Transfer of Siegel cusp forms of degree 2

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ABSTRACT. Let π be the automorphic representation of $\mathrm{GSp}_4(\mathbb{A})$ generated by a full level cuspidal Siegel eigenform that is not a Saito-Kurokawa lift, and τ be an arbitrary cuspidal, automorphic representation of $\mathrm{GL}_2(\mathbb{A})$. Using Furusawa's integral representation for $\mathrm{GSp}_4 \times \mathrm{GL}_2$ combined with a pullback formula involving the unitary group $\mathrm{GU}(3,3)$, we prove that the L-functions $L(s, \pi \times \tau)$ are "nice". The converse theorem of Cogdell and Piatetski-Shapiro then implies that such representations π have a functorial lifting to a cuspidal representation of $\mathrm{GL}_4(\mathbb{A})$. Combined with the exterior-square lifting of Kim, this also leads to a functorial lifting of π to a cuspidal representation of $\mathrm{GL}_5(\mathbb{A})$. As an application, we obtain analytic properties of various L-functions related to full level Siegel cusp forms. We also obtain special value results for $\mathrm{GSp}_4 \times \mathrm{GL}_1$ and $\mathrm{GSp}_4 \times \mathrm{GL}_2$.

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Introduction

We will start by giving some general background on Siegel modular forms and automorphic representations and then go on to explain the contents of this work.

Siegel modular forms

Classical elliptic modular forms, which are holomorphic functions on the complex upper half plane with certain transformation properties, can be generalized in various directions. One such generalization is the theory of *Siegel modular forms*, which includes the elliptic case as the *degree one* case. General references for Siegel modular forms are [17] and [43]. Just as in the elliptic case, Siegel modular forms come with a *weight* and a *level*. Parts of the theory generalize to the Siegel case in a straightforward way. For example, the space of Siegel modular forms of fixed weight and level is finite-dimensional. Siegel modular forms have Fourier expansions similar to that of elliptic modular forms. Also, there is a theory of Hecke operators, hence a notion of eigenform, and each Siegel modular form is a linear combination of eigenforms. The most interesting Siegel modular forms are the *cusp forms*, characterized by the vanishing of certain Fourier coefficients.

Beyond the elliptic case, much work has concentrated on Siegel modular forms of *degree two*. For example, Igusa wrote two famous papers [36] and [37] in the 1960's, where he determined the structure of the ring of Siegel modular forms of degree two with respect to the full modular group $\mathrm{Sp}_4(\mathbb{Z})$ (in analogy to the statement that the ring of modular forms with respect to $\mathrm{SL}_2(\mathbb{Z})$ is generated by the algebraically independent Eisenstein series of weight 4 and weight 6). Another milestone came about a decade later, when Andrianov [1] associated a degree-4 Euler product $L(s, F, \mathrm{spin})$, now known as the *spin L-function*, to a Siegel modular form F with respect to $\mathrm{Sp}_4(\mathbb{Z})$ (assumed to be an eigenform for all Hecke operators) and proved its basic analytic properties: meromorphic continuation, functional equation, and control over the possible poles.

A few years after this, Saito and Kurokawa [49] independently discovered the existence of degree two Siegel modular forms that violated the naive generalization of the Ramanujan conjecture: the statement that the roots of the Hecke polynomials in the denominator of the spin L-function of cusp forms have absolute value 1. In a series of papers [55], [56], [57], [2], [91] Maass, Andrianov and Zagier showed that such Siegel modular forms are precisely those that "come from" elliptic modular forms.[2] More precisely, there is a construction, now known as the *Saito-Kurokawa lifting*, which associates to an elliptic eigenform f for $\mathrm{SL}_2(\mathbb{Z})$ a Siegel eigenform F for $\mathrm{Sp}_4(\mathbb{Z})$, such that the spin L-function of F is the product of the Hecke L-function of f times two zeta factors. More

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[2]There exist cuspidal automorphic representations of $\mathrm{Sp}_4(\mathbb{A})$ (and of $\mathrm{GSp}_4(\mathbb{A})$) which violate the Ramanujan conjecture but are *not* lifts from elliptic modular forms, such as the examples constructed by Howe and Piatetski-Shapiro [34]. Such representations, however, cannot arise from holomorphic Siegel modular forms (of any level) with weight $k > 2$; see Corollary 4.5 of [67].

precisely, with appropriate normalization and disregarding archimedean factors,

$$L(s, F, \text{spin}) = L(s, f)\zeta(s + 1/2)\zeta(s - 1/2). \tag{1}$$

The zeta factors produce poles, and in fact Evdokimov [16] and Oda [62] have shown that the existence of a pole in the spin L-function characterizes Saito-Kurokawa liftings. The book [15] by Eichler and Zagier gives a streamlined account of the construction of Saito-Kurokawa lifts via the theory of Jacobi forms. If a holomorphic eigenform for $\text{Sp}_4(\mathbb{Z})$ is not Saito-Kurokawa, then in fact it satisfies the Ramanujan conjecture. This was proved by Weissauer [90].

Many questions that have been answered in the elliptic case remain open for higher degree Siegel modular forms. For example, there is as of yet no good theory of old- and newforms resembling the classical theory of Atkin and Lehner. Questions of level aside, even the case of Siegel modular forms with respect to the full modular group has many challenges remaining. For instance, it is not yet known if a Siegel modular form for $\text{Sp}_4(\mathbb{Z})$, assumed to be an eigenform for all Hecke operators, is determined by its Hecke eigenvalues (a statement known as *multiplicity one*). This has to do with the difficulty of relating Hecke eigenvalues and Fourier coefficients (unlike in the elliptic case, where the Hecke eigenvalues *are* the Fourier coefficients). Indeed, the Fourier coefficients of Siegel cusp forms of degree 2 are mysterious arithmetic objects, which are conjecturally related to central L-values of twisted spin L-functions; see [20] for a good discussion. The precise form of this relationship is known as *Böcherer's conjecture*, and one of us has observed [74] that a version of this conjecture implies multiplicity one.

Automorphic representations

It is well known that the theory of elliptic modular forms embeds into the more general theory of automorphic forms on the group $\text{GL}_2(\mathbb{A})$. Here, $\mathbb{A}$ denotes the ring of adeles of the number field $\mathbb{Q}$. The details of this process are explained in [7] and [22], and are roughly as follows. Given an elliptic cusp form f of weight k and level N that is an eigenfunction of the Hecke operators $T(p)$ for all but finitely many primes p, one may associate to f a complex-valued function Φ_f on $\text{GL}_2(\mathbb{A})$ satisfying certain invariance properties. In particular, Φ_f is left-invariant under the group of rational points $\text{GL}_2(\mathbb{Q})$, right invariant under a compact-open subgroup of the finite adeles depending on N, and transforms according to the character $e^{2\pi i k x}$ of the group $\text{SO}(2) \cong \mathbb{R}/\mathbb{Z}$ at the archimedean place. Let V be the space of functions spanned by right translates of Φ_f. Then V carries a representation of the group $\text{GL}_2(\mathbb{A})$. We denote this representation by π_f and call it the *automorphic representation attached to f*. Using the hypothesis that f is an eigenform, one can prove that π_f is irreducible. Like any irreducible representation of $\text{GL}_2(\mathbb{A})$, it factors as a restricted tensor product $\otimes_v \pi_v$, where π_v is an irreducible, admissible representation of the local group $\text{GL}_2(\mathbb{Q}_v)$. The product extends over all places v of $\mathbb{Q}$, and for $v = \infty$ we understand that $\mathbb{Q}_v = \mathbb{R}$. The original modular form f, or rather its adelic version Φ_f, can be recovered as a special vector in the representation π. In fact, if f is a newform, then Φ_f is a pure tensor $\otimes_v \phi_v$, where each ϕ_p is a *local newform* in π_p for finite p, and ϕ_∞ is a lowest weight vector in π_∞, a discrete series representation.

This procedure generalizes to Siegel modular forms F of degree n for the full modular group. The group GL_2 is to be replaced by the symplectic similitude group GSp_{2n}; see [3] for details. The resulting representation π_∞ of $\text{GSp}_{2n}(\mathbb{R})$ is a holomorphic discrete series representation with scalar minimal K-type determined by the weight of F. Unfortunately, for Siegel modular forms with level, the corresponding procedure is not quite as nice, due to the lack of a good theory of *local newforms* for representations of the group $\text{GSp}_{2n}(\mathbb{Q}_p)$, and our lack of knowledge of global multiplicity one.

However, see [71] for a treatment of adelization for Siegel cusp forms of arbitrary level that suffices for many applications.

Once a modular form is realized as a special vector in an automorphic representation, can the considerable machinery available for such representations be used to gain new insights into the classical theory? Sometimes, the answer is yes. For example, a method of Langlands, formulated for automorphic representations, was used in [3] to prove that the spin L-functions of Siegel modular forms of degree three have meromorphic continuation to the entire complex plane. As an example in the elliptic modular forms case, one might use the Langlands-Shahidi method to deduce analytic properties of several *symmetric power L-functions* $L(s, f, \mathrm{sym}^n)$ attached to an elliptic cuspidal eigenform f.

There is however a serious limiting factor to the applicability of automorphic methods to Siegel modular forms of higher degree. Namely, if F is an eigen-cuspform of degree $n > 1$, assumed to be of full level for simplicity, then the associated automorphic representation π_F of $\mathrm{GSp}_{2n}(\mathbb{A})$ is *non-generic* (meaning it has no Whittaker model). The obstruction comes from the archimedean place: If $\pi_F = \otimes_v \pi_v$, then the archimedean component π_∞, which is a holomorphic discrete series representation, is not generic. If π_F were generic, one could, for example, apply the Langlands-Shahidi method, and immediately obtain the analytic properties of a series of L-functions. Also, at least for the degree two case, questions of multiplicity one could be answered immediately; see [40].

Functoriality

Langlands' principle of functoriality, a central conjecture in the theory of automorphic representations, describes the relationships between automorphic objects living on two different algebraic groups. More precisely, let G and H be reductive, algebraic groups, which for simplicity we assume to be defined over $\mathbb{Q}$ and split. Attached to G and H are *dual groups* $\hat{G}$ and $\hat{H}$, which are complex reductive Lie groups whose root systems are dual to those of G and H, respectively. Then, according to the principle, every homomorphism of Lie groups $\hat{G} \to \hat{H}$ should give rise to a "lifting", or "transfer", of automorphic representations of $G(\mathbb{A})$ to automorphic representations of $H(\mathbb{A})$. For example, in [23], Gelbart and Jacquet proved the existence of the *symmetric square lifting* for $G = \mathrm{GL}_2$ to $H = \mathrm{GL}_3$. Here, $\hat{G} = \mathrm{GL}_2(\mathbb{C})$ and $\hat{H} = \mathrm{GL}_3(\mathbb{C})$, and $\hat{G} \to \hat{H}$ is an irreducible three-dimensional representation of $\hat{G}$. A more recent example, and one that we will use in this work, is Kim's exterior square lifting [41] from $G = \mathrm{GL}_4$ to $H = \mathrm{GL}_6$. Here, $\hat{G} = \mathrm{GL}_4(\mathbb{C})$, $\hat{H} = \mathrm{GL}_6(\mathbb{C})$, and $\hat{G} \to \hat{H}$ is the irreducible six-dimensional representation of $\hat{G}$ given as the exterior square of the four-dimensional standard representation.

What exactly qualifies as a "lifting" is often formulated in terms of the L-functions attached to automorphic representations. Let G be as above, and let $\pi = \otimes_v \pi_v$ be an automorphic representation of $G(\mathbb{A})$. As additional ingredient we need a finite-dimensional complex representation ρ of $\hat{G}$. Attached to this data is an Euler product

$$L(s, \pi, \rho) = \prod_v L(s, \pi_v, \rho),$$

where s is a complex parameter. We ignore for a moment the fact that in many situations the local factors $L(s, \pi_v, \rho)$ may not be known; at least for almost all primes the factors *are* known, and are of the form $Q(p^{-s})^{-1}$, where $Q(X)$ is a polynomial of degree equal to the dimension of ρ

and satisfying $Q(0) = 1$. It is also known that the product converges in some right half plane. If ρ is a natural "standard" representation, it is often omitted from the notation. Now if H is a second group, and if $\varphi : \hat{G} \to \hat{H}$ is a homomorphism of Lie groups, then the associated "lifting" maps automorphic representations $\pi = \otimes \pi_v$ of $G(\mathbb{A})$ to automorphic representations $\Pi = \otimes \Pi_v$ of $H(\mathbb{A})$ in such a way that

$$L(s, \Pi, \rho) = L(s, \pi, \rho \circ \varphi) \tag{2}$$

for all finite-dimensional representations ρ of $\hat{H}$. Sometimes one can only prove that the Euler products coincide for almost all primes, in which case one may speak of a *weak lifting*. For example, in [41], Kim proved the equality of the relevant Euler products for the exterior square lifting at all primes outside 2 and 3. Later, Henniart [33] showed the equality for the remaining primes, proving that Kim's lifting is in fact *strong*.

It seems worthwhile to emphasize here that each instance of lifting discovered so far has had numerous applications to number theory. Functoriality is a magic wand that forces additional constraints on the automorphic representations being lifted and allows us to prove various desirable local and global properties for them. To give just one example, Kim's *symmetric fourth lifting* [41] from GL_2 to GL_5 has provided the best known bound towards the Ramanujan conjecture for cuspidal automorphic representations of GL_2. By using these bounds for the case of GL_2 over a totally real field, Cogdell, Piatetski-Shapiro and Sarnak [13] were able to confirm the last remaining case of Hilbert's eleventh problem.

We mention that the Saito-Kurokawa lifting also fits into the framework of Langlands functoriality. Recall that this lifting maps elliptic modular forms to Siegel modular forms of degree 2, so one would expect the relevant groups to be $G = GL_2$ and $H = GSp_4$, or rather, since all representations involved have trivial central character, the projective groups $G = PGL_2$ and $H = PGSp_4$. But in fact, one should really take $G = PGL_2 \times PGL_2$; see [51] and [78]. The associated dual groups are $\hat{G} = SL_2(\mathbb{C}) \times SL_2(\mathbb{C})$ and $\hat{H} = Sp_4(\mathbb{C})$, and the homomorphism of dual groups is given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} \longmapsto \begin{bmatrix} a & & b & \\ & a' & & b' \\ c & & d & \\ & c' & & d' \end{bmatrix}. \tag{3}$$

The first factor PGL_2 carries the representation π_f associated to an elliptic eigenform f. The second, "hidden" factor PGL_2 carries an *anomalous* representation π_{an}, which is a certain constituent of a parabolically induced representation. It is clear from (3) that the lifting Π of the pair (π_f, π_{an}) has the property $L(s, \Pi) = L(s, \pi)L(s, \pi_{an})$. Looking only at finite places, this identity is precisely the equality (1). Hence, it is the presence of the anomalous representation π_{an} that accounts for the pole in the L-function, and the violation of the Ramanujan conjecture. Inside Π one can find (the adelization of) the Siegel modular form F that is the Saito-Kurokawa lifting of f.

The transfer of Siegel modular forms to GL_4 and GL_5

Again we consider Siegel modular forms of degree 2, restricting our attention to cusp forms and full level. If F is an eigenform, we can attach to it a cuspidal, automorphic representation $\pi_F = \otimes_v \pi_v$ of $GSp_4(\mathbb{A})$. Now, the dual group of GSp_4 is $GSp_4(\mathbb{C})$, which sits inside $GL_4(\mathbb{C})$. Interpreting the latter as the dual group of GL_4, we see that the principle of functoriality predicts the existence of a lifting from GSp_4 to GL_4. In particular, we should be able to lift our modular form F (or

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rather π_F) to an automorphic representation Π of $\mathrm{GL}_4(\mathbb{A})$. If F is of Saito-Kurokawa type, it is obvious, but not very exciting, how to construct the lifting; the result will be a representation globally induced from the $(2,2)$-parabolic of $\mathrm{GL}_4(\mathbb{A})$. In particular, the lifting is not cuspidal. It is much more intricate to lift non-Saito-Kurokawa forms; this is, in fact, the main theme of this work.

Let us move our focus away from π_F and consider for a moment all cuspidal representations of $\mathrm{GSp}_4(\mathbb{A})$. What is the current status of the lifting from GSp_4 to GL_4 predicted by Langlands functoriality? In [4] Asgari and Shahidi have achieved the lifting for all *generic* cuspidal, automorphic representations of $\mathrm{GSp}_4(\mathbb{A})$. The reason for the restriction to generic representations lies in their use of the Langlands-Shahidi method. As emphasized already, Siegel cusp forms correspond to non-generic representations; so this method cannot be used to lift them.

Another commonly used tool to prove functoriality is the trace formula. Trace formula methods have the potential to prove the existence of liftings for *all* automorphic representations. This method has been much developed by Arthur, but for specific situations is still subject to various versions of the *fundamental lemma*. At the time of this writing it is unclear to the authors whether all the necessary ingredients for the lifting from GSp_4 to GL_4 are in place. Certainly, a complete proof is not yet published.

