4.6 Autoregressive Moving Average Model ARMA(1,1)

This section is an introduction to a wide class of models ARMA(p,q) which we will consider in more detail later in this course. The special case, ARMA(1,1), is defined by linear difference equations with constant coefficients as follows.

Definition 4.8. *A TS* $\{X_t\}$ *is an* ***ARMA(1,1) process*** *if it is stationary and it satisfies*

$$X_t - \phi X_{t-1} = Z_t + \theta Z_{t-1} \quad \text{for every } t, \tag{4.31}$$

where $\{Z_t\} \sim WN(0, \sigma^2)$ *and* $\phi + \theta \neq 0$.

Such a model may be viewed as a generalization of the two previously introduced models: AR(1) and MA(1). Compare

AR(1): $X_t = \phi X_{t-1} + Z_t$

MA(1): $X_t = Z_t + \theta Z_{t-1}$

ARMA(1,1): $X_t - \phi X_{t-1} = Z_t + \theta Z_{t-1}$

Hence, when $\phi = 0$ then ARMA(1,1) $\equiv$ MA(1) and we denote such a process as ARMA(0,1). Similarly, when $\theta = 0$ then ARMA(1,1) $\equiv$ AR(1) and we denote such process as ARMA(1,0).

Here, as in the MA and AR models, we can use the backshift operator to write the ARMA model more concisely as

$$\phi(B)X_t = \theta(B)Z_t, \tag{4.32}$$

where $\phi(B)$ and $\theta(B)$ are the linear filters:

$$\phi(B) = 1 - \phi B, \quad \theta(B) = 1 + \theta B.$$

4.6.1 Causality and invertibility of ARMA(1,1)

For what values of the parameters ϕ and θ does the stationary ARMA(1,1) exist and is useful? To answer this question we will look at the two properties of TS, causality and invertibility.

The solution to 4.31, or to 4.32, can be written as

$$X_t = \frac{1}{\phi(B)}\theta(B)Z_t.$$

However, for $|\phi| < 1$ we have (see Remark 4.13)

$$\begin{aligned}
\frac{1}{\phi(B)}\theta(B) &= (1 + \phi B + \phi^2 B^2 + \phi^3 B^3 + \ldots)(1 + \theta B) \\
&= 1 + \phi B + \phi^2 B^2 + \phi^3 B^3 + \ldots + \theta B + \phi\theta B^2 + \phi^2\theta B^3 + \phi^3\theta B^4 + \ldots \\
&= 1 + (\phi + \theta)B + (\phi^2 + \phi\theta)B^2 + (\phi^3 + \phi^2\theta)B^3 + \ldots \\
&= 1 + (\phi + \theta)B + (\phi + \theta)\phi B^2 + (\phi + \theta)\phi^2 B^3 + \ldots \\
&= \sum_{j=0}^{\infty} \psi_j B^j,
\end{aligned}$$

where $\psi_0 = 1$ and $\psi_j = (\phi + \theta)\phi^{j-1}$ for $j = 1, 2, \ldots$. Thus, we can write the solution to 4.32 in the form of an MA(∞) model, i.e.,

$$X_t = Z_t + (\phi + \theta)\sum_{j=1}^{\infty} \phi^{j-1} Z_{t-j}. \tag{4.33}$$

This is a stationary unique process.

Now, suppose that $|\phi| > 1$. Then, by similar arguments as in the AR(1) model, it can be shown that

$$X_t = -\theta\phi^{-1} Z_t - (\phi + \theta)\sum_{j=1}^{\infty} \phi^{-j-1} Z_{t+j}.$$

Here too, we obtained a noncausal process which depends on future noise values, hence of no practical value.

If $|\phi| = 1$ then there is no stationary solution to 4.32.

While causality means that the process $\{X_t\}$ is expressible in terms of past values of $\{Z_t\}$, the dual property of invertibility means that the process $\{Z_t\}$ is expressible in the past values of $\{X_t\}$. Is ARMA(1,1) invertible?

