tau \leq p$ $\phi_{pp} \neq 0$ and $\phi_{11}, \ldots, \phi_{p-1,p-1}$ are not necessarily zero.

Remark 6.6. **The PACF of MA(q)**
Let

$$X_t = Z_t + \theta_1 Z_{t-1} + \ldots + \theta_q Z_{t-q}, \quad Z_t \sim WN(0, \sigma^2)$$

be an invertible MA(q) process, i.e., roots of $\theta(z)$ lie outside the unit circle. Then its linear representation is

$$X_t = -\sum_{j=1}^{\infty} \pi_j X_{t-j} + Z_t.$$

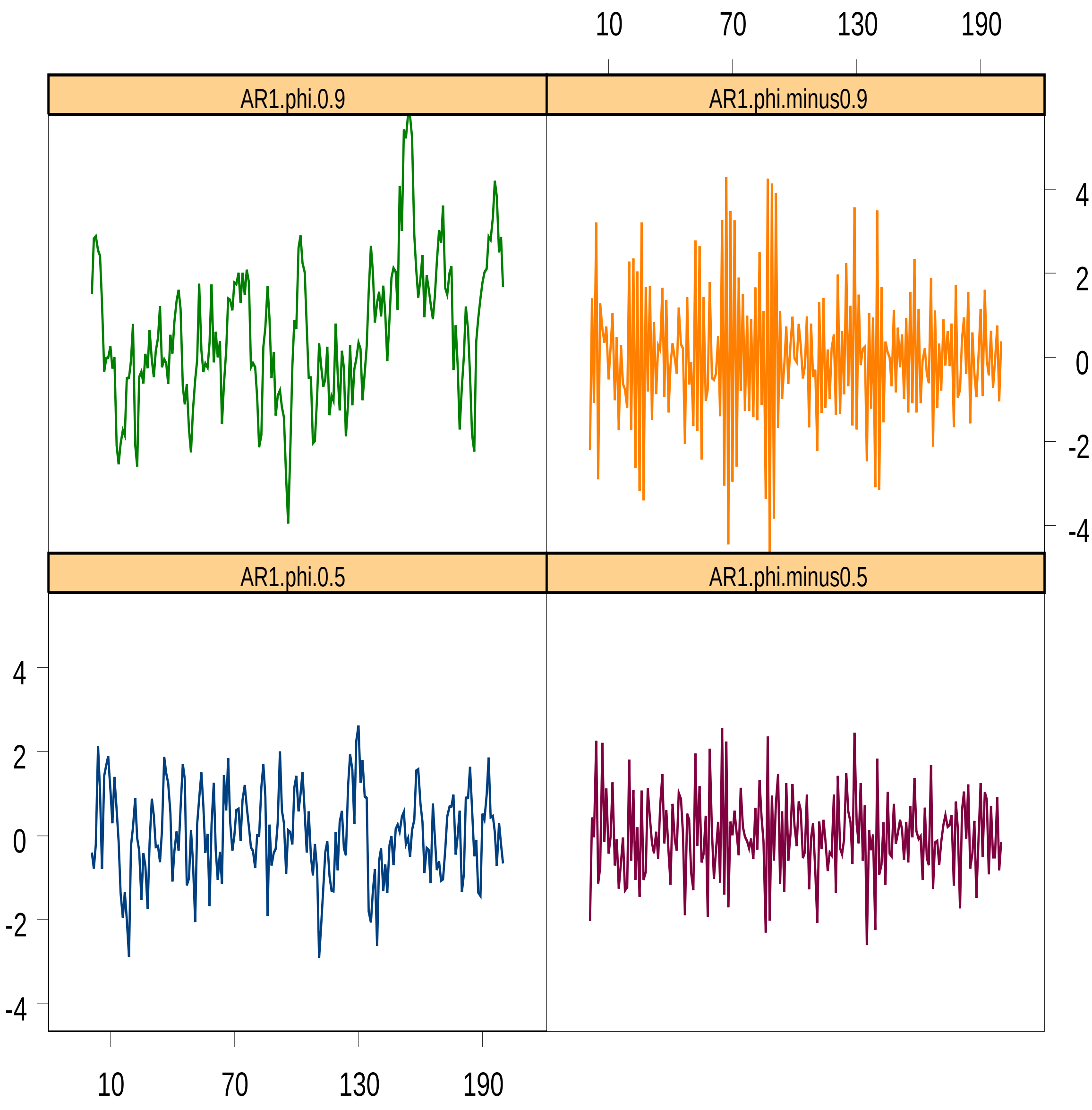

Figure 6.4: AR(1) for various values of the parameters $\phi = 0.9, -0.9, 0.5, -0.5$.