In this work we use the Converse Theorem to prove that full-level Siegel cusp forms of degree two can be lifted to GL_4. To the best of our knowledge, the Converse Theorem has not been used before to prove functorial transfer for a non-generic representation on a quasi-split group. Given F and π_F as above, it is easy enough to predict what the lifting $\Pi = \otimes \Pi_v$ to GL_4 should be. In fact, we can *define* Π_v, which is an irreducible, admissible representation of $\mathrm{GL}_4(\mathbb{Q}_v)$, for all places v in such a way that the required lifting condition (2) is automatically satisfied. The only question is then: Is the representation Π of $\mathrm{GL}_4(\mathbb{A})$ thus defined automorphic? This is the kind of question the Converse Theorem is designed to answer. According to the version of the Converse Theorem given in [11], the answer is affirmative if the twisted (Rankin-Selberg) L-functions $L(s, \Pi \times \tau)$ are "nice" for *all* automorphic representations τ of $\mathrm{GL}_2(\mathbb{A})$, or alternatively, for all *cuspidal* automorphic representations of $\mathrm{GL}_2(\mathbb{A})$ and $\mathrm{GL}_1(\mathbb{A})$. The GL_1 twists are not a serious problem, so our main task will be to prove niceness for the GL_2 twists. Recall that "nice" means the L-functions can be analytically continued to entire functions, satisfy a functional equation, and are bounded in vertical strips.

Once we establish the transfer of π_F to GL_4, we will go one step further and lift π_F to the group GL_5 as well. Recall that π_F is really a representation of the projective group $\mathrm{PGSp}_4(\mathbb{A})$, the dual group of which is $\mathrm{Sp}_4(\mathbb{C})$. The lifting to GL_4 comes from the inclusion $\mathrm{Sp}_4(\mathbb{C}) \to \mathrm{GL}_4(\mathbb{C})$, or, in other words, the natural representation ρ_4 of $\mathrm{Sp}_4(\mathbb{C})$ on $\mathbb{C}^4$. The "next" irreducible representation of $\mathrm{Sp}_4(\mathbb{C})$ is five-dimensional, and we denote it by ρ_5. Interpreting ρ_5 as a homomorphism of dual groups $\mathrm{Sp}_4(\mathbb{C}) \to \mathrm{GL}_5(\mathbb{C})$, the principle of functoriality predicts the existence of a lifting from PGSp_4 to GL_5. Using Kim's exterior square lifting [41], one can in fact show that the transfer of π_F to GL_5 exists. To summarize, we will prove the following lifting theorem.

Theorem A: *Let F be a cuspidal Siegel modular form of degree 2 with respect to $\mathrm{Sp}_4(\mathbb{Z})$. Assume that F is an eigenform for all Hecke operators, and not of Saito-Kurokawa type. Let π_F be the associated cuspidal, automorphic representation of $\mathrm{GSp}_4(\mathbb{A})$. Then π_F admits a strong lifting to an automorphic representation Π_4 of $\mathrm{GL}_4(\mathbb{A})$, and a strong lifting to an automorphic representation Π_5 of $\mathrm{GL}_5(\mathbb{A})$. Both Π_4 and Π_5 are cuspidal.*

For more precise statements of these results, see Theorem 5.1.2 and Theorem 5.1.5.

Bessel models

We have yet to explain how to prove "niceness" for the L-functions relevant for the Converse Theorem. Before doing so, let us digress and explain the important notion of *Bessel model* for representations of GSp_4. These models can serve as a substitute for the often missing Whittaker models. We start by explaining the local, non-archimedean notion of Bessel model. Thus, let F be a p-adic field. We fix a non-trivial character ψ of F. Recall that the *Siegel parabolic subgroup* of GSp_4 is the standard maximal parabolic subgroup whose radical U is abelian. Let S be a non-degenerate, symmetric matrix with coefficients in F. Then S defines a character θ of $U(F)$ via

$$\theta(\begin{bmatrix} 1 & X \\ & 1 \end{bmatrix}) = \psi(\mathrm{tr}(SX)).$$

Let T be the identity component of the subgroup of the Levi component of the Siegel parabolic fixing θ. Hence, the elements t of $T(F)$ satisfy $\theta(tut^{-1}) = \theta(u)$ for all $u \in U(F)$. The group T turns out to be abelian. In fact, it is a two-dimensional torus which is split exactly if $-\det(S)$ is a square in $F^\times$. The semidirect product $R = TU$ is called the *Bessel subgroup* of GSp_4 with respect to θ. Every character Λ of $T(F)$ gives rise to a character $\Lambda \otimes \theta$ of $R(F)$ via $(\Lambda \otimes \theta)(tu) = \Lambda(t)\theta(u)$.

Now, let (π, V) be an irreducible, admissible representation of $\mathrm{GSp}_4(F)$. Let θ and Λ be as above. We consider functionals $\beta : V \to \mathbb{C}$ with the property

$$\beta(\pi(tu)v) = \Lambda(t)\theta(u)\beta(v)$$

for $t \in T(F)$, $u \in U(F)$ and $v \in V$. A non-zero such functional is called a (Λ, θ)-*Bessel functional* for π. It is known that, for fixed θ and Λ, there can be at most one such functional up to multiples; see [61]. It is also known that, unless π is one-dimensional, there always exists a Bessel functional for *some* choice of θ and Λ. If θ and Λ are such that a (Λ, θ)-Bessel functional for π exists, then π can be realized as a subspace of the space of functions $B : \mathrm{GSp}_4(F) \to \mathbb{C}$ with the transformation property

$$B(tuh) = \Lambda(t)\theta(u)B(h) \qquad \text{for all } t \in T(F),\ u \in U(F),\ h \in \mathrm{GSp}_4(F), \tag{4}$$

with the action of $\mathrm{GSp}_4(F)$ on this space given by right translation. Sugano [83] has determined the explicit formula in the above realization for the spherical vector in an unramified representation π.

Similar definitions can be made, and similar statements hold, in the archimedean context. See [65] for explicit formulas for Bessel models for a class of lowest weight representations of $\mathrm{GSp}_4(\mathbb{R})$. All we will need in this work are formulas for holomorphic discrete series representations with scalar minimal K-type. These have already been determined in [83].

Next, consider global Bessel models. Given S as above but with entries in $\mathbb{Q}$, we obtain a character θ of $U(\mathbb{A})$ via a fixed non-trivial character ψ of $\mathbb{Q}\backslash\mathbb{A}$. The resulting torus T can be adelized, and is then isomorphic to the group of ideles $\mathbb{A}_L^\times$ of a quadratic extension L of $\mathbb{Q}$. We assume that $-\det(S)$ is not a square in $\mathbb{Q}^\times$, so that L is a field (and not isomorphic to $\mathbb{Q} \oplus \mathbb{Q}$). Let $\pi = \otimes\pi_v$ be a cuspidal, automorphic representation of $\mathrm{GSp}_4(\mathbb{A})$, and let V be the space of automorphic forms realizing π. Assume that a Hecke character Λ of $\mathbb{A}_L^\times$ is chosen such that the restriction of Λ to $\mathbb{A}^\times$ coincides with the central character of π. For each $\phi \in V$ consider the corresponding Bessel

function

$$B_\phi(g) = \int\limits_{Z_H(\mathbb{A})R(\mathbb{Q})\backslash R(\mathbb{A})} (\Lambda \otimes \theta)(r)^{-1}\phi(rg)\,dr, \tag{5}$$

where Z_H is the center of $H := \mathrm{GSp}_4$. If these integrals are non-zero, then we obtain a model of π consisting of functions on $\mathrm{GSp}_4(\mathbb{A})$ with a left transformation property analogous to (4). In this case, we say that π has a *global Bessel model* of type (S, Λ, ψ). It implies that the corresponding *local* Bessel models exist for all places v of $\mathbb{Q}$. The local uniqueness of Bessel models implies global uniqueness. However, if we are given $\pi = \otimes\pi_v$ and some triple (S, Λ, ψ) such that π_v has a local Bessel model of type (S_v, Λ_v, ψ_v) for each place v, then it does *not* necessarily imply that π has a global Bessel model of type (S, Λ, ψ). Indeed, conjecturally, a global Bessel model exists in the above case if and only if a certain central L-value is non-vanishing; see [68] for a discussion of this point.

So, what can be said about the *existence* of global Bessel models for those automorphic representations coming from Siegel modular forms? Assume that $\pi_F = \otimes\pi_v$ is attached to a full level Siegel cusp form, as above. Then π_p, for each prime p, is a spherical representation. Such representations admit Bessel models for which the character Λ_p is unramified. We would like our global Bessel data to be as unramified as possible. So the question arises, can we find a global Bessel model for which the Hecke character Λ is unramified *everywhere*? Not only that, we would like L to be an *imaginary* quadratic extension, and would like the archimedean component of Λ, which is a character of $\mathbb{C}^\times$, to be trivial. The existence of such a Λ turns out to be related to the non-vanishing of certain Fourier coefficients of F; see Lemma 5.1.1 for a precise statement. Using analytic methods and half-integral weight modular forms, the second author has recently proved [75] that the required non-vanishing condition is always satisfied. Hence, a particularly nice global Bessel model always exists for π_F. This removes assumption (0.1) of [19], and will make our results hold unconditionally for all cuspidal Siegel eigenforms of full level.

Furusawa's integrals

We now return from the world of Bessel models to the problem of proving "niceness" for the L-functions relevant for the Converse Theorem. Recall that, in order to apply the Converse Theorem for GL_4, the essential task is to control the Rankin-Selberg L-functions $L(s, \Pi \times \tau)$, where Π is the predicted transfer to $\mathrm{GL}_4(\mathbb{A})$, and τ is an arbitrary cuspidal, automorphic representation of $\mathrm{GL}_2(\mathbb{A})$. By the very definition of Π, this L-function equals $L(s, \pi \times \tau)$, where $\pi = \pi_F$ is the cuspidal representation of $\mathrm{GSp}_4(\mathbb{A})$ attached to F. For this type of Rankin-Selberg product, Furusawa [19] has pioneered an *integral representation*, which we now explain. This integral representation involves an Eisenstein series on a unitary similitude group $\mathrm{GU}(2, 2)$. Unitary groups are defined with respect to a quadratic extension L. Here, *the appropriate quadratic field extension $L/\mathbb{Q}$ is the one coming from a global Bessel model for π.* Hence, given the cusp form F, we first find a particularly good Bessel model for π, with the Hecke character $\Lambda = \otimes\Lambda_v$ unramified everywhere, as explained above. The quadratic extension is then $L = \mathbb{Q}(\sqrt{-\det(S)})$. For the precise definition of $\mathrm{GU}(2, 2)$ see (19). Note that this group contains GSp_4, which we henceforth abbreviate by H.

Let us next explain the Eisenstein series $E(h, s; f)$ appearing in Furusawa's integral representation. Eisenstein series come from sections in global parabolically induced representations. The relevant parabolic P of $\mathrm{GU}(2, 2)$ is the *Klingen parabolic subgroup*, i.e., the maximal parabolic subgroup with non-abelian radical. There is a natural map from $\mathbb{A}_L^\times \times \mathbb{A}_L^\times \times \mathrm{GL}_2(\mathbb{A})$ to the adelized Levi

component of P. Therefore, via suitably chosen Hecke characters χ_0 and χ of $\mathbb{A}_L^\times$, the $\mathrm{GL}_2(\mathbb{A})$ cuspidal representation τ can be extented to a representation of the Levi component of $P(\mathbb{A})$. Parabolic induction to all of $\mathrm{GU}(2,2)(\mathbb{A})$ then yields representations $I(s, \chi, \chi_0, \tau)$, where s is a complex parameter (see Sect. 1.1 for details). The Eisenstein series is constructed from an analytic section f in this family of induced representations via a familiar summation process; see (141). By general theory, $E(h, s; f)$ is convergent in some right half plane, has meromorphic continuation to all of $\mathbb{C}$ and satisfies a functional equation.

With the Eisenstein series in place, Furusawa considers integrals of the form

$$Z(s, f, \phi) = \int\limits_{H(\mathbb{Q}) Z_H(\mathbb{A}) \backslash H(\mathbb{A})} E(h, s; f) \phi(h) \, dh. \tag{6}$$

Here, as before, Z_H denotes the center of $H = \mathrm{GSp}_4$. The function ϕ is a vector in the space of automorphic forms realizing π. Furusawa's "basic identity" (here equation (143)) shows that, if all the data are factorizable, then the integral $Z(s, f, \phi)$ is Eulerian, i.e., it factors into a product of local zeta integrals. More precisely, assume that $f = \otimes f_v$, with f_v an analytic section in the local induced representation $I(s, \chi_v, \chi_{0,v}, \tau_v)$. Assume also that $\phi = \otimes \phi_v$, a pure tensor in $\pi = \otimes \pi_v$. Then the local zeta integrals are of the form

$$Z(s, W_{f_v}, B_{\phi_v}) = \int\limits_{R(\mathbb{Q}_v) \backslash H(\mathbb{Q}_v)} W_{f_v}(\eta h, s) B_{\phi_v}(h) \, dh. \tag{7}$$

Here, η is a certain element in $\mathrm{GU}(2,2)(\mathbb{Q}_v)$ defined in (16). The function W_{f_v} is a Whittaker-type function depending on f_v, and B_{ϕ_v} is the vector corresponding to ϕ_v in the local (Λ_v, θ_v)-Bessel model of π_v. We see how important it is that all the local Bessel models exist. We also see the usefulness of explicit formulas for B_{ϕ_v} in order to be able to evaluate the integrals (7).

Furusawa has calculated the local integrals (7) in the non-archimedean case when all the local data is unramified. The result is

$$Z(s, W_{f_p}, B_{\phi_p}) = \frac{L(3s + \frac{1}{2}, \tilde{\pi}_p \times \tilde{\tau}_p)}{L(6s + 1, \chi_p|_{\mathbb{Q}_p^\times}) L(3s + 1, \tau_p \times \mathcal{AI}(\Lambda_p) \times \chi_p|_{\mathbb{Q}_p^\times})}, \tag{8}$$

where $\tilde{\pi}$ and $\tilde{\tau}$ are the contragredient representations, and where χ_p and Λ_p are the local components of the Hecke characters χ and Λ mentioned above. The symbol $\mathcal{AI}$ denotes automorphic induction; thus, the second L-factor in the denominator is a factor for $\mathrm{GL}_2 \times \mathrm{GL}_2$. By taking the product of (8) over all unramified places, it follows that the quantity $Z(s, f, \phi)$ given by the integral (6) is essentially equal to the global L-function $L(3s + \frac{1}{2}, \tilde{\pi} \times \tilde{\tau})$ divided by some well-understood global L-functions for GL_1 and $\mathrm{GL}_2 \times \mathrm{GL}_2$ (here, "essentially" means that we ignore a finite number of local factors corresponding to the ramified places). Consequently, if we can control the local factors at these bad (ramified) places, the integral (6) can be used to study $L(s, \pi \times \tau)$. In the end, $L(s, \pi \times \tau)$ will inherit analytic properties, like meromorphic continuation and functional equation, from the Eisenstein series appearing in (6). This is the essence of the method of integral representations.

The art of choosing distinguished vectors in local representations

Recall that the identity (8) for the local zeta integrals holds only if *all* the local ingredients are unramified, including π_p, τ_p, Λ_p and χ_p. The representations π_p are always unramified since the

modular form F has full level. The character Λ_p is also unramified by our choice of Bessel model. But, in order to apply the Converse Theorem, we need to be able to twist by arbitrary GL_2 representations, meaning that τ_p could be any irreducible, admissible, infinite-dimensional representation of $\mathrm{GL}_2(\mathbb{Q}_p)$.

There is a natural choice for the function ϕ appearing in (6), namely, the adelization of the modular form F. The local vectors ϕ_p are then unramified at each finite place, and a lowest weight vector at the archimedean place. By the discussion at the end of the previous subsection, the quantity $Z(s, f, \phi)$ is equal to a ratio of global L-functions up to a finite number of factors coming from the bad places. We need to be able to explicitly evaluate these bad factors, and in particular, make sure that the local zeta integrals are all non-zero. This is where the correct choice of local data entering the zeta integrals becomes very important. In short, for each place v where τ_v is not an unramified principal series, we have to make a choice of local section f_v defining the Eisenstein series, and a wrong choice of f_v may lead to integrals that are not computable, or worse, that are zero. There are in fact two important requirements that f_v must satisfy:

i) f_v must be such that the local zeta integral $Z(s, W_{f_v}, B_{\phi_v})$ is non-zero and explicitly computable.

ii) f_v should be uniquely characterized by *right transformation properties.*

This second requirement is important in view of the calculation of *local intertwining operators*, which are essential for obtaining the functional equation of $L(s, \pi \times \tau)$. The local intertwining operators map each f_v to a vector in a similar parabolically induced representation via an explicit integral. This integral involves left transformations of f_v, and hence preserves all right transformation properties. If f_v is indeed characterized by its properties on the right, we know a priori that the result of applying an intertwining operator is the function analogous to f_v. By uniqueness, the intertwining operator can then be calculated by evaluating at a single point.