ARMA(1,1) model is

$$\phi(B)X_t = \theta(B)Z_t$$

and so writing the solution for Z_t we have

$$Z_t = \frac{1}{\theta(B)}\phi(B)X_t = \frac{1}{1 + \theta B}(1 - \phi B)X_t. \tag{4.34}$$

Now, if $|\theta| < 1$ then the power series expansion of the function $f(x) = \frac{1}{1+\theta x}$ is

$$f(x) = \frac{1}{1+\theta x} = \sum_{j=0}^{\infty}(-\theta)^j x^j,$$

what in terms of the backshift operator B can be written as

$$\frac{1}{1+\theta B} = \sum_{j=0}^{\infty}(-\theta)^j B^j.$$

Applied to 4.34 it gives

$$\begin{aligned} Z_t &= \sum_{j=0}^{\infty}(-\theta)^j B^j (1-\phi B) X_t \\ &= X_t - (\phi+\theta)\sum_{j=1}^{\infty}(-\theta)^{j-1} X_{t-j}. \end{aligned}$$

The conclusion is that ARMA(1,1) is invertible if $|\theta| < 1$. Otherwise it is noninvertible.

The two properties, causality and invertibility, determine the admissible region for the values of parameters ϕ and θ, which is the square

$$\begin{aligned} &-1 < \phi < 1 \\ &-1 < \theta < 1. \end{aligned}$$

4.6.2 ACVF and ACF of ARMA(1,1)

The fact that we can express ARMA(1,1) as a linear process of the form

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j},$$

where Z_t is a white noise, is very helpful in deriving the ACVF and ACF of the process. By Corollary 4.1 we have

$$\gamma(\tau) = \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+\tau}.$$

For ARMA(1,1) the coefficients ψ_j are

$$\begin{aligned} &\psi_0 = 1 \\ &\psi_j = (\phi+\theta)\phi^{j-1} \quad \text{for } j = 1, 2, \ldots \end{aligned}$$

and we can easily derive expressions for $\gamma(0)$ and $\gamma(1)$.

$$\begin{aligned}\gamma(0) &= \sigma^2 \sum_{j=0}^{\infty} \psi_j^2 \\ &= \sigma^2 \left[1 + (\phi+\theta)^2 \sum_{j=1}^{\infty} \phi^{2(j-1)}\right] \\ &= \sigma^2 \left[1 + (\phi+\theta)^2 \sum_{j=0}^{\infty} \phi^{2j}\right] \\ &= \sigma^2 \left[1 + \frac{(\phi+\theta)^2}{1-\phi^2}\right].\end{aligned}$$

and

$$\begin{aligned}\gamma(1) &= \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+1} \\ &= \sigma^2 \left[1(\phi+\theta) + (\phi+\theta)(\phi+\theta)\phi + (\phi+\theta)\phi(\phi+\theta)\phi^2 + (\phi+\theta)\phi^2(\phi+\theta)\phi^3 + \ldots\right] \\ &= \sigma^2 \left[(\phi+\theta) + (\phi+\theta)^2\phi(1+\phi^2+\phi^4+\ldots)\right] \\ &= \sigma^2 \left[(\phi+\theta) + (\phi+\theta)^2\phi \sum_{j=0}^{\infty} \phi^{2j}\right] \\ &= \sigma^2 \left[(\phi+\theta) + \frac{(\phi+\theta)^2\phi}{1-\phi^2}\right]\end{aligned}$$

Similar derivations for $\tau \geq 2$ give

$$\gamma(\tau) = \phi^{\tau-1}\gamma(1). \tag{4.35}$$

Hence, we can calculate the autocorrelation function $\rho(\tau)$. For $\tau = 1$ we obtain

$$\rho(1) = \frac{\gamma(1)}{\gamma(0)} = \frac{(\phi+\theta)(1+\phi\theta)}{1+2\phi\theta+\theta^2} \tag{4.36}$$

and for $\tau \geq 2$ we have

$$\rho(\tau) = \phi^{\tau-1}\rho(1). \tag{4.37}$$

From these formulae we can see that when $\phi = -\theta$ the ACF $\rho(\tau) = 0$ for $\tau = 1, 2, \ldots$ and the process is just a white noise. Graph 4.14 shows the admissible region for the parameters ϕ and θ and indicates the regions when we have special cases of ARMA(1,1), which are white noise, AR(1) and MA(1).