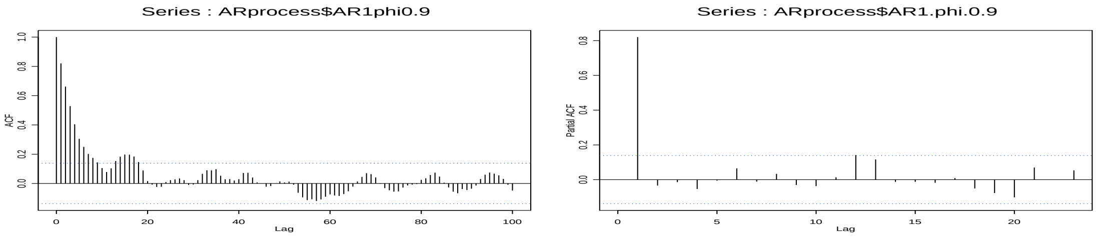

Figure 6.5: ACF and PACF of the AR(1) process $x_t = 0.9x_{t-1} + z_t$.

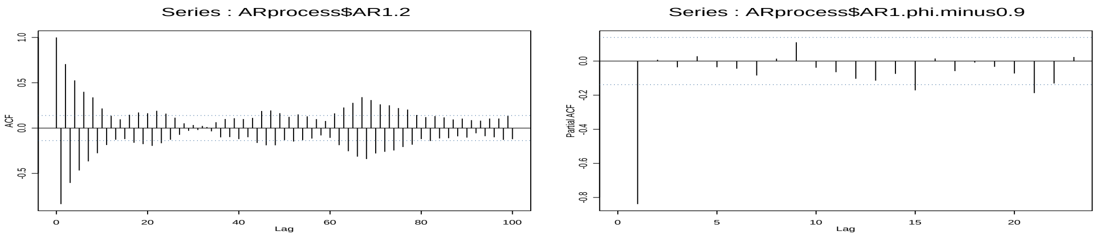

Figure 6.6: ACF and PACF of the AR(1) process $x_t = -0.9x_{t-1} + z_t$.

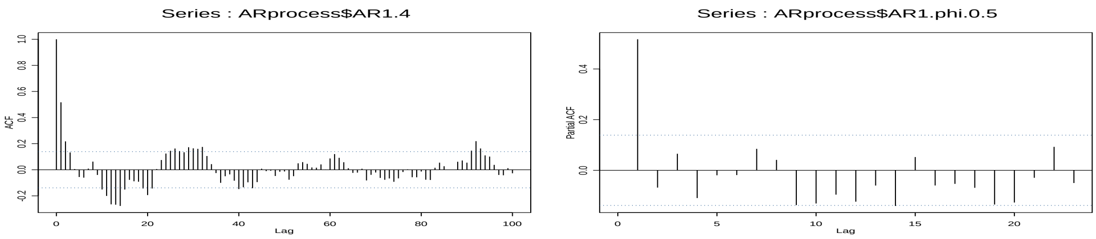

Figure 6.7: ACF and PACF of the AR(1) process $x_t = 0.5x_{t-1} + z_t$.

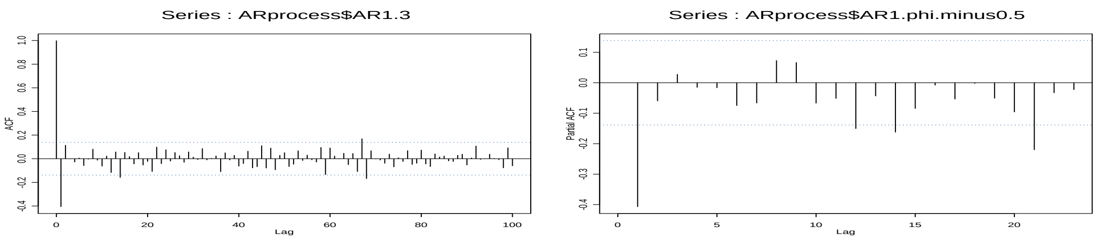

Figure 6.8: ACF and PACF of the AR(1) process $x_t = -0.5x_{t-1} + z_t$.