For a finite prime p, it turns out that the local induced representations admit a *local newform theory*. This will be the topic of Sect. 1.2. In particular, there is a distinguished vector in $I(s, \chi_p, \chi_{0,p}, \tau_p)$, unique up to multiples and characterized by being invariant under a certain congruence subgroup. Suitably normalized, this vector is a good and natural choice for f_p.

The choice of f_v for $v = \infty$ is rather intricate and is the topic of Sect. 1.3. It comes down to finding a suitable function on $\mathrm{GU}(2,2)(\mathbb{C})$ with certain transformation properties on the left and on the right. Moreover, one has to assure that this function is K-finite, where K is the maximal compact subgroup of $\mathrm{GU}(2,2)(\mathbb{C})$. We will cook up an appropriate function as a certain polynomial in matrix coefficients; see Proposition 1.3.4.

Having defined all the local sections in this way, it is then possible to calculate the local zeta integrals at all places. The result is a formula similar to (8), namely

$$Z(s, W_{f_v}, B_{\phi_v}) = \frac{L(3s + \frac{1}{2}, \tilde{\pi}_v \times \tilde{\tau}_v)}{L(6s + 1, \chi_v|_{\mathbb{Q}_v^\times}) L(3s + 1, \tau_v \times \mathcal{AI}(\Lambda_v) \times \chi_v|_{\mathbb{Q}_v^\times})} Y_v(s), \tag{9}$$

with an explicitly given correction factor $Y_v(s)$. The details are given in Theorem 2.2.1 for the non-archimedean case and Corollary 2.2.3 for the archimedean case.

We would like to remark that the kind of careful selection of distinguished local vectors as described above is quite typical when one wants to precisely understand automorphic representations at highly ramified places. We will have to play a similar game again later, when we prove the pullback formula.

The global integral representation

Let us summarize what we have so far. We started with a cuspidal Siegel eigenform F for $\mathrm{Sp}_4(\mathbb{Z})$. Its adelization ϕ generates an irreducible, cuspidal, automorphic representation $\pi = \pi_F$ of $\mathrm{GSp}_4(\mathbb{A})$. Using a non-vanishing theorem for the Fourier coefficients of F, we can find a particularly nice global Bessel model for π. The involved quadratic extension $L/\mathbb{Q}$ gives rise to a unitary group $\mathrm{GU}(2,2)$. For $\tau = \otimes\tau_p$ an arbitrary cuspidal, automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ and some auxiliary characters χ and χ_0, we consider the representation $I(s, \chi, \chi_0, \tau)$ induced from the Klingen parabolic subgroup of $\mathrm{GU}(2,2)$. It is possible to choose sections f_v in the local representations $I(s, \chi_v, \chi_{0,v}, \tau_v)$, for *all* places v, so that the identity (9) holds with an explicit factor $Y_v(s)$. Via Furusawa's "basic identity", the product of all the local zeta integrals equals the integral $Z(s, f, \phi)$ in (6). Hence, we obtain the *global integral representation*

$$Z(s, f, \phi) = \frac{L(3s + \frac{1}{2}, \tilde{\pi} \times \tilde{\tau})}{L(6s + 1, \chi|_{\mathbb{A}^\times})L(3s + 1, \tau \times \mathcal{AI}(\Lambda) \times \chi|_{\mathbb{A}^\times})} Y(s), \tag{10}$$

with an explicitly known function $Y(s)$. At this stage we obtain our first result about $L(s, \pi \times \tau)$, namely, that this L-function has meromorphic continuation to all of $\mathbb{C}$. This is because the same is true for the Eisenstein series appearing in $Z(s, f, \phi)$, and for the other functions in (10) as well.

The integral representation (10) may also be used to prove the expected functional equation satisfied by $L(s, \pi \times \tau)$. Since the functional equations for the other L-functions in (10) are known, all one needs is the functional equation of the Eisenstein series $E(h, s; f)$. This in turn comes down to a calculation of local intertwining operators, which we carry out in Sects. 1.4 (non-archimedean case) and 1.5 (archimedean case). As already mentioned, the characterization of our local sections by right transformation properties means that the intertwining operators need to be evaluated only at one specific point. We caution however that this evaluation is *very difficult*, and our description in Sects. 1.4 and 1.5 is essentially an overview that hides the actual length of the calculations involved. The determination of the functional equation, given the results of the intertwining operator calculations, is carried out in Sect. 2.4. The result is exactly as it should be:

Theorem B: *The L-function $L(s, \pi \times \tau)$ has meromorphic continuation to all of $\mathbb{C}$ and satisfies the functional equation*

$$L(s, \pi \times \tau) = \varepsilon(s, \pi \times \tau)L(1 - s, \tilde{\pi} \times \tilde{\tau}), \tag{11}$$

where $\varepsilon(s, \pi \times \tau)$ is the global ε-factor attached to the representation $\pi \times \tau$ via the local Langlands correspondence at every place.

In Theorem 2.4.3 this result is actually obtained under a mild hypothesis on the ramification of τ, which however will be removed later in Theorem 5.2.2.

The pullback formula

As mentioned earlier, we need to prove that the L-functions $L(s, \pi \times \tau)$ are nice, i.e., they can be analytically continued to entire functions, satisfy a functional equation, and are bounded in

vertical strips. So far, using the global integral representation (10), we have proved meromorphic continuation and functional equation for $L(s, \pi \times \tau)$. It turns out that boundedness in vertical strips follows from a general theorem of Gelbart and Lapid [24] once entireness is known. So it all boils down to showing that $L(s, \pi \times \tau)$ has no poles anywhere in the complex plane. Unfortunately, the global integral representation (10) cannot be directly used to control the poles of this L-function. The reason is that the analytic properties of the Klingen-type Eisenstein series $E(h, s; f)$ are not understood to the required extent.

To control the poles, we will prove a *pullback formula* and express our Eisenstein series $E(h, s; f)$ as an integral of a GL_2 automorphic form against a restricted Eisenstein series on a larger group $\mathrm{U}(3, 3)$. Let us briefly describe the general philosophy behind pullback formulas. Let G_1, G_2 and G_3 be semisimple groups such that there is an embedding $G_1 \times G_2 \to G_3$. Suppose that we want to understand a complicated Eisenstein series $E(g_2, s; f)$ on G_2 for which the inducing data f essentially comes from an automorphic representation σ on G_1. Then, one can often find a simpler (degenerate) Eisenstein series $E(g, s; \Upsilon)$ on the larger group G_3 such that there is a precise formula of the form

$$\int_{G_1(\mathbb{Q}) \backslash G_1(\mathbb{A})} E((g_1, g_2), s; \Upsilon) \Psi(g_1) dg_1 = T(s) E(g_2, s; f) \tag{12}$$

where Ψ is a suitable vector in the space of σ and $T(s)$ is an explicitly determined correction factor.

Pullback formulas have a long history. Garrett [21] used pullback formulas for Eisenstein series on symplectic groups to study the triple product L-function, as well as to establish the algebraicity of certain ratios of inner products of Siegel modular forms. Pullback formulas for Eisenstein series on unitary groups were first proved in a classical setting by Shimura [82]. Unfortunately, Shimura only considers certain special types of Eisenstein series in his work, which do not include ours except in the very specific case when the local data is unramified everywhere.
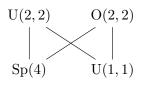
In Theorem 3.5.1 we prove a pullback formula in the form (12) when $G_i = \mathrm{U}(i, i)$ (for $i = 1, 2, 3$), σ is essentially the representation $\chi_0 \times \tau$ and $E(g_2, s; f)$ is (the restriction from $\mathrm{GU}(2, 2)$ to $\mathrm{U}(2, 2)$ of) the Eisenstein series involved in (6). This results in a *second global integral representation* for $L(s, \pi \times \tau)$ involving $E((g_1, g_2), s; \Upsilon)$; see Theorem 3.6.1. Since $E((g_1, g_2), s; \Upsilon)$ is a degenerate Siegel type Eisenstein series on $\mathrm{U}(3, 3)$, its analytic properties are better understood. Indeed, by the work of Tan [85], we deduce that $L(s, \pi \times \tau)$ has at most one possible pole in $\mathrm{Re}(s) \geq 1/2$, namely at the point $s = 1$ (Proposition 4.1.4). The proof of holomorphy at this point requires additional arguments.

We have not yet discussed how one goes about proving a formula like (12). There are two main ingredients involved. The first ingredient is combinatorial and involves the computation of a certain double coset space. In our case, this has already been done by Shimura [82]; see the proof of Theorem 3.5.1. The second ingredient is local and involves a careful choice of vectors in local representations. Indeed, the double coset computation reduces the task of proving the pullback formula to making a delicate choice for the local sections Υ_v at all archimedean and non-archimedean places, and then proving certain identities ("local pullback formulas") involving local zeta integrals. See Sect. 3.1 for the definition of local sections in the non-archimedean case(s) and Sect. 3.3 for the definition in the archimedean case. The local zeta integrals are calculated in Sect. 3.2 (non-archimedean case) and Sect. 3.4 (archimedean case).

The Siegel-Weil formula, entireness, functoriality

We have so far proved that $L(s, \pi \times \tau)$ has only one possible pole in $\mathrm{Re}(s) \geq 1/2$, namely at the point $s = 1$. In order to prove the holomorphy at this point, it suffices to show (because of the second integral representation) that the residue of the $\mathrm{U}(3,3)$ Eisenstein series $E((g_1, g_2), s; \Upsilon)$ at a relevant point s_0, when integrated against the adelization ϕ of our Siegel cusp form F, vanishes.

To do this, we employ the regularized Siegel-Weil formula for $\mathrm{U}(n, n)$ due to Ichino [35], which asserts that this residue of $E((g_1, g_2), s; \Upsilon)$ at s_0 is equal to a regularized theta integral. Consequently, if $L(s, \pi \times \tau)$ has a pole, then the integral of the (adelized) Siegel modular form F against a regularized theta integral is non-zero (Proposition 4.3.2). An argument using the *seesaw diagram*

$$
\begin{array}{ccc}
\mathrm{U}(2,2) & & \mathrm{O}(2,2) \\
| & \times & | \\
\mathrm{Sp}(4) & & \mathrm{U}(1,1)
\end{array}
$$

then shows that π_1, the cuspidal, automorphic representation of $\mathrm{Sp}_4(\mathbb{A})$ generated by F, participates in the theta correspondence with a split orthogonal group $\mathrm{O}(2,2)$. But this is impossible by explicit knowledge of the archimedean local theta correspondence [69]. This proves the holomorphy of $L(s, \pi \times \tau)$ at the point $s = 1$.

Thus, $L(s, \pi \times \tau)$ has no poles in the region $\mathrm{Re}(s) \geq 1/2$. By the functional equation, it follows that it has no poles in the region $\mathrm{Re}(s) \leq 1/2$. We thus obtain Theorem 4.1.1, which states that $L(s, \pi \times \tau)$ is an entire function. As observed earlier, the theorem of Gelbart and Lapid [24] now implies boundedness in vertical strips. We have finally achieved our goal of proving the "niceness" — analytic continuation to an entire function that satisfies the functional equation and is bounded in vertical strips — of $L(s, \pi \times \tau)$. By the Converse Theorem and Kim's exterior square lifting, our main lifting theorem (Theorem A) now follows.

Applications

Having established the liftings, we now turn to applications. Applying a backwards lifting from GL_4 to $\mathrm{SO}_5(\mathbb{A}) \cong \mathrm{PGSp}_4(\mathbb{A})$, we prove in Theorem 5.1.4 the existence of a globally generic representation on $\mathrm{GSp}_4(\mathbb{A})$ in the same L-packet as π. Also thanks to our liftings, the machinery of Rankin-Selberg L-functions on $\mathrm{GL}_n \times \mathrm{GL}_m$ is available for the study of L-functions related to Siegel modular forms.

All this is exploited in Sect. 5.2. We obtain the niceness of a host of L-functions associated to Siegel cusp forms, including L-functions for $\mathrm{GSp}_4 \times \mathrm{GL}_n$ for any n, and for $\mathrm{GSp}_4 \times \mathrm{GSp}_4$; here, on the GSp_4-factors, we can have the 4-dimensional or the 5-dimensional representation of the dual group. We also obtain niceness for the degree 10 (adjoint) L-function of Siegel modular forms, as well as some analytic properties for the degree 14 and the degree 16 L-functions. For the precise results, see Theorems 5.2.1, 5.2.2 and 5.2.3.

To give a flavor of the results obtained, we restate below part of the $\mathrm{GSp}_4 \times \mathrm{GSp}_4$ result in classical language.

Theorem C: *Let F and G be Siegel cusp forms of full level and weights k, l respectively, and suppose that neither of them is a Saito-Kurokawa lift. Assume further that F and G are eigenfunctions for*

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all the Hecke operators $T(n)$, with eigenvalues $\lambda_F(n)$ and $\lambda_G(n)$ respectively. Let $L(s, F, \mathrm{spin})$ and $L(s, G, \mathrm{spin})$ denote their spin L-functions, normalized so that the functional equation takes s to $1 - s$. Concretely, these L-functions are defined by Dirichlet series

$$L(s, F, \mathrm{spin}) = \zeta(2s+1) \sum_{n=1}^{\infty} \frac{\lambda_F(n)}{n^{s+k-3/2}}, \quad L(s, G, \mathrm{spin}) = \zeta(2s+1) \sum_{n=1}^{\infty} \frac{\lambda_G(n)}{n^{s+l-3/2}},$$

that analytically continue to entire functions, and possess Euler products,

$$L(s, F, \mathrm{spin}) = \prod_p \prod_{i=1}^{4} \left(1 - \beta_{F,p}^{(i)} p^{-s}\right)^{-1}, \quad L(s, G, \mathrm{spin}) = \prod_p \prod_{i=1}^{4} \left(1 - \beta_{F,p}^{(i)} p^{-s}\right)^{-1}.$$

Define the degree 16 convolution L-function $L(s, F \times G)$ by the following Euler product:

$$L(s, F \times G) = \prod_p \prod_{i=1}^{4} \prod_{j=1}^{4} \left(1 - \beta_{F,p}^{(i)} \beta_{G,p}^{(j)} p^{-s}\right)^{-1}.$$

Then $L(s, F \times G)$ is absolutely convergent for $\mathrm{Re}(s) > 1$, has meromorphic continuation to the entire complex plane, and is non-vanishing on $\mathrm{Re}(s) = 1$. Moreover, $L(s, F \times G)$ is entire, except in the special case $k = l$ and $\lambda_F(n) = \lambda_G(n)$ for all n, when it has a simple pole at $s = 1$.

By combining our lifting results with the results of [53], we also prove that $L(1/2, F, \mathrm{spin}) \geq 0$. We prove similar non-negativity results for the "spin $\times$ standard" L-function as well as for suitable L-functions on $\mathrm{GSp}_4 \times \mathrm{GL}_2$ and $\mathrm{GSp}_4 \times \mathrm{GL}_3$; see Theorem 5.2.4 for the precise statement.

We also obtain critical value results in the spirit of Deligne's conjecture for $\mathrm{GSp}_4 \times \mathrm{GL}_1$ (Theorem 5.3.3) and for $\mathrm{GSp}_4 \times \mathrm{GL}_2$ (Theorem 5.3.7). Theorem 5.3.3 follows by combining our lifting theorem with a critical value result for $\mathrm{GL}_4 \times \mathrm{GL}_1$ proved by Grobner and Raghuram [30]. Theorem 5.3.7, on the other hand, follows directly from the second global integral representation (Theorem 3.6.1) using the methods of [73].

Further remarks

As for related works, the transfer from GSp_4 to GL_4 for *all* cuspidal, automorphic representations should eventually follow from the trace formula. At the time of this writing, we do not know whether all the necessary elements for this far reaching program of Arthur's have been completed. The existence of a globally generic representation of $\mathrm{GSp}_4(\mathbb{A})$ in the same L-packet as π (Theorem 5.1.4) is also proved in [89] using theta liftings and the topological trace formula. We hope, however, that our present work is of independent interest, both because it provides a "proof of concept" that certain cases of non-generic transfer can be established without resorting to trace formula arguments, and because the explicit nature of our integral representation makes it a useful tool to attack other problems related to Siegel cusp forms. As an example of the latter, we would like to mention Gross-Prasad type questions for $\mathrm{GSp}_4 \times \mathrm{GL}_2$ as a potential future application. Also, the above mentioned special value result for $\mathrm{GSp}_4 \times \mathrm{GL}_2$, which is an application of our integral representation, does not immediately follow from the transfer obtained via the trace formula.