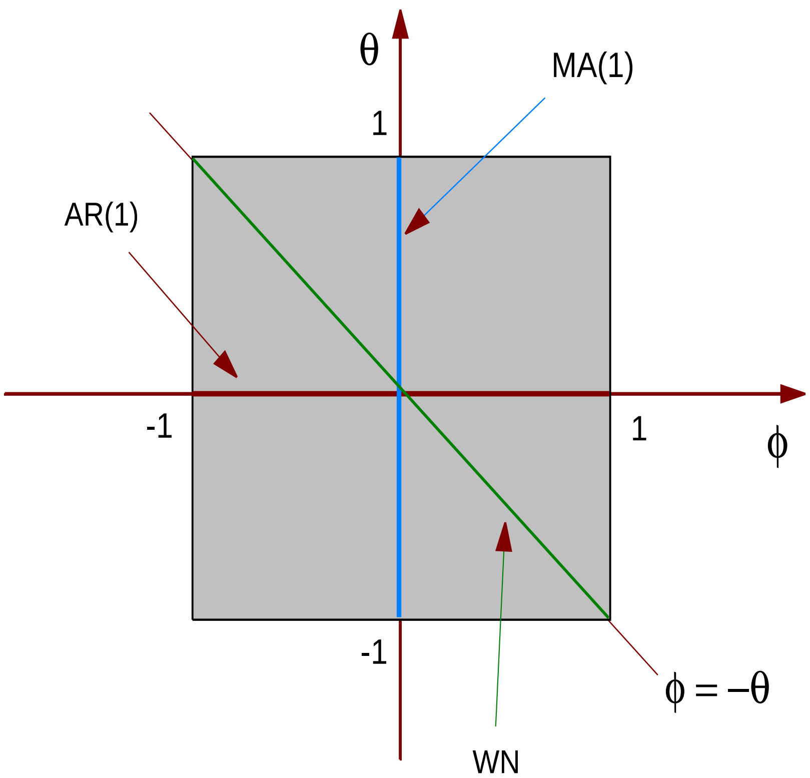

Figure 4.14: Admissible parameter region for ARMA(1,1)