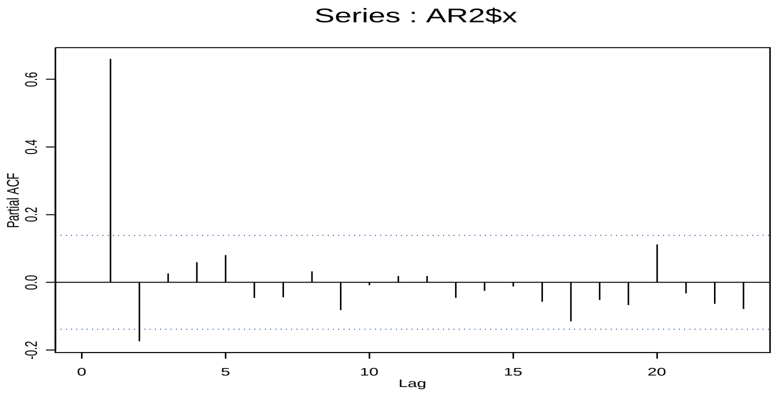

Figure 6.9: The PACF for AR(2) $x_t - 0.7x_{t-1} + 0.1x_{t-2} = z_t$.

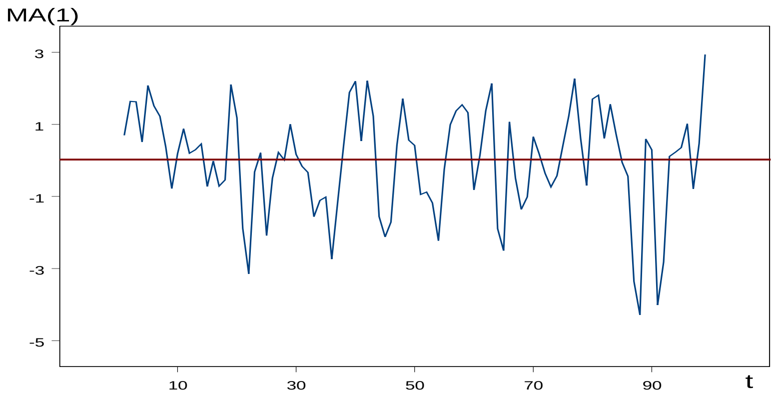

Figure 6.10: MA(1) process $x_t = z_t + 0.9z_{t-1}$.

This is an AR(∞) representation ($p = \infty$) and the PACF will never cut off as for the AR(p) with finite p.

The PACF of MA models behaves like ACF for AR models and PACF for AR models behaves like ACF for MA models.

It can be shown that

$$\phi_{\tau\tau} = \frac{(-\theta)^\tau(1-\theta^2)}{1-\theta^{2(\tau+1)}}, \quad \tau \geq 1.$$

Remark 6.7. **The PACF of ARMA(p,q)**
An invertible ARMA model has an infinite AR representation, hence the PACF will not cut off.

The following table summarizes the behaviour of the PACF of the causal and invertible ARMA models (see R.H.Shumway and Stoffer (2000)).

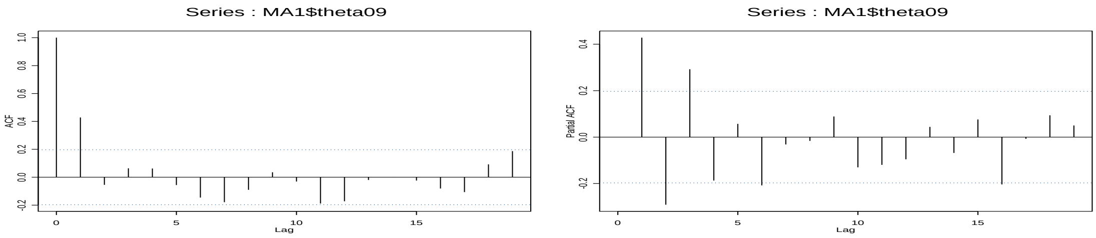

Figure 6.11: ACF and PACF of the MA(1) process $x_t = z_t + 0.9z_{t-1}$.

	AR(p)	MA(q)	ARMA(p,q)
ACF	Tails off	Cuts off after lag q	Tails off
PACF	Cuts off after lag p	Tails off	Tails off