Acknowledgements

Notation

Basic objects

i) The symbols $\mathbb{Z}$, $\mathbb{Z}_{\geq 0}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}_p$ and $\mathbb{Q}_p$ have the usual meanings. The symbol $\mathbb{A}_F$ denotes the ring of adeles of an algebraic number field F, and $\mathbb{A}_F^\times$ denotes its group of ideles. The symbols $\mathbb{A}$ and $\mathbb{A}^\times$ will always denote $\mathbb{A}_{\mathbb{Q}}$ and $\mathbb{A}_{\mathbb{Q}}^\times$ respectively.

ii) For any commutative ring R and positive integer n, let $\mathrm{Mat}_{n,n}(R)$ denote the ring of $n \times n$ matrices with entries in R, and let $\mathrm{GL}_n(R)$ denote the group of invertible elements in $\mathrm{Mat}_{n,n}(R)$. We use $R^\times$ to denote $\mathrm{GL}_1(R)$. If $A \in \mathrm{Mat}_{n,n}(R)$, we let ${}^tA$ denote its transpose.

iii) Define $J_n \in \mathrm{Mat}_{n,n}(\mathbb{Z})$ by
$$J_n = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}.$$

iv) In this paper, all non-archimedean local fields will be understood to be of characteristic zero. If F is such a field, let $\mathfrak{o}$ be its ring of integers and $\mathfrak{p}$ be the maximal ideal of $\mathfrak{o}$. Let ϖ be a generator of $\mathfrak{p}$, and let q be the cardinality of the residue class field $\mathfrak{o}/\mathfrak{p}$.

v) Let F be as above. If L is a quadratic field extension of F, or $L = F \oplus F$, let $\left(\frac{L}{\mathfrak{p}}\right)$ be the Legendre symbol. By definition, $\left(\frac{L}{\mathfrak{p}}\right) = -1$ if L/F is an unramified field extension (the *inert case*), $\left(\frac{L}{\mathfrak{p}}\right) = 0$ if L/F is a ramified field extension (the *ramified case*), and $\left(\frac{L}{\mathfrak{p}}\right) = 1$ if $L = F \oplus F$ (the *split case*). In the field case, let $\bar{x}$ denote the Galois conjugate of $x \in L$. In the split case, let $\overline{(x,y)} = (y,x)$. In all cases, the norm is defined by $N(x) = x\bar{x}$. If L is a field, then let $\mathfrak{o}_L$ be its ring of integers. If $L = F \oplus F$, then let $\mathfrak{o}_L = \mathfrak{o} \oplus \mathfrak{o}$. Let ϖ_L be a generator of $\mathfrak{p}_L$ if L is a field, and set $\varpi_L = (\varpi, 1)$ if L is not a field. We fix the following ideal in $\mathfrak{o}_L$,
$$\mathfrak{P} := \mathfrak{p}\mathfrak{o}_L = \begin{cases} \mathfrak{p}_L & \text{if } \left(\frac{L}{\mathfrak{p}}\right) = -1, \\ \mathfrak{p}_L^2 & \text{if } \left(\frac{L}{\mathfrak{p}}\right) = 0, \\ \mathfrak{p} \oplus \mathfrak{p} & \text{if } \left(\frac{L}{\mathfrak{p}}\right) = 1. \end{cases} \tag{13}$$

Here, $\mathfrak{p}_L$ is the maximal ideal of $\mathfrak{o}_L$ when L is a field extension. Note that $\mathfrak{P}$ is prime only if $\left(\frac{L}{\mathfrak{p}}\right) = -1$. We have $\mathfrak{P}^n \cap \mathfrak{o} = \mathfrak{p}^n$ for all $n \geq 0$.

vi) We fix additive characters once and for all, as follows. If F is a non-archimedean local field, ψ is required to have conductor $\mathfrak{o}$. If $F = \mathbb{R}$, then $\psi(x) = e^{-2\pi i x}$. For any $a \in F$, let $\psi^a(x) = \psi(ax)$.

The quadratic extension

Let F be a non-archimedean local field of characteristic zero, or $F = \mathbb{R}$. The unitary groups we shall be working with are defined with respect to a quadratic extension L/F. We shall now

explain the conventions for this quadratic extension. We fix three elements $\mathbf{a}, \mathbf{b}, \mathbf{c} \in F$ such that $\mathbf{d} := \mathbf{b}^2 - 4\mathbf{ac} \neq 0$. Then let

$$L = \begin{cases} F(\sqrt{\mathbf{d}}) & \text{if } \mathbf{d} \notin F^{\times 2}, \\ F \oplus F & \text{if } \mathbf{d} \in F^{\times 2}. \end{cases} \tag{14}$$

We shall make the following **assumptions**.

- If F is non-archimedean, assume that $\mathbf{a}, \mathbf{b} \in \mathfrak{o}$ and $\mathbf{c} \in \mathfrak{o}^{\times}$. Assume moreover that if $\mathbf{d} \notin F^{\times 2}$, then $\mathbf{d}$ is the generator of the discriminant of L/F, and if $\mathbf{d} \in F^{\times 2}$, then $\mathbf{d} \in \mathfrak{o}^{\times}$.

- If $F = \mathbb{R}$, assume that $S = \begin{bmatrix} \mathbf{a} & \mathbf{b}/2 \\ \mathbf{b}/2 & \mathbf{c} \end{bmatrix} \in \mathrm{Mat}_{2,2}(\mathbb{R})$ is a positive definite matrix. Equivalently, $\mathbf{c} > 0$ and $\mathbf{d} < 0$.

Hence, if $F = \mathbb{R}$, we always assume that $L = \mathbb{C}$. In all cases let

$$\alpha = \begin{cases} \dfrac{\mathbf{b} + \sqrt{\mathbf{d}}}{2\mathbf{c}} & \text{if } L \text{ is a field,} \\[2ex] \left(\dfrac{\mathbf{b} + \sqrt{\mathbf{d}}}{2\mathbf{c}}, \dfrac{\mathbf{b} - \sqrt{\mathbf{d}}}{2\mathbf{c}}\right) & \text{if } L = F \oplus F. \end{cases} \tag{15}$$

An important role will be played by the matrix

$$\eta = \begin{bmatrix} 1 & 0 & & \\ \alpha & 1 & & \\ & & 1 & -\bar{\alpha} \\ & & 0 & 1 \end{bmatrix}. \tag{16}$$

We further define

$$\eta_0 = \begin{cases} \eta & \text{if } F \text{ is } p\text{-adic,} \\[1ex] \dfrac{1}{\sqrt{2}} \begin{bmatrix} 1 & i & & \\ i & 1 & & \\ & & 1 & i \\ & & i & 1 \end{bmatrix} & \text{if } F = \mathbb{R}. \end{cases} \tag{17}$$

Algebraic groups

For simplicity we will make all definitions over the local field F, but it is clear how to define the corresponding global objects.

i) Let $H = \mathrm{GSp}_4$ and $G_j = \mathrm{GU}(j, j; L)$ be the algebraic F-groups whose F-points are given by

$$H(F) = \{g \in \mathrm{GL}_4(F) \mid {}^t g J_2 g = \mu_2(g) J_2, \, \mu_2(g) \in F^{\times}\}, \tag{18}$$

$$G_j(F) = \{g \in \mathrm{GL}_{2j}(L) \mid {}^t \bar{g} J_j g = \mu_j(g) J_j, \, \mu_j(g) \in F^{\times}\}. \tag{19}$$

ii) We define, for $\zeta \in L^{\times}$ and $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G_1(F)$,

$$m_1(\zeta) = \begin{bmatrix} \zeta & & & \\ & 1 & & \\ & & \bar{\zeta}^{-1} & \\ & & & 1 \end{bmatrix}, \qquad m_2(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = \begin{bmatrix} 1 & & & \\ & a & & b \\ & & \bar{a}d - b\bar{c} & \\ & c & & d \end{bmatrix}. \tag{20}$$

iii) Let P be the standard maximal parabolic subgroup of $G_2(F)$ with a non-abelian unipotent radical. Let $P = MN$ be the Levi decomposition of P. We have $M = M^{(1)}M^{(2)}$, where

$$M^{(1)}(F) = \{m_1(\zeta) \mid \zeta \in L^\times\}, \tag{21}$$

$$M^{(2)}(F) = \{m_2(g) \mid g \in G_1(F)\}, \tag{22}$$

$$N(F) = \left\{ \begin{bmatrix} 1 & z & & \\ & 1 & & \\ & & 1 & \\ & & -\bar{z} & 1 \end{bmatrix} \begin{bmatrix} 1 & & x & y \\ & 1 & \bar{y} & \\ & & 1 & \\ & & & 1 \end{bmatrix} \middle| x \in F,\ y, z \in L \right\}. \tag{23}$$

The modular factor of the parabolic P is given by

$$\delta_P(m_1(\zeta)m_2(g)) = |N(\zeta)\mu_1^{-1}(g)|^3, \tag{24}$$

where $|\cdot|$ is the normalized absolute value on F.

iv) Let P_{12} be the maximal parabolic subgroup of G_3, defined by

$$P_{12} = G_3 \cap \begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ & & & * & * & * \\ & & & * & * & * \\ & & & * & * & * \end{bmatrix}. \tag{25}$$

Let $P_{12} = M_{12}N_{12}$ be the Levi decomposition, with

$$M_{12}(F) := \left\{ m(A, v) = \begin{bmatrix} A & 0 \\ 0 & v\,{}^t\bar{A}^{-1} \end{bmatrix} \middle| A \in \mathrm{GL}_3(L),\ v \in F^\times \right\},$$

$$N_{12}(F) := \left\{ \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \middle| b \in \mathrm{Mat}_{3,3}(L),\ {}^t\bar{b} = b \right\}.$$

The modular function of P_{12} is given by

$$\delta_{12}\left(\begin{bmatrix} A & \\ & v\,{}^t\bar{A}^{-1} \end{bmatrix} \right) = |v^{-3}N(\det(A))|^3, \qquad v \in F^\times,\ A \in \mathrm{GL}_3(L). \tag{26}$$

v) Let ι be the embedding of $\{(g_1, g_2) \in G_1(F) \times G_2(F) : \mu_1(g_1) = \mu_2(g_2)\}$ into $G_3(F)$ defined by

$$\iota\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right) = \begin{bmatrix} A & & B & \\ & a & & -b \\ C & & D & \\ & -c & & d \end{bmatrix}. \tag{27}$$

Congruence subgroups

Assuming that F is p-adic, we will use the following notation for congruence subgroups,

$$K^{(0)}(\mathfrak{P}^n) = G_1(\mathfrak{o}) \cap \begin{bmatrix} \mathfrak{o}_L & \mathfrak{o}_L \\ \mathfrak{P}^n & \mathfrak{o}_L \end{bmatrix}, \tag{28}$$

$$K^{(1)}(\mathfrak{P}^n) = G_1(\mathfrak{o}) \cap \begin{bmatrix} 1 + \mathfrak{P}^n & \mathfrak{o}_L \\ \mathfrak{P}^n & \mathfrak{o}_L \end{bmatrix}, \tag{29}$$

$$\begin{aligned} K_1^{(1)}(\mathfrak{P}^n) &= U(1,1;L)(\mathfrak{o}) \cap K^{(1)}(\mathfrak{P}^n) \\ &= U(1,1;L)(\mathfrak{o}) \cap \begin{bmatrix} 1 + \mathfrak{P}^n & \mathfrak{o}_L \\ \mathfrak{P}^n & \mathfrak{o}_L \end{bmatrix} = U(1,1;L)(\mathfrak{o}) \cap \begin{bmatrix} 1 + \mathfrak{P}^n & \mathfrak{o}_L \\ \mathfrak{P}^n & 1 + \mathfrak{P}^n \end{bmatrix}, \end{aligned} \tag{30}$$

$$K^{(1)}(\mathfrak{p}^n) = \mathrm{GL}_2(\mathfrak{o}) \cap \begin{bmatrix} 1 + \mathfrak{p}^n & \mathfrak{o} \\ \mathfrak{p}^n & \mathfrak{o} \end{bmatrix}. \tag{31}$$

If τ is an irreducible, admissible representation of $\mathrm{GL}_n(F)$, we let $a(\tau)$ be the non-negative integer such that $\mathfrak{p}^{a(\tau)}$ is the conductor of τ; see Theorem 1.2.2 for a characterization in the GL_2 case. If χ is a character of $F^\times$, then $a(\chi)$ is the smallest non-negative integer such that χ is trivial on $\mathfrak{o}^\times \cap (1 + \mathfrak{p}^{a(\chi)})$.

Representations of $\mathrm{GL}_2(\mathbb{R})$

If p is a positive integer and $\mu \in \mathbb{C}$, we let $\mathcal{D}_{p,\mu}$ be the irreducible representation of $\mathrm{GL}_2(\mathbb{R})$ with minimal weight $p + 1$ and central character satisfying $a \mapsto a^{2\mu}$ for $a > 0$. Every other irreducible, admissible representation of $\mathrm{GL}_2(\mathbb{R})$ is of the form $\beta_1 \times \beta_2$ with characters β_1, β_2 of $\mathbb{R}^\times$; see (37). Note that, if $\mu \in i\mathbb{R}$, then $\mathcal{D}_{p,\mu}$ is a discrete series representation.

1 Distinguished vectors in local representations

In this section we will develop some local theory, both archimedean and non-archimedean, which will be utilized in subsequent sections on global integral representations. Recall the definitions of the groups $G_j = \mathrm{GU}(j,j;L)$ from (19). The local theory will exhibit distinguished vectors in certain parabolically induced representations of $G_2(F)$, where $F = \mathbb{R}$ or F is p-adic. We will also study the behavior of these vectors under local intertwining operators. Since the distinguished vectors are characterized by right transformation properties, the intertwining operators map distinguished vectors to distinguished vectors. This fact will later be applied to obtain the functional equation of global L-functions.

Unless otherwise noted, F is a non-archimedean local field of characteristic zero, or $F = \mathbb{R}$. We let L, α, η be as in (14), (15), (16), respectively.

1.1 Parabolic induction to $\mathrm{GU}(2,2)$

Let (τ, V_τ) be an irreducible, admissible, infinite-dimensional representation of $\mathrm{GL}_2(F)$, and let χ_0 be a character of $L^\times$ such that $\chi_0\big|_{F^\times}$ coincides with ω_τ, the central character of τ. Then the pair (χ_0, τ) defines a representation of $G_1(F) \cong M^{(2)}(F)$ on the same space V_τ via

$$\tau(\lambda g) = \chi_0(\lambda)\tau(g), \qquad \lambda \in L^\times, \; g \in \mathrm{GL}_2(F). \tag{32}$$

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We denote this representation by $\chi_0 \times \tau$. Every irreducible, admissible representation of $G_1(F)$ is of the form (32). If V_τ is a space of functions on $\mathrm{GL}_2(F)$ on which $\mathrm{GL}_2(F)$ acts by right translation, then $\chi_0 \times \tau$ can be realized as a space of functions on $M^{(2)}(F)$ on which $M^{(2)}(F)$ acts by right translation. This is accomplished by extending every $W \in V_\tau$ to a function on $M^{(2)}(F)$ via

$$W(\lambda g) = \chi_0(\lambda)W(g), \qquad \lambda \in L^\times,\ g \in \mathrm{GL}_2(F). \tag{33}$$

If s is a complex parameter, χ is any character of $L^\times$ and $\chi_0 \times \tau$ is a representation of $M^{(2)}(F)$ as above, we denote by $I(s, \chi, \chi_0, \tau)$ the induced representation of $G_2(F)$ consisting of functions $f : G_2(F) \to V_\tau$ with the transformation property

$$f(m_1(\zeta)m_2(b)ng) = \left|N(\zeta)\mu_1^{-1}(b)\right|^{3(s+\frac{1}{2})}\chi(\zeta)(\chi_0 \times \tau)(b)f(g) \tag{34}$$

for $\zeta \in L^\times$ and $b \in G_1(F)$.