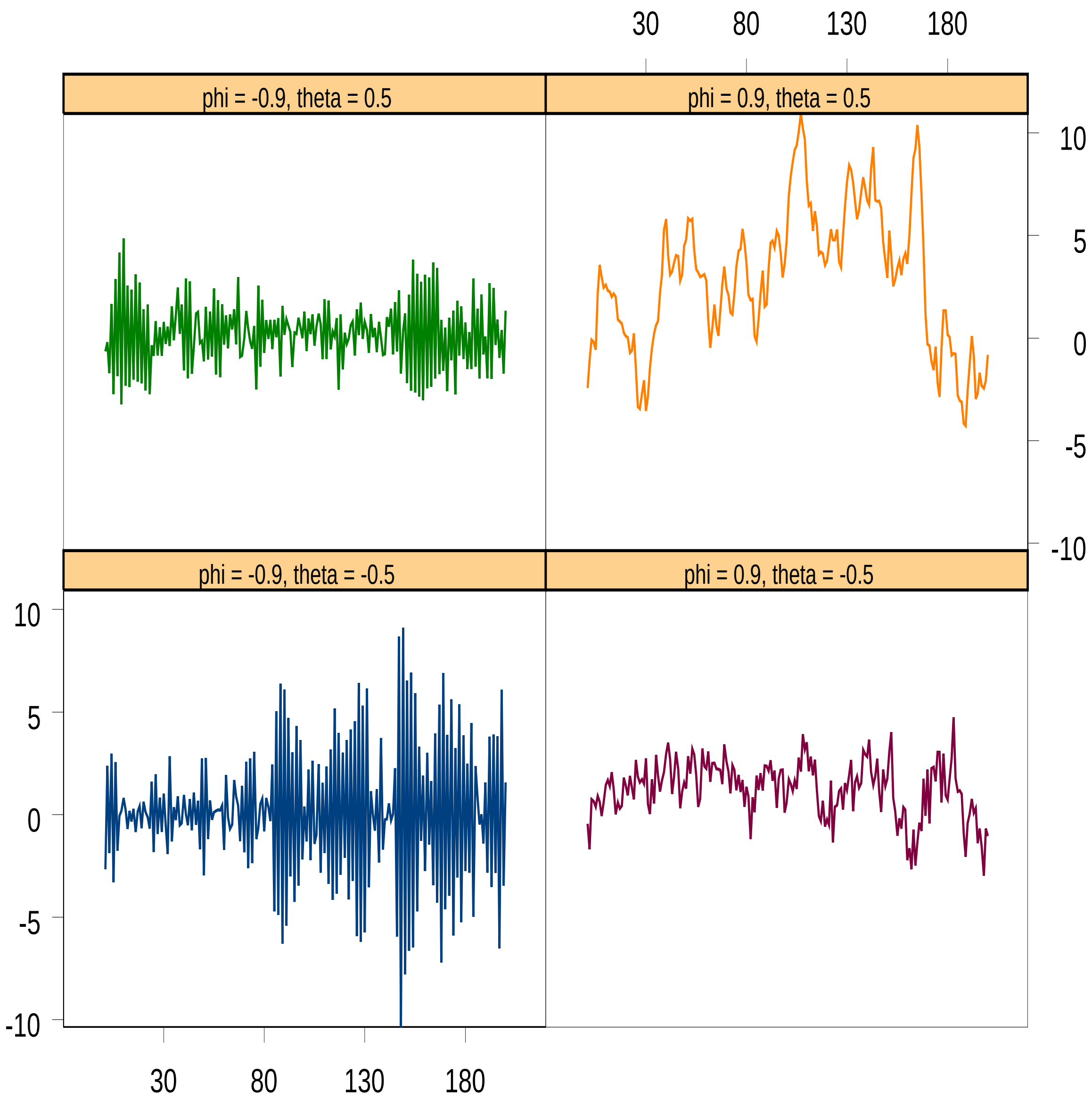

Figure 4.15: ARMA(1,1) for various values of the parameters ϕ and θ.

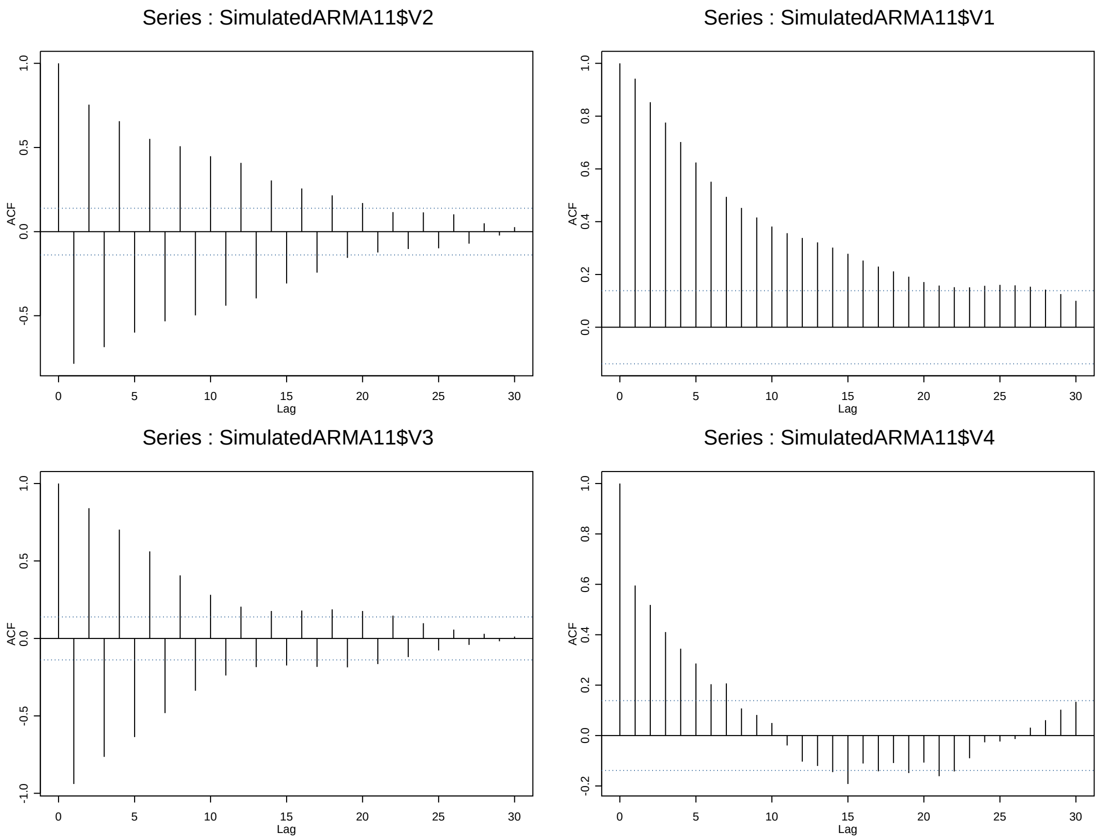

Figure 4.16: ACF of the ARMA(1,1) processes with the parameter values as in Figure 4.15, respectively.