Now taking V_τ to be the Whittaker model of τ with respect to the character ψ, if we associate to each f as above the function on $G_2(F)$ given by $W_f(g) = f(g)(1)$, then we obtain another model $I_W(s, \chi, \chi_0, \tau)$ of $I(s, \chi, \chi_0, \tau)$ consisting of functions $W : G_2(F) \to \mathbb{C}$. These functions satisfy

$$W(m_1(\zeta)m_2(\begin{bmatrix} \lambda & \\ & \lambda \end{bmatrix})g) = |N(\zeta\lambda^{-1})|^{3(s+\frac{1}{2})}\chi(\zeta)\chi_0(\lambda)W(g), \qquad \zeta, \lambda \in L^\times, \tag{35}$$

and

$$W(\begin{bmatrix} 1 & z & & \\ & 1 & & \\ & & 1 & \\ & & -\bar{z} & 1 \end{bmatrix}\begin{bmatrix} 1 & & x & y \\ & 1 & \bar{y} & w \\ & & 1 & \\ & & & 1 \end{bmatrix}g) = \psi(w)W(g), \qquad w, x \in F,\ y, z \in L. \tag{36}$$

Assume on the other hand that τ is a parabolically induced representation $\beta_1 \times \beta_2$, not necessarily irreducible, with characters $\beta_1, \beta_2 : F^\times \to \mathbb{C}^\times$. The standard model of $\beta_1 \times \beta_2$ consists of functions $\varphi : \mathrm{GL}_2(F) \to \mathbb{C}$ with the transformation property

$$\varphi(\begin{bmatrix} a & b \\ & d \end{bmatrix}g) = |ad^{-1}|^{1/2}\beta_1(a)\beta_2(d)\varphi(g) \qquad \text{for all } a, d \in F^\times,\ b \in F,\ g \in \mathrm{GL}_2(F). \tag{37}$$

If we associate to f as in (34), now taking values in the standard model of $\beta_1 \times \beta_2$, the function Φ_f on $G_2(F)$ given by $\Phi_f(g) = f(g)(1)$, then we obtain another model of $I(s, \chi, \chi_0, \tau)$, which we denote by $I_\Phi(s, \chi, \chi_0, \tau)$. It consists of functions $\Phi : G_2(F) \to \mathbb{C}$ with the transformation property

$$\Phi(\begin{bmatrix} \zeta & * & * & * \\ & \lambda & * & * \\ & & \bar{\zeta}^{-1}N(\lambda) & \\ & & * & \lambda \end{bmatrix}\begin{bmatrix} 1 & & & \\ & a & & \\ & & ad & \\ & & & d \end{bmatrix}g)$$

$$= |N(\zeta\lambda^{-1})|^{3(s+\frac{1}{2})}|a|^{-3s-1}|d|^{-3s-2}\chi(\zeta)\chi_0(\lambda)\beta_1(a)\beta_2(d)\Phi(g) \tag{38}$$

for all $\zeta, \lambda \in L^\times$, $a, d \in F^\times$.

Intertwining operators

Assume that τ is generic, and let χ, χ_0 be as above. For $f \in I(s, \chi, \chi_0, \tau)$ with $\mathrm{Re}(s)$ large enough, the local intertwining operator is defined by

$$(M(s)f)(g) = \int_{N(F)} f(w_1ng)\,dn, \qquad w_1 = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ 1 & & & \end{bmatrix}. \tag{39}$$

Calculations show that $M(s)$ defines an intertwining map

$$M(s): \ I(s, \chi, \chi_0, \tau) \longrightarrow I(-s, \bar{\chi}^{-1}, \chi\bar{\chi}\chi_0, \chi\tau), \tag{40}$$

where by $\chi\tau$ we mean the twist $(\chi|_{F^\times}) \otimes \tau$. It is easily checked that the above formula (39) also defines intertwining operators $M(s)$ from $I_\Phi(s, \chi, \chi_0, \tau)$ to $I_\Phi(-s, \bar{\chi}^{-1}, \chi\bar{\chi}\chi_0, \chi\tau)$ and from $I_W(s, \chi, \chi_0, \tau)$ to $I_W(-s, \bar{\chi}^{-1}, \chi\bar{\chi}\chi_0, \chi\tau)$. In Corollary 1.2.4 (non-archimedean case) and Corollary 1.3.7 (archimedean case) we will identify a distinguished element

$$W^\# = W^\#(\,\cdot\,, s, \chi, \chi_0, \tau)$$

in $I_W(s, \chi, \chi_0, \tau)$. This distinguished function will have the property

$$M(s)W^\#(\,\cdot\,, s, \chi, \chi_0, \tau) = K(s)W^\#(\,\cdot\,, -s, \bar{\chi}^{-1}, \chi\bar{\chi}\chi_0, \chi\tau). \tag{41}$$

with a "constant" $K(s)$ (independent of $g \in G_2(F)$, but dependent on s, as well as χ, χ_0 and τ). In most cases $K(s)$ exists because $W^\#$ is characterized, up to scalars, by *right* transformation properties. An exception is the archimedean "different parity" Case C, defined in Sect. 1.3, in which case said right transformation properties characterize a *two*-dimensional space. In this case the existence of the function $K(s)$ such that (41) holds will follow from explicit calculations. Note that if $\eta_0 \in G_2(F)$ is such that $W^\#(\eta_0) = 1$, then we obtain the formula

$$K(s) = \int\limits_{N(F)} W^\#(w_1 n\eta_0, s, \chi, \chi_0, \tau)\, dn \tag{42}$$

by evaluating at η_0. Explicitly,

$$K(s) = \int\limits_L \int\limits_L \int\limits_F W^\#(w_1 \begin{bmatrix} 1 & z & & \\ & 1 & & \\ & & 1 & \\ & & -\bar{z} & 1 \end{bmatrix} \begin{bmatrix} 1 & & x & y \\ & 1 & \bar{y} & \\ & & 1 & \\ & & & 1 \end{bmatrix} \eta_0)\, dx\, dy\, dz. \tag{43}$$

Our goal in Sects. 1.4 and 1.5 will be to calculate the function $K(s)$. We will then also be more precise about the measures on F and L used in (43).

1.2 Distinguished vectors: non-archimedean case

In this section let F be a non-archimedean local field of characteristic zero. Let τ be any irreducible, admissible representation of $\mathrm{GL}_2(F)$, and let χ_0 be a character of $L^\times$ such that $\chi_0|_{F^\times} = \omega_\tau$, the central character of τ. Let Λ be an unramified character of $L^\times$, and let χ be the character of $L^\times$ defined by

$$\chi(\zeta) = \Lambda(\bar{\zeta})^{-1}\chi_0(\bar{\zeta})^{-1}. \tag{44}$$

For a complex parameter s, let $I(s, \chi, \chi_0, \tau)$ be as in Sect. 1.1. Let $K^{G_2} = G_2(F) \cap \mathrm{GL}_4(\mathfrak{o}_L)$, a maximal compact subgroup. We define the principal congruence subgroups $\Gamma(\mathfrak{P}^r) := \{g \in G_2(F) \mid g \equiv 1 \pmod{\mathfrak{P}^r}\}$ with $\mathfrak{P}$ as in (13). For $r = 0$ we understand that $\Gamma(\mathfrak{P}^r) = K^{G_2}$. For any $m \geq 0$, we let

$$\eta_m = \begin{bmatrix} 1 & 0 & & \\ \alpha\varpi^m & 1 & & \\ & & 1 & -\bar{\alpha}\varpi^m \\ & & 0 & 1 \end{bmatrix}. \tag{45}$$

For systematic reasons, we let η_∞ be the identity matrix. Note that $\eta_0 = \eta$; see (16).

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1.2.1 Proposition. *For any $r \geq 0$ the following disjoint double coset decompositions hold,*

$$G_2(F) = \bigsqcup_{0 \leq m \leq \infty} P(F)\eta_m K^H = \bigsqcup_{0 \leq m \leq r} P(F)\eta_m K^H \Gamma(\mathfrak{P}^r). \tag{46}$$

Moreover, for any $0 \leq m < r$, we have

$$P(F)\eta_m K^H \Gamma(\mathfrak{P}^r) = P(F)\eta_m K^H. \tag{47}$$

Proof. Using the Iwasawa decomposition, (46) follows from

$$K^{G_2} = \bigsqcup_{0 \leq m \leq \infty} P(\mathfrak{o})\eta_m K^H = \bigsqcup_{0 \leq m \leq r} P(\mathfrak{o})\eta_m K^H \Gamma(\mathfrak{P}^r). \tag{48}$$

One can show that the double cosets on the right hand side of (48) are disjoint by observing that the function

$$K^{G_2} \ni g \mapsto \min\big(v((gJ\,{}^tg)_{3,2}), v((gJ\,{}^tg)_{3,4})\big)$$

takes different values on the double cosets. We take the above function modulo $\mathfrak{P}^r$ for the disjointness of the double cosets involving $\Gamma(\mathfrak{P}^r)$. Knowing disjointness, one obtains the second equality in (48) by multiplying the double cosets by $\Gamma(\mathfrak{P}^r)$ on the right. We only sketch a proof of the first equality in (48). The first step consists in showing that $K^{G_2} = P(\mathfrak{o})K^H \Gamma(\mathfrak{P}) \sqcup P(\mathfrak{o})\eta K^H \Gamma(\mathfrak{P})$, which can be done explicitly by considering the three cases – inert, ramified and split – separately. Then, for $g \in P(\mathfrak{o})\gamma_0 K^H$ with $\gamma_0 \in \Gamma(\mathfrak{P})$ or $\gamma_0 \in \eta\Gamma(\mathfrak{P})$, observe that

$$\gamma_0 \in G_2(F) \cap \begin{bmatrix} \mathfrak{o}_L^\times & \mathfrak{P} & \mathfrak{o}_L & \mathfrak{o}_L \\ \mathfrak{o}_L & \mathfrak{o}_L^\times & \mathfrak{o}_L & \mathfrak{o}_L \\ \mathfrak{P} & \mathfrak{P} & \mathfrak{o}_L^\times & \mathfrak{o}_L \\ \mathfrak{P} & \mathfrak{P} & \mathfrak{P} & \mathfrak{o}_L^\times \end{bmatrix}.$$

Using appropriate matrix identities one can show, for any γ_0 in this set, that there exist $p \in P(\mathfrak{o})$, $h \in K^H$ and a unique $m \in \{0, 1, 2, \cdots, \infty\}$ such that $\gamma_0 = p\eta_m h$; this completes the proof of (46).

To prove (47), we rewrite (46) as

$$G_2(F) = \bigsqcup_{0 \leq m < r} P(F)\eta_m K^H \sqcup X, \qquad X = \bigsqcup_{r \leq m \leq \infty} P(F)\eta_m K^H,$$

and also

$$G_2(F) = \bigsqcup_{0 \leq m < r} P(F)\eta_m K^H \Gamma(\mathfrak{P}^r) \sqcup Y, \qquad Y = P(F)\,\eta_r K^H \Gamma(\mathfrak{P}^r).$$

For $m \geq r$, we have $\eta_m \in P(F)K^H \Gamma(\mathfrak{P}^r) = P(F)\eta_r K^H \Gamma(\mathfrak{P}^r)$. Hence $X \subset Y$. Evidently, for $m < r$, we have $P(F)\eta_m K^H \subset P(F)\eta_m K^H \Gamma(\mathfrak{P}^r)$. It follows that $P(F)\eta_m K^H = P(F)\eta_m K^H \Gamma(\mathfrak{P}^r)$. ∎

We recall the standard newform theory for GL_2. Let the congruence subgroup $K^{(1)}(\mathfrak{p}^n)$ be as in (31). The following result is well known (see [9], [14]).

1.2.2 Theorem. *Let (τ, V_τ) be a generic, irreducible, admissible representation of $\mathrm{GL}_2(F)$. Then the spaces*

$$V_\tau(n) = \{v \in V_\tau \mid \tau(g)v = v \text{ for all } g \in K^{(1)}(\mathfrak{p}^n)\}$$

are non-zero for n large enough. If n is minimal with $V_\tau(n) \neq 0$, then $\dim(V_\tau(n)) = 1$. For $r \geq n$, we have $\dim(V_\tau(r)) = r - n + 1$.

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If n is minimal such that $V_\tau(n) \neq 0$, then $\mathfrak{p}^n$ is called the *conductor* of τ, and we write $n = a(\tau)$. Any non-zero vector in $V_\tau(a(\tau))$ is called a *local newform*. The following theorem is a local newforms result for the induced representations $I(s, \chi, \chi_0, \tau)$ with respect to the congruence subgroups $K^H \Gamma(\mathfrak{P}^r)$.

1.2.3 Theorem. *Let (τ, V_τ) be a generic, irreducible, admissible representation of $\mathrm{GL}_2(F)$ with central character ω_τ and conductor $\mathfrak{p}^n$. Let χ_0 be a character of $L^\times$ such that $\chi_0\big|_{F^\times} = \omega_\tau$ and $\chi_0((1 + \mathfrak{P}^n) \cap \mathfrak{o}_L^\times) = 1$, and let χ be the character of $L^\times$ defined by (44), where Λ is unramified. Let*

$$V(r) := \{\phi \in I(s, \chi, \chi_0, \tau) \mid \phi(g\gamma, s) = \phi(g, s) \text{ for all } g \in G(F), \gamma \in K^H \Gamma(\mathfrak{P}^r)\}$$

for a non-negative integer r. Then

$$\dim(V(r)) = \begin{cases} \dfrac{(r - n + 1)(r - n + 2)}{2} & \text{if } r \geq n, \\ 0 & \text{if } r < n. \end{cases}$$

Proof. Let $\phi \in V(r)$. By Proposition 1.2.1, ϕ is completely determined by its values on η_m, $0 \leq m \leq r$. For such m, and any $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in K^{(1)}(\mathfrak{p}^{r-m})$, we have $A := m_2(g) \in M(F)N(F) \cap \eta_m K^H \Gamma(\mathfrak{P}^r)\eta_m^{-1}$. It follows that $\phi(\eta_m) = \phi(A\eta_m) = \tau(g)\phi(\eta_m)$. Hence, for $0 \leq m \leq r$, the vector $v_m := \phi(\eta_m)$ lies in $V_\tau(r - m)$. Since the conductor of τ is $\mathfrak{p}^n$, we conclude that $v_m = 0$ if $r - m < n$. Therefore $\dim(V(r)) = 0$ for all $r < n$.

Now suppose that $r \geq n$. We will show that, for any m such that $r - m \geq n$, if v_m is chosen to be any vector in $V_\tau(r - m)$, then we obtain a well-defined function ϕ in $V(r)$. For $m = r$ this is easy to check, since in this case $n = 0$ and all the data is unramified. Assume therefore that $r > m$. We have to show that for $m_1 n_1 \eta_m k_1 \gamma_1 = m_2 n_2 \eta_m k_2 \gamma_2$, with $m_i \in M(F)$, $n_i \in N(F)$, $k_i \in K^H$ and $\gamma_i \in \Gamma(\mathfrak{P}^r)$,

$$|N(\zeta_1) \cdot \mu_1^{-1}|^{3(s+1/2)} \chi(\zeta_1)\,(\chi_0 \times \tau)(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix})v_m = |N(\zeta_2) \cdot \mu_2^{-1}|^{3(s+1/2)} \chi(\zeta_2)\,(\chi_0 \times \tau)(\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix})v_m. \tag{49}$$

We have $\eta_m^{-1} m_2^{-1} m_1 n^* \eta_m \in K^H \Gamma(\mathfrak{P}^r)$, where $n^* \in N(F)$ depends on m_1, m_2, n_1, n_2. Write

$$m_2^{-1} m_1 = \begin{bmatrix} \zeta & & \\ & \tilde{a} & & \tilde{b} \\ & & \mu\bar{\zeta}^{-1} & \\ & \tilde{c} & & \tilde{d} \end{bmatrix}.$$

By definition, $\zeta_1 = \zeta_2 \zeta$ and $\mu_1 = \mu_2 \mu$. We have $\zeta \in \mathfrak{o}_L^\times$ and $\mu \in \mathfrak{o}^\times$. Hence (49) is equivalent to

$$\chi(\zeta)\,(\chi_0 \times \tau)(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix})v_m = (\chi_0 \times \tau)(\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix})v_m. \tag{50}$$

One can check that $\tilde{a}\bar{\zeta}^{-1} \in 1 + \mathfrak{P}^{r-m}$ and $\tilde{c} \in \mathfrak{P}^{r-m}$. Hence, using the definition of χ, χ_0 (with unramified Λ) and the fact that $v_m \in V_\tau(r - m)$,

$$\chi(\zeta)\,(\chi_0 \times \tau)(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix})v_m = \chi(\zeta)\,(\chi_0 \times \tau)(\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\begin{bmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{bmatrix})v_m$$

$$= \chi(\zeta)\chi_0(\tilde{a})\,(\chi_0 \times \tau)\left(\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\begin{bmatrix} 1 & \tilde{b}/\tilde{a} \\ \tilde{c}/\tilde{a} & \tilde{d}/\tilde{a} \end{bmatrix}\right)v_m$$

$$= \chi_0(\bar{\zeta}^{-1})\chi_0(\tilde{a})\,(\chi_0 \times \tau)\left(\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right)v_m$$

$$= (\chi_0 \times \tau)\left(\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right)v_m,$$

as claimed. Now, using the formula for $\dim(V_\tau(r-m))$ from Theorem 1.2.2 completes the proof of the theorem. ∎

Assume that $W^{(0)}$ is the newform in the Whittaker model of τ with respect to an additive character of conductor $\mathfrak{o}$. Then it is known that $W^{(0)}(1) \neq 0$, so that this function can be normalized by $W^{(0)}(1) = 1$. The following is an immediate consequence of the above theorem (and its proof).

1.2.4 Corollary. *There exists a unique element $W^{\#}(\,\cdot\,,s)$ in $I_W(s,\chi,\chi_0,\tau)$ with the properties*

$$W^{\#}(gk,s) = W^{\#}(g,s) \qquad \text{for } g \in G_2(F),\ k \in K^H\Gamma(\mathfrak{P}^n), \tag{51}$$

and

$$W^{\#}(\eta_0,s) = 1, \tag{52}$$

where $\eta_0 = \eta$ as in (17). The function $W^{\#}(\,\cdot\,,s)$ is supported on $P(F)\eta_0 K^H\Gamma(\mathfrak{P}^n)$. On this double coset,

$$W^{\#}(m_1(\zeta)m_2(g)\eta_0,s) = |N(\zeta)\cdot\mu_1^{-1}(g)|^{3(s+1/2)}\chi(\zeta)\,W^{(0)}(g), \tag{53}$$

where $\zeta \in L^\times$, $g \in G_1(F)$, and $W^{(0)}$ is the newform in V_τ, normalized by $W^{(0)}(1) = 1$, and extended to a function on $G_1(F)$ via the character χ_0 (see (33)).

1.3   Distinguished vectors: archimedean case

Let $F = \mathbb{R}$ and $L = \mathbb{C}$, and $G_2 = \mathrm{GU}(2,2;\mathbb{C})$ as in the notations. Consider the symmetric domains $\mathbb{H}_2 := \{Z \in \mathrm{Mat}_{2,2}(\mathbb{C}) \mid i(^t\bar{Z} - Z) \text{ is positive definite}\}$ and $\mathfrak{h}_2 := \{Z \in \mathbb{H}_2 \mid {}^tZ = Z\}$. The group $G_2^+(\mathbb{R}) := \{g \in G_2(\mathbb{R}) \mid \mu_2(g) > 0\}$ acts on $\mathbb{H}_2$ via $(g,Z) \mapsto g\langle Z\rangle$, where

$$g\langle Z\rangle = (AZ+B)(CZ+D)^{-1}, \text{ for } g = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in G_2^+(\mathbb{R}), Z \in \mathbb{H}_2.$$

Under this action, $\mathfrak{h}_2$ is stable by $H^+(\mathbb{R}) = \mathrm{GSp}_4^+(\mathbb{R})$. The group $K_\infty^{G_2} = \{g \in G_2^+(\mathbb{R}) : \mu_2(g) = 1, g\langle i_2\rangle = i_2\}$ is a maximal compact subgroup of $G_2^+(\mathbb{R})$. Here, $i_2 = \begin{bmatrix} i & \\ & i \end{bmatrix} \in \mathbb{H}_2$. By the Iwasawa decomposition

$$G_2(\mathbb{R}) = M^{(1)}(\mathbb{R})M^{(2)}(\mathbb{R})N(\mathbb{R})K_\infty^{G_2}, \tag{54}$$

where $M^{(1)}(\mathbb{R})$, $M^{(2)}(\mathbb{R})$ and $N(\mathbb{R})$ are as defined in (21), (22) and (23). For $g \in G_2^+(\mathbb{R})$ and $Z \in \mathbb{H}_2$, let $J(g,Z) = CZ + D$ be the automorphy factor. Then, for any integer l, the map

$$k \longmapsto \det(J(k,i_2))^l \tag{55}$$

defines a character $K_\infty^{G_2} \to \mathbb{C}^\times$. Let $K_\infty^H = K_\infty \cap H^+(\mathbb{R})$. Then K_∞^H is a maximal compact subgroup of $H^+(\mathbb{R})$.

Let (τ, V_τ) be a generic, irreducible, admissible representation of $\mathrm{GL}_2(\mathbb{R})$ with central character ω_τ. Let l_2 be an integer of the same parity as the weights of τ (the precise value of l_2 is largely irrelevant, and we will later make a specific choice). Let χ_0 be the character of $\mathbb{C}^\times$ such that $\chi_0\big|_{\mathbb{R}^\times} = \omega_\tau$ and $\chi_0(\zeta) = \zeta^{l_2}$ for $\zeta \in \mathbb{C}^\times$, $|\zeta| = 1$. Let χ be the character of $\mathbb{C}^\times$ given by

$$\chi(\zeta) = \chi_0(\bar{\zeta})^{-1}. \tag{56}$$

We interpret χ as a character of $M^{(1)}(\mathbb{R}) \cong \mathbb{C}^\times$. We extend τ to a representation of $G_1(\mathbb{R})$ as in (32). *In the archimedean case, we can always realize τ as a subrepresentation of a parabolically induced representation $\beta_1 \times \beta_2$, with characters $\beta_1, \beta_2 : \mathbb{R}^\times \to \mathbb{C}^\times$ (see (37)).* We define the complex numbers t_1, t_2, p, q by

$$\beta_1(a) = a^{t_1}, \qquad \beta_2(a) = a^{t_2}, \qquad p = t_1 - t_2, \qquad q = t_1 + t_2 \tag{57}$$

for $a > 0$.

Remark: Evidently, q is related to the central character of τ via $\omega_\tau(a) = a^q$ for $a > 0$. The number p could also be more intrinsically defined via the eigenvalue of the Laplace operator. Note that if τ belongs to the principal series and β_1 and β_2 are interchanged, then p changes sign; this ambiguity will be irrelevant. We also note that if τ is a discrete series representation of lowest weight l_1, then $p = l_1 - 1$.

The induced representation $I_\Phi(s, \chi, \chi_0, \tau)$ is now a subrepresentation of $I_\Phi(s, \chi, \chi_0, \beta_1 \times \beta_2)$. Any $\Phi \in I_\Phi(s, \chi, \chi_0, \tau)$ satisfies the transformation property (38); in view of the Iwasawa decomposition, Φ is determined by its restriction to $K_\infty^{G_2}$. Conversely, given a function $\Phi : K_\infty^{G_2} \to \mathbb{C}$, it can be extented to a function on $G_2(\mathbb{R})$ with the property (38) if and only if

$$\Phi(\hat{\zeta}_1 k) = \zeta^{l_2}\, \Phi(k), \qquad \Phi(\hat{\zeta}_2 k) = \zeta^{l_2}\, \Phi(k), \tag{58}$$

for all $\zeta \in S^1$ and $k \in K_\infty^{G_2}$. Here, we used the notation

$$\hat{\zeta}_1 = \begin{bmatrix} \zeta & & & \\ & 1 & & \\ & & \zeta & \\ & & & 1 \end{bmatrix}, \qquad \hat{\zeta}_2 = \begin{bmatrix} 1 & & & \\ & \zeta & & \\ & & 1 & \\ & & & \zeta \end{bmatrix}.$$

We will therefore study certain spaces of functions on $K_\infty^{G_2}$ with the property (58).

The spaces W_{m,l,l_2}^Δ

We begin by describing the Lie algebra and the finite-dimensional representations of $K_\infty^{G_2}$. Let $\mathfrak{g}$ be the Lie algebra of $U(2,2)$. Let $X \mapsto -{}^t\bar{X}$ be the Cartan involution on $\mathfrak{g}$. Let $\mathfrak{k}$ be the $+1$ eigenspace and let $\mathfrak{p}$ be the -1 eigenspace of the Cartan involution. We denote by $\mathfrak{k}_\mathbb{C}$ and $\mathfrak{p}_\mathbb{C}$ the complexifications of $\mathfrak{k}$ and $\mathfrak{p}$, respectively. Then

$$\mathfrak{k}_\mathbb{C} = \{ \begin{bmatrix} A & B \\ -B & A \end{bmatrix} \mid A, B \in \mathrm{Mat}_{2,2}(\mathbb{C}) \}, \qquad \mathfrak{p}_\mathbb{C} = \{ \begin{bmatrix} A & B \\ B & -A \end{bmatrix} \mid A, B \in \mathrm{Mat}_{2,2}(\mathbb{C}) \}.$$

Hence $\mathfrak{g}_\mathbb{C} = \mathfrak{k}_\mathbb{C} \oplus \mathfrak{p}_\mathbb{C} = \mathfrak{gl}(4, \mathbb{C})$. The following eight elements constitute a convenient basis for $\mathfrak{k}_\mathbb{C}$.

$$U_1 = \tfrac{1}{2} \begin{bmatrix} 1 & & & -i \\ i & 0 & & \\ & & 1 & \\ & & & 0 \end{bmatrix}, \qquad U_2 = \tfrac{1}{2} \begin{bmatrix} 0 & & & \\ & 1 & & -i \\ & i & & \\ & & 1 & 0 \end{bmatrix}, \qquad V_1 = \tfrac{1}{2} \begin{bmatrix} -1 & & & -i \\ i & 0 & & -1 \\ & & & 0 \end{bmatrix}, \qquad V_2 = \tfrac{1}{2} \begin{bmatrix} 0 & & & \\ & -1 & & -i \\ & i & & \\ & & -1 & 0 \end{bmatrix},$$

$$P_+ = \frac{1}{2}\begin{bmatrix} 0 & 1 & & -i \\ & 0 & & \\ & i & 0 & 1 \\ & & & 0 \end{bmatrix}, \qquad P_- = \frac{1}{2}\begin{bmatrix} 0 & & & \\ 1 & 0 & & -i \\ & & 0 & \\ i & & 1 & 0 \end{bmatrix}, \qquad Q_+ = \frac{1}{2}\begin{bmatrix} 0 & & & \\ -1 & 0 & & -i \\ & & 0 & \\ i & & -1 & 0 \end{bmatrix}, \qquad Q_- = \frac{1}{2}\begin{bmatrix} 0 & -1 & & -i \\ & 0 & & \\ & i & 0 & -1 \\ & & & 0 \end{bmatrix}.$$

We have

$$\mathfrak{k}_{\mathbb{C}} = \langle U_1, U_2, P_+, P_- \rangle \oplus \langle V_1, V_2, Q_+, Q_- \rangle \cong \mathfrak{gl}(2, \mathbb{C}) \oplus \mathfrak{gl}(2, \mathbb{C}). \tag{59}$$

The center of $\mathfrak{k}$ is 2-dimensional, spanned by

$$i(U_1 + U_2 + V_1 + V_2) = \begin{bmatrix} & 1 & 1 \\ -1 & & \\ & -1 & \end{bmatrix} \qquad \text{and} \qquad i(U_1 + U_2 - V_1 - V_2) = \begin{bmatrix} i & & \\ & i & \\ & & i \end{bmatrix}. \tag{60}$$

The Casimir operators for the two copies of $\mathfrak{sl}(2, \mathbb{C})$ are given by

$$\Delta_1 = (U_1 - U_2)^2 + 2(P_+ P_- + P_- P_+), \qquad \Delta_2 = (V_1 - V_2)^2 + 2(Q_+ Q_- + Q_- Q_+).$$

The irreducible representations of $\mathfrak{k}_{\mathbb{C}}$ which lift to representations of $K_\infty^{G_2}$ are parametrized by four integers,

$$m_1: \text{ highest weight of } \langle U_1 - U_2, P_+, P_- \rangle, \quad n_1: \text{ eigenvalue of } U_1 + U_2,$$
$$m_2: \text{ highest weight of } \langle V_1 - V_2, Q_+, Q_- \rangle, \quad n_2: \text{ eigenvalue of } V_1 + V_2,$$

subject to the condition that they all have the same parity and that $m_1, m_2 \geq 0$. The parity condition is a consequence of overlapping one-parameter subgroups generated by $U_1 \pm U_2$ and $V_1 \pm V_2$. Let $\rho_{m_1, n_1, m_2, n_2}$ be the irreducible representation of $K_\infty^{G_2}$ corresponding to (m_1, n_1, m_2, n_2). Then $\dim \rho_{m_1, n_1, m_2, n_2} = (m_1 + 1)(m_2 + 1)$, and the contragredient representation is given by $\tilde{\rho}_{m_1, n_1, m_2, n_2} = \rho_{m_1, -n_1, m_2, -n_2}$.

1.3.1 Lemma. *Let m_1, n_1, m_2, n_2 be integers of the same parity with $m_1, m_2 \geq 0$.*

 i) *Any vector v in $\rho_{m_1, n_1, m_2, n_2}$ satisfies*

$$\Delta_1 v = m_1(m_1 + 2)v, \quad \Delta_2 v = m_2(m_2 + 2)v, \quad (U_1 + U_2)v = n_1 v, \quad (V_1 + V_2)v = n_2 v. \tag{61}$$

 ii) *The representation $\rho_{m_1, n_1, m_2, n_2}$ of $K_\infty^{G_2}$ contains the trivial representation of K_∞^H if and only if $m_1 = m_2$ and $n_1 = -n_2$. If these conditions are satisfied, then the trivial representation of K_∞^H appears in $\rho_{m_1, n_1, m_2, n_2}$ exactly once.*

Proof. Equation (61) holds by definition of $\rho_{m_1, n_1, m_2, n_2}$ and because the Casimir operator acts as the scalar $m(m + 2)$ on the irreducible representation of $\mathfrak{sl}(2, \mathbb{C})$ of dimension $m + 1$. The complexification of the Lie algebra of K_∞^H is given by $\mathfrak{k}_{\mathbb{C}}^H = \langle U_1 + V_1, U_2 + V_2, P_+ + Q_+, P_- + Q_- \rangle$. This Lie algebra is isomorphic to $\mathfrak{gl}(2, \mathbb{C})$ and sits diagonally in the product of the two copies of $\mathfrak{gl}(2, \mathbb{C})$ in (59). It follows that the restriction of the representation $\rho_{m_1, n_1, m_2, n_2}$ to $\mathfrak{k}_{\mathbb{C}}^H$ is given by $\rho_{m_1, n_2} \otimes \rho_{m_2, n_2}$, with each factor being a representation of $\mathfrak{k}_{\mathbb{C}}^H \cong \mathfrak{gl}(2, \mathbb{C})$. Such a tensor product contains the trivial representation (and then with multiplicity one) if and only if the second factor is the contragredient of the first, i.e., if and only if $(m_2, n_2) = (m_1, -n_1)$. ∎

Let m be a non-negative integer, and l and l_2 be any integers. Recall that l_2 determines the extension of the central character of τ to $\mathbb{C}^\times$. In our later applications l will indicate the scalar minimal K-type of a lowest weight representation of $\mathrm{GSp}_4(\mathbb{R})$, but for now l is just an integer. Let W_{m,l,l_2}^Δ be the space of smooth, $K_\infty^{G_2}$-finite functions $\Phi : K_\infty^{G_2} \to \mathbb{C}$ with the properties

$$\Phi(\hat{\zeta}_1 g) = \Phi(\hat{\zeta}_2 g) = \zeta^{l_2} \Phi(g) \qquad \text{for } g \in K_\infty^{G_2}, \ \zeta \in S^1, \tag{62}$$

$$\Phi(gk) = \det(J(k, i_2))^{-l}\Phi(g) \qquad \text{for } g \in K_\infty^{G_2}, \ k \in K_\infty^H, \tag{63}$$

$$\Delta_1\Phi = \Delta_2\Phi = m(m+2)\Phi. \tag{64}$$

In (64), the Casimir elements Δ_i are understood to act by right translation. As noted above, property (62) is required to extend Φ to an element of $I_\Phi(s, \chi, \chi_0, \tau)$. Property (63) will become important when we evaluate local zeta integrals in Sect. 2.2. Imposing the additional condition (64) will result in a certain uniqueness which is useful for calculating intertwining operators; see Sect. 1.5. Evidently, the group consisting of all elements

$$\hat{r}(\theta) := m_2\left(\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}\right), \qquad \theta \in \mathbb{R},$$

acts on W_{m,l,l_2}^Δ by left translation. Let W_{m,l,l_2,l_1}^Δ be the subspace of W_{m,l,l_2}^Δ consisting of Φ with the additional property

$$\Phi(\hat{r}(\theta)g) = e^{il_1\theta}\Phi(g) \qquad \text{for } g \in G_2(\mathbb{R}), \ \theta \in \mathbb{R}. \tag{65}$$

Then

$$W_{m,l,l_2}^\Delta = \bigoplus_{l_1 \in \mathbb{Z}} W_{m,l,l_2,l_1}^\Delta. \tag{66}$$

Let D be the function on $K_\infty^{G_2}$ given by $D(g) = \det(J(g, i_2))$. It is easily verified that

$$(U_1 - U_2)D = (V_1 - V_2)D = P_\pm D = Q_\pm D = 0.$$

Hence $\Delta_i D = 0$ for $i = 1, 2$, and consequently $\Delta_i(fD^l) = (\Delta_i f)D^l$ for any smooth function f on $K_\infty^{G_2}$. It is then easy to see that the map $\Phi \mapsto \Phi D^l$ provides isomorphisms

$$W_{m,l,l_2}^\Delta \xrightarrow{\sim} W_{m,0,l_2+l}^\Delta \qquad \text{and} \qquad W_{m,l,l_2,l_1}^\Delta \xrightarrow{\sim} W_{m,0,l_2+l,l_1-l}^\Delta. \tag{67}$$

Let $L^2(K_\infty^{G_2})_{\text{fin}}$ be the space of smooth, $K_\infty^{G_2}$-finite functions $K_\infty^{G_2} \to \mathbb{C}$. It is a module for $K_\infty^{G_2} \times K_\infty^{G_2}$ via $((g_1, g_2).f)(h) = f(g_1^{-1}hg_2)$. By the Peter-Weyl theorem, as $K_\infty^{G_2} \times K_\infty^{G_2}$-modules,

$$L^2(K_\infty^{G_2})_{\text{fin}} \cong \bigoplus_\rho (\tilde{\rho} \otimes \rho) \qquad \text{(algebraic direct sum)},$$

where ρ runs through all equivalence classes of irreducible representations of $K_\infty^{G_2}$, and where $\tilde{\rho}$ denotes the contragredient. Evidently,

$$W_{m,0,l_2+l,l_1-l}^\Delta = \bigoplus_\rho \left(W_{m,0,l_2+l,l_1-l}^\Delta \cap (\tilde{\rho} \otimes \rho)\right), \tag{68}$$

and analogously for $W_{m,0,l_2+l}^\Delta$.

1.3.2 Lemma. *Let m be a non-negative integer, and l and l_2 be any integers.*

i) Let $\rho = \rho_{m_1,n_1,m_2,n_2}$. Then, for $l_1 \in \mathbb{Z}$,

$$\dim\left(W_{m,0,l_2+l,l_1-l}^\Delta \cap (\tilde{\rho} \otimes \rho)\right) = \begin{cases} 1 & \text{if } m_1 = m_2 = m, \ n_1 = l_2 + l, \ n_2 = -(l_2 + l), \\ & |l_1 - l| \leq m, \ l_1 - l \equiv l_2 + l \equiv m \bmod 2, \\ 0 & \text{otherwise.} \end{cases}$$

ii) For $l_1 \in \mathbb{Z}$,

$$\dim \left(W^{\Delta}_{m,0,l_2+l,l_1-l} \right) = \begin{cases} 1 & \text{if } |l_1 - l| \leq m, \ l_1 - l \equiv l_2 + l \equiv m \text{ mod } 2, \\ 0 & \text{otherwise.} \end{cases}$$

iii)

$$\dim \left(W^{\Delta}_{m,l,l_2} \right) = \begin{cases} m+1 & \text{if } l_2 + l \equiv m \text{ mod } 2, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. i) By the right K^H_∞-invariance of functions in $W^{\Delta}_{m,0,l_1-l,l_2+l}$ and Lemma 1.3.1 ii), if $\tilde{\rho} \otimes \rho$ contributes to $W^{\Delta}_{m,0,l_2+l,l_1-l}$, then necessarily $m_1 = m_2$ and $n_1 = -n_2$. Condition (64) forces $m_1 = m_2 = m$. Assume all of this is satisfied, say $\rho = \rho_{m,n,m,-n}$. Then, by Lemma 1.3.1 ii), there exists a non-zero vector $v_0 \in \rho$, unique up to multiples, such that v_0 is fixed by K^H. Hence, any element $w \in W^{\Delta}_{m,0,l_2+l,l_1-l} \cap (\tilde{\rho} \otimes \rho)$ is of the form $w = v \otimes v_0$ for some $v \in \tilde{\rho} = \rho_{m,-n,m,n}$. Taking into account that the first element of the center of $\mathfrak{k}_{\mathbb{C}}$ in (60) acts trivially on $W^{\Delta}_{m,0,l_2+l,l_1-l}$, any element Φ of this space has the following transformation properties under left translation L,

$$L(U_1 - U_2)\Phi = (l_1 - l)\Phi, \qquad L(V_1 - V_2)\Phi = (l_1 - l)\Phi, \qquad (69)$$
$$L(U_1 + U_2)\Phi = -(l_2 + l)\Phi, \qquad L(V_1 + V_2)\Phi = (l_2 + l)\Phi. \qquad (70)$$

Since $U_1 + U_2$ and $V_1 + V_2$ are in the center, (70) implies that $R(U_1 + U_2)\Phi = (l_2 + l)\Phi$ and $R(V_1 + V_2)\Phi = -(l_2 + l)\Phi$, where R is right translation. It follows that $n = l_2 + l$. This number must have the same parity as m. From (69) we conclude that v is a vector of weight $(l_1 - l, l_1 - l)$ in $\tilde{\rho}$. There exists such a vector v in $\tilde{\rho}$ if and only if $-m \leq l_1 - l \leq m$ and $l_1 - l \equiv m$ mod 2, and in this case v is unique up to multiples.

ii) follows from i) and (68).

iii) For $l = 0$ the statement follows from ii) and (66). For other values of l, it follows from the $l = 0$ case and (67). ∎

Our next task will be to find an explicit formula for the function spanning the one-dimensional space W^{Δ}_{m,l,l_2,l_1}. We define, for $g \in K^{G_2}_\infty$,

$$\hat{a}(g) = (1,1)\text{--coefficient of } J(g\,^t g, i_2), \qquad \hat{b}(g) = (1,2)\text{--coefficient of } J(g\,^t g, i_2),$$
$$\hat{c}(g) = (2,1)\text{--coefficient of } J(g\,^t g, i_2), \qquad \hat{d}(g) = (2,2)\text{--coefficient of } J(g\,^t g, i_2).$$

Since they can be written in terms of matrix coefficients, these are $K^{G_2}_\infty$-finite functions. It is not difficult to calculate the action of $P_\pm, Q_\pm$, the torus elements and the Casimir elements on the functions $\hat{a}, \hat{b}, \hat{c}, \hat{d}$ under left and right translation explicitly. The following lemma summarizes the results.

1.3.3 Lemma. *Let m be a non-negative integer.*

i) *If $f = \hat{a}^{i_1} \hat{b}^{i_2} \hat{c}^{i_3} \hat{d}^{i_4}$ with non-negative integers $i_1, \ldots, i_4$ such that $i_1 + i_2 + i_3 + i_4 = m$, then, under right translation,*

$$\Delta_i f^m = m(m+2)f^m \qquad \text{for } i = 1, 2.$$

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ii) The functions f as in i) are contained in $\tilde{\rho} \otimes \rho$ with $\rho = \rho_{m,m,m,-m}$ and are right invariant under K_∞^H.

iii) Let $f = \hat{b}^{m-j} \hat{c}^j$ with $0 \leq j \leq m$. Then, with L being left translation,

$$L(U_1 - U_2)f = (m - 2j)f, \qquad L(V_1 - V_2)f = (m - 2j)f, \tag{71}$$
$$L(U_1 + U_2)f = -mf, \qquad L(V_1 + V_2)f = mf. \tag{72}$$

Using this lemma, it is easy to verify that the function

$$\hat{b}^{\frac{m+l_1-l}{2}} \hat{c}^{\frac{m-l_1+l}{2}} (\hat{a}\hat{d} - \hat{b}\hat{c})^{\frac{l_2+l-m}{2}}$$

lies in $W^\Delta_{m,0,l_2+l,l_1-l}$, provided all exponents are integers and the first two are non-negative. In view of (67), we obtain the following result.

1.3.4 Proposition. *Let m be a non-negative integer, and l, l_1, l_2 be any integers. We assume that $|l_1 - l| \leq m$ and $l_1 - l \equiv l_2 + l \equiv m \bmod 2$, so that the space W^Δ_{m,l,l_2,l_1} is one-dimensional. Then this space is spanned by the function*

$$\Phi^\#_{m,l,l_2,l_1} := (-i)^m \, \hat{b}^{\frac{m+l_1-l}{2}} \, \hat{c}^{\frac{m-l_1+l}{2}} (\hat{a}\hat{d} - \hat{b}\hat{c})^{\frac{l_2+l-m}{2}} D^{-l}, \tag{73}$$

where $D(g) = \det(J(g, i_2))$. This function has the property that

$$\Phi^\#_{m,l,l_2,l_1}(\eta_0) = 1, \tag{74}$$

with η_0 as in (17).

Special vectors in $I(s, \chi, \chi_0, \tau)$

We return to the induced representation $I_\Phi(s, \chi, \chi_0, \tau)$, considered a subspace of the Borel induced representation $I_\Phi(s, \chi, \chi_0, \beta_1 \times \beta_2)$. Since the functions $\Phi^\#_{m,l,l_2,l_1}$ defined in Proposition 1.3.4 satisfy condition (58), they extend to elements of $I_\Phi(s, \chi, \chi_0, \beta_1 \times \beta_2)$. We use the same notation for the extended functions.

1.3.5 Lemma. *The function $\Phi^\#_{m,l,l_2,l_1}$ belongs to $I_\Phi(s, \chi, \chi_0, \tau)$ if and only if the weight l_1 occurs in τ.*

Proof. As a subspace of $I_\Phi(s, \chi, \chi_0, \beta_1 \times \beta_2)$, the representation $I_\Phi(s, \chi, \chi_0, \tau)$ consists of all functions $\Phi : G_2 \to \mathbb{C}$ of the form

$$\Phi(m_1 m_2 nk) = \delta_P(m_1 m_2)^{s+1/2} \chi(m_1) \varphi(m_2) J(k), \qquad m_i \in M_i(\mathbb{R}), \, n \in N(\mathbb{R}), \, k \in K_\infty^{G_2},$$

where φ lies in $\chi_0 \times \tau$, and where J is an appropriate function on $K_\infty^{G_2}$. It follows that $\Phi \in I_\Phi(s, \chi, \chi_0, \beta_1 \times \beta_2)$ lies in $I_\Phi(s, \chi, \chi_0, \tau)$ if and only if the function

$$M_2(\mathbb{R}) \ni m_2 \longmapsto \Phi(m_2)\delta_P(m_2)^{-s-1/2}$$

belongs to $\chi_0 \times \tau$. Since $\Phi^\#_{m,l,l_2,l_1}$ satisfies (65), the function $m_2 \mapsto \Phi^\#_{m,l,l_2,l_1}(m_2)$ has weight l_1. The assertion follows. ∎

For simplicity, we will from now on let $\mathbf{c} = 1$ for the rest of this section; this is all we need for the global application. Let the classical Whittaker function $W_{k,m}$ be the same as in [7, p. 244] or [58, 7.1.1]. We fix a point $t^+ \in \mathbb{R}_{>0}$, depending on p, such that

$$W_{\pm \frac{l_1}{2}, \frac{p}{2}}(t^+) \neq 0 \qquad \text{for all } l_1 \in \mathbb{Z}. \tag{75}$$

Note that, if p is a positive integer (corresponding to τ being a discrete series representation), we can choose $t^+ = 1$, since $W_{\pm \frac{l_1}{2}, \frac{p}{2}}$ is essentially an exponential function. Let W_{l_1} be the vector of weight l_1 in the ψ^{-1} Whittaker model of τ. Using differential operators and solving differential equations, one can show that there exist constants $a^+, a^- \in \mathbb{C}$ such that

$$W_{l_1}(\begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix}) = \begin{cases} a^+ \omega_\tau((4\pi t)^{1/2}) W_{\frac{l_1}{2}, \frac{p}{2}}(4\pi t) & \text{if } t > 0, \\ a^- \omega_\tau((-4\pi t)^{1/2}) W_{-\frac{l_1}{2}, \frac{p}{2}}(-4\pi t) & \text{if } t < 0. \end{cases} \tag{76}$$

Our choice of additive character implies that a^+ is non-zero as long as $l_1 > 0$. We will normalize the constant $a^+ = a^+_{l_1, p, q}$ such that

$$W_{l_1}(\begin{bmatrix} t^+ & \\ & 1 \end{bmatrix}) = 1.$$

i.e.,

$$a^+ = a^+_{l_1, p, q} = \left(\omega_\tau((4\pi t^+)^{1/2}) W_{\frac{l_1}{2}, \frac{p}{2}}(4\pi t^+) \right)^{-1} = (4\pi t^+)^{-q/2} W_{\frac{l_1}{2}, \frac{p}{2}}(4\pi t^+)^{-1}. \tag{77}$$

Consider the Whittaker realization $I_W(s, \chi, \chi_0, \tau)$ of $I(s, \chi, \chi_0, \tau)$, with τ given in its ψ^{-1} Whittaker model (see (35), (36)). We extent W_{l_1} to a function on $G_1(\mathbb{R})$ via the character χ_0; see (33). Using the Iwasawa decomposition, we define a function $W^\#_{m,l,l_2,l_1}$ in $I_W(s, \chi, \chi_0, \tau)$ by

$$W^\#_{m,l,l_2,l_1}(m_1 m_2 nk, s) = (t^+)^{q/2} \delta_P^{s+1/2}(m_1 m_2) \chi(m_1) W_{l_1}(m_2) \Phi^\#_{m,l,l_2,l_1}(k), \tag{78}$$

where $m_1 \in M^{(1)}(\mathbb{R})$, $m_2 \in M^{(2)}(\mathbb{R})$, $n \in N(\mathbb{R})$ and $k \in K^{G_2}_\infty$, and where $\Phi^\#_{m,l,l_2,l_1}$ is the same function as in (73); this is well-defined by the transformation properties of W_{l_1} and $\Phi^\#_{m,l,l_2,l_1}$. Note that

$$W^\#_{m,l,l_2,l_1}(\text{diag}(\sqrt{t^+}, t^+, \sqrt{t^+}, 1)\eta_0, s) = 1.$$

There is an intertwining operator $\Phi \mapsto W_\Phi$ from $I_\Phi(s, \chi, \chi_0, \tau)$ to $I_W(s, \chi, \chi_0, \tau)$, which, in the region of convergence, is given by

$$W_\Phi(g) = \int_\mathbb{R} e^{-2\pi i x} \Phi(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ -1 & & & \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & x \\ & & 1 & \\ & & & 1 \end{bmatrix} g)\, dx. \tag{79}$$

Outside the region of convergence the intertwining operator is given by the analytic continuation of this integral. This operator is simply an extension of a standard intertwining operator for the underlying $GL_2(\mathbb{R})$ representation $\beta_1 \times \beta_2$. It is easy to see that under this intertwining operator the function $\Phi^\#_{m,l,l_2,l_1}$ maps to a multiple of $W^\#_{m,l,l_2,l_1}$. Let $\kappa_{l_1,p,q}$ be the constant such that

$$W_{\Phi^\#_{m,l,l_2,l_1}} = \kappa_{l_1,p,q} W^\#_{m,l,l_2,l_1} \tag{80}$$

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We will distinguish three disjoint cases A, B, C according to the type of τ and the constellation of its weights relative to the integer l (which later will be a minimal GSp_4 weight).

- **Case A**: Neither the weight l nor the weight $l-1$ occur in τ.
- **Case B**: The weight l occurs in τ.
- **Case C**: The weight $l-1$ occurs in τ.

(81)

Note that in Case A necessarily $\tau = \mathcal{D}_{p,q/2}$, a discrete series representation with Harish-Chandra parameter $p \geq l$ (and central character satisfying $a \mapsto a^q$ for $a > 0$). In this case let us set $l_1 = p+1$, which is the minimal weight. It satisfies $l_1 \geq 2$. In each of the three cases we will define a non-negative integer m and a distinguished function $\Phi^\#$ as a linear combination of certain $\Phi^\#_{m,l,l_2,l_1}$ as in (73). The definition is as in the following table. The last column of the table shows $W^\#$, by definition the image of $\Phi^\#$ under the intertwining operator $\Phi \mapsto W_\Phi$.

Case	m	$\Phi^\#$	$W^\#$
A	$l_1 - l$	$\Phi^\#_{m,l,l_2,l_1}$	$\kappa_{l_1,p,q} W^\#_{m,l,l_2,l_1}$
B	0	$\Phi^\#_{m,l,l_2,l}$	$\kappa_{l,p,q} W^\#_{m,l,l_2,l}$
C	1	$\Phi^\#_{m,l,l_2,l+1} + \left(3s - \frac{p+q}{2}\right)\Phi^\#_{m,l,l_2,l-1}$	$\kappa_{l+1,p,q} W^\#_{m,l,l_2,l+1}$ $+\left(3s - \frac{p+q}{2}\right)\kappa_{l-1,p,q} W^\#_{m,l,l_2,l-1}$

(82)

In all cases, by Lemma 1.3.5, the function $\Phi^\#$ lies in $I_\Phi(s,\chi,\chi_0,\tau)$.

1.3.6 Theorem. *Let (τ, V_τ) be a generic, irreducible, admissible representation of $\mathrm{GL}_2(\mathbb{R})$ with central character ω_τ. We realize τ as a subrepresentation of an induced representation $\beta_1 \times \beta_2$, and define $p, q \in \mathbb{C}$ by (57). Let l_2 be an integer of the same parity as the weights of τ. Let χ_0 be the character of $\mathbb{C}^\times$ such that $\chi_0\big|_{\mathbb{R}^\times} = \omega_\tau$ and $\chi_0(\zeta) = \zeta^{l_2}$ for $\zeta \in S^1$, and let χ be the character of $\mathbb{C}^\times$ defined by (56). Assume that l is a positive integer. Let m and $\Phi^\#$ be chosen according to table (82).*

i) *The function $\Phi^\#$ satisfies*

$$\Phi^\#(gk) = \det(J(k,i_2))^{-l}\Phi^\#(g) \qquad \text{for } g \in G_2(\mathbb{R}),\ k \in K_\infty^H \tag{83}$$

and

$$\Delta_1 \Phi^\# = \Delta_2 \Phi^\# = m(m+2)\Phi^\#. \tag{84}$$

ii) *Assume we are in Case A or B. Then, up to scalars, $\Phi^\#$ is the unique $K_\infty^{G_2}$-finite element of $I_\Phi(s,\chi,\chi_0,\tau)$ with the properties (83) and (84).*

iii) *Assume we are in Case C. Then the space of $K_\infty^{G_2}$-finite functions in $I_\Phi(s,\chi,\chi_0,\tau)$ with the properties (83) and (84) is two-dimensional, spanned by $\Phi^\#_{m,l,l_2,l-1}$ and $\Phi^\#_{m,l,l_2,l+1}$.*

Proof. i) is obvious, since $\Phi^\#$ lies in W^Δ_{m,l,l_2}.

ii) Assume first we are in Case A. By our hypotheses, $0 < l < l_1$. Assume that $\Phi \in I_\Phi(s,\chi,\chi_0,\tau)$ is $K_\infty^{G_2}$-finite and satisfies (83) and (84). Then, evidently, the restriction of Φ to $K_\infty^{G_2}$ lies in W^Δ_{m,l,l_2}.

By (66) and Proposition 1.3.4,

$$W^{\Delta}_{m,l,l_2} = \bigoplus_{\substack{j \in \mathbb{Z} \\ |j-l| \leq m \\ j-l \equiv m \bmod 2}} \mathbb{C}\Phi^{\#}_{m,l,l_2,j}.$$

If a $\Phi^{\#}_{m,l,l_2,j}$ occurring in this direct sum is an element of $I_{\Phi}(s,\chi,\chi_0,\tau)$, then, by Lemma 1.3.5, the weight j occurs in τ. Since τ has minimal weight l_1, this implies $j \leq -l_1$ or $j \geq l_1$. The first inequality leads to a contradiction, and the second inequality implies $j = l_1$. This proves the uniqueness in Case A. In Case B, as before, the restriction of any $K^{G_2}_{\infty}$-finite $\Phi \in I_{\Phi}(s,\chi,\chi_0,\tau)$ satisfying (83) and (84) to $K^{G_2}_{\infty}$ lies in W^{Δ}_{m,l,l_2}. By Lemma 1.3.2, this space is one-dimensional.

iii) Again, the restriction of any $K^{G_2}_{\infty}$-finite $\Phi \in I_{\Phi}(s,\chi,\chi_0,\tau)$ satisfying (83) and (84) to $K^{G_2}_{\infty}$ lies in W^{Δ}_{m,l,l_2}. By Lemma 1.3.2, this space is two-dimensional. ∎

Since the functions $\Phi^{\#}$ and $W^{\#}$ have the same right transformation properties, the following is an immediate consequence of Theorem 1.3.6.

1.3.7 Corollary. *Let the non-negative integer m and the function $W^{\#}$ in $I_W(s,\chi,\chi_0,\tau)$ be chosen according to table (82).*

i) *The function $W^{\#}$ satisfies*

$$W^{\#}(gk) = \det(J(k,i_2))^{-l} W^{\#}(g) \qquad \text{for } g \in G_2(\mathbb{R}), \ k \in K^H_{\infty} \tag{85}$$

and

$$\Delta_1 W^{\#} = \Delta_2 W^{\#} = m(m+2)W^{\#}. \tag{86}$$

ii) *Assume we are in Case A or B. Then, up to scalars, $W^{\#}$ is the unique $K^{G_2}_{\infty}$-finite element of $I_W(s,\chi,\chi_0,\tau)$ with the properties (85) and (86).*

iii) *Assume we are in Case C. Then the space of $K^{G_2}_{\infty}$-finite functions in $I_W(s,\chi,\chi_0,\tau)$ with the properties (85) and (86) is two-dimensional, spanned by $W^{\#}_{m,l,l_2,l-1}$ and $W^{\#}_{m,l,l_2,l+1}$.*

A relation between unknown constants

In this section we defined the constants $\kappa_{l_1,p,q}$ and $a^+_{l_1,p,q}$; see (80) and (77). Note that these constants also depend on the choice of the point t^+, which is not reflected in the notation. We do not know the explicit value of any of these constants. However, the following lemma describes a relation between these constants which will become important in the proof of Lemma 2.4.2.

1.3.8 Lemma. *Let β_1 and β_2 be characters of $\mathbb{R}^{\times}$ such that the induced representation $\beta_1 \times \beta_2$ is irreducible. Let $p,q \in \mathbb{C}$ be as in (57). Then, for any integer l whose parity is different from the parity of the weights of $\beta_1 \times \beta_2$,*

$$\frac{\kappa_{l-1,p,q}\, a^+_{l-1,p,q}}{\kappa_{l+1,p,q}\, a^+_{l+1,p,q}} = -\frac{1}{2}(p+l). \tag{87}$$

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Proof. We consider the intertwining operator $\varphi \mapsto W_\varphi$ from τ to the Whittaker model $\mathcal{W}(\beta_1 \times \beta_2, \psi^{-\mathbf{c}})$ which, in the region of convergence, is given by

$$W_\varphi(g) = \int_{\mathbb{R}} e^{-2\pi i x} \varphi\left(\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}\begin{bmatrix} 1 & x \\ & 1 \end{bmatrix} g\right) dx. \tag{88}$$

For a weight k occurring in $\beta_1 \times \beta_2$ let φ_k be the element of $\beta_1 \times \beta_2$ of weight k satisfying $\varphi_k(1) = 1$, and let W_k be the element of $\mathcal{W}(\beta_1 \times \beta_2, \psi^{-1})$ of weight k satisfying $W_k(\begin{bmatrix} t^+ & \\ & 1 \end{bmatrix}) = 1$. Then $W_{\varphi_k} = \kappa_{k,p,q}(t^+)^{q/2} W_k$ with the same $\kappa_{k,p,q}$ as in (80).

Recall that the constants $a^+_{k,p,q}$ defined in (77) were designed so that

$$w_k(t) := W_k(\begin{bmatrix} t & \\ & 1 \end{bmatrix}) = a^+_{k,p,q}\,(4\pi t)^{q/2}\, W_{\frac{k}{2}, \frac{p}{2}}(4\pi t) \tag{89}$$

satisfies $w_k(t^+) = 1$. If L denotes the Lie algebra element $\frac{1}{2}\begin{bmatrix} 1 & -i \\ -i & -1 \end{bmatrix}$, then straightforward calculations show that

$$\tau(L)\varphi_k = \frac{p+1-k}{2}\varphi_{k-2}, \qquad (\tau(L)W_k)(\begin{bmatrix} t & \\ & 1 \end{bmatrix}) = \left(-\frac{q}{2} - \frac{k}{2} + 2\pi t\right)w_k(t) + t w_k'(t) \tag{90}$$

(where τ stands for the right translation action on both $\beta_1 \times \beta_2$ and its Whittaker model). For k one of the weights appearing in $\beta_1 \times \beta_2$, define constants $\lambda_{k,p,q}$ and $\mu_{k,p,q}$ by

$$\tau(L)\varphi_k = \lambda_{k,p,q}\,\varphi_{k-2} \qquad \text{and} \qquad \tau(L)W_k = \mu_{k,p,q}W_{k-2}.$$

By our normalizations, $\lambda_{k,p,q} = (\tau(L)\varphi_k)(1)$ and $\mu_{k,p,q} = (\tau(L)W_k)(\begin{bmatrix} t^+ & \\ & 1 \end{bmatrix})$. Hence, by (90),

$$\lambda_{k,p,q} = \frac{p+1-k}{2}, \qquad \mu_{k,p,q} = -\frac{q}{2} - \frac{k}{2} + 2\pi t^+ + t^+ w_k'(t^+). \tag{91}$$

Following the function φ_k through the commutative diagram

$$
\begin{array}{ccc}
\beta_1 \times \beta_2 & \xrightarrow{\ \tau(L)\ } & \beta_1 \times \beta_2 \\
\big\downarrow & & \big\downarrow \\
\mathcal{W}(\tau, \psi^{-1}) & \xrightarrow[\tau(L)]{} & \mathcal{W}(\tau, \psi^{-1})
\end{array}
$$

we get the identity

$$\kappa_{k,p,q}\,\mu_{k,p,q} = \lambda_{k,p,q}\,\kappa_{k-2,p,q}. \tag{92}$$

To further calculate the constant $\mu_{k,p,q}$, we will take the derivative of the function w_k defined in (89). We will make use of the following identity for Whittaker functions,

$$z\,W_{k,b}'(z) = \left(k - \frac{z}{2}\right)W_{k,b}(z) - \left(b^2 - \left(k - \frac{1}{2}\right)^2\right)W_{k-1,b}(z) \tag{93}$$

(see [58, 7.2.1]). Using this, one obtains from (77), (89) and (91) that

$$\mu_{k,p,q} = -\frac{p^2 - (k-1)^2}{4}\,\frac{a^+_{k,p,q}}{a^+_{k-2,p,q}}.$$

Substituting the values of $\lambda_{k,p,q}$ and $\mu_{k,p,q}$ into (92) and setting $k = l + 1$ proves the asserted identity. ∎

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1.4 Intertwining operator: non-archimedean case

In this section let F be p-adic. We use the notation from Theorem 1.2.3. In addition, we will assume that $\Lambda\big|_{F^\times} = 1$; this will be sufficient for our global applications. In this section we will calculate the function $K(s)$ given in (42).

Let us be precise about the measure on $N(F)$. Recall that $N(F)$ consists of one copy of F and two copies of L. The measure on F is the one that is self-dual with respect to the character ψ, and the measure on L is the one that is self-dual with respect to the character $\psi \circ \mathrm{tr}_{L/F}$. Since we are assuming that ψ has conductor $\mathfrak{o}$, it follows (see Sect. 2.2 of [86]) that

$$\mathrm{vol}(\mathfrak{o}) = 1 \qquad \text{and} \qquad \mathrm{vol}(\mathfrak{o}_L) = N(\mathbf{d})^{-1/2}. \tag{94}$$

Recall here that the norm of the different is the discriminant, and that $\mathbf{d} = \mathbf{b}^2 - 4\mathbf{ac}$ generates the discriminant of L/F by our conventions. If we let $\mathbf{d}\mathfrak{o} = \varpi^\delta \mathfrak{o}$ (where $\delta = 0$ unless L/F is a ramified field extension), then $\mathrm{vol}(\mathfrak{o}_L) = q^{-\delta/2}$. This explains the factor $q^{-\delta}$ in the following result.

1.4.1 Proposition. (Gindikin-Karpelevich Formula) *Let δ be the valuation of the discriminant of L/F if L/F is a ramified field extension, and $\delta = 0$ otherwise. If τ is unramified, then*

$$K(s) = q^{-\delta} \frac{L(6s, \chi\big|_{F^\times})L(3s, \tau \times \mathcal{AI}(\Lambda) \times \chi\big|_{F^\times})}{L(6s+1, \chi\big|_{F^\times})L(3s+1, \tau \times \mathcal{AI}(\Lambda) \times \chi\big|_{F^\times})}.$$

This formula can be obtained by a straightforward integral calculation; we omit the details. For non-spherical τ it will be necessary to distinguish the inert, split and ramified cases. For our global applications it turns out that explicit knowledge of $K(s)$ at finitely many finite places is not necessary. Thus, we will only calculate $K(s)$ in the inert and split cases.

We will first assume that L/F is an unramified field extension. We write the explicit formula (43) as $I_1 + I_2$, where in I_1 the z-integration is restricted to the set $\mathfrak{o}_L$, and in I_2 the z-integration is restricted to $L \setminus \mathfrak{o}_L$. After some changes of variables, we get

$$I_1 = \int_{\mathfrak{o}_L} \int_L \int_F W^{\#} \left(\begin{bmatrix} 1 \\ y & 1 \\ x & & 1 & -\bar{y} \\ & & & 1 \end{bmatrix} w_1 \begin{bmatrix} 1 & z \\ & 1 \\ & & 1 \\ & & -\bar{z} & 1 \end{bmatrix} \eta, s \right) dz\, dy\, dx \tag{95}$$

and

$$I_2 = \int_{L \setminus \mathfrak{o}_L} \int_L \int_F |z\bar{z}|\, W^{\#} \left(\begin{bmatrix} \bar{z}^{-1} & & & 1 \\ & z^{-1} & 1 \\ & z \\ & & & \bar{z} \end{bmatrix} \begin{bmatrix} 1 \\ & 1 & & xz\bar{z} - \bar{y}z - y\bar{z} \\ & & 1 \\ & & & 1 \end{bmatrix} \right.$$
$$\left. \times \begin{bmatrix} 1 \\ y & 1 \\ x & & 1 & -\bar{y} \\ & & & 1 \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & -1 \\ & -1 \\ -1 & & & \end{bmatrix} \begin{bmatrix} 1 \\ z^{-1} & 1 \\ & & 1 & -\bar{z}^{-1} \\ & & & 1 \end{bmatrix} \eta, s \right) dx\, dy\, dz. \tag{96}$$

The argument of $W^{\#}$ needs to be written as pk, where $p \in P(F)$ and $k \in K^{G_2}$. For both I_1 and I_2 the key is decomposing the matrix $g = \begin{bmatrix} 1 \\ y & 1 \\ x & & 1 & -\bar{y} \\ & & & 1 \end{bmatrix}$ in this way. There are five cases depending

on the values of x and y. For instance, if $x \in \mathfrak{o}, y \in \mathfrak{o}_L$ then g already lies in K^{G_2}. On the other hand if $x \in \mathfrak{o}, y \notin \mathfrak{o}_L$ then

$$g = \begin{bmatrix} -y^{-1} & -1 & & \\ & -y & & \\ & & -\bar{y} & \\ & & 1 & -\bar{y}^{-1} \end{bmatrix} \begin{bmatrix} & & 1 & \\ -1 & -y^{-1} & & \\ -x\bar{y}^{-1} & & -\bar{y}^{-1} & 1 \\ -x & & -1 & \end{bmatrix}. \tag{97}$$

Similar matrix identities (which we omit for reasons of brevity) exist in the three remaining cases

$$x \notin \mathfrak{o},\ y \in \mathfrak{o}_L, \qquad x \notin \mathfrak{o},\ y \notin \mathfrak{o}_L,\ yx^{-1} \in \mathfrak{o}_L, \qquad x \notin \mathfrak{o},\ y \notin \mathfrak{o}_L,\ yx^{-1} \notin \mathfrak{o}_L.$$

We now have ten cases, five for $z \in \mathfrak{o}_L$ and five for $z \in L \setminus \mathfrak{o}_L$. In each case let k denote the K^{G_2} component of the argument of $W^{\#}$. Using the fact that $W^{\#}$ is supported on $P(F)\eta_0 K^H \Gamma(\mathfrak{P}^n)$ gives the following conditions on k. The notation is such that $y = y_1 + \alpha y_2$ and $z = z_1 + \alpha z_2$ with $y_1, y_2, z_1, z_2 \in \mathfrak{o}$.

case	x	y	z	yx^{-1}	condition for k to be in the support of $W^{\#}$
$i)$	$\in \mathfrak{o}$	$\in \mathfrak{o}_L$	$\in \mathfrak{o}_L$		$y_2 + (x + yz + \bar{y}\bar{z}) \in \mathfrak{o}^{\times}$ or $z_2 - z\bar{z} \in \mathfrak{o}^{\times}$
$ii)$	$\in \mathfrak{o}$	$\notin \mathfrak{o}_L$	$\in \mathfrak{o}_L$		$z_2 - z\bar{z} \in \mathfrak{o}^{\times}$
$iii)$	$\notin \mathfrak{o}$	$\in \mathfrak{o}_L$	$\in \mathfrak{o}_L$		always
$iv)$	$\notin \mathfrak{o}$	$\notin \mathfrak{o}_L$	$\in \mathfrak{o}_L$	$\in \mathfrak{o}_L$	$\frac{y_2}{x} - \frac{y\bar{y}}{x^2}z_2 + \left(\frac{x+yz}{x}\right)\left(\frac{x+\bar{y}\bar{z}}{x}\right) \in \mathfrak{o}^{\times}$
$v)$	$\notin \mathfrak{o}$	$\notin \mathfrak{o}_L$	$\in \mathfrak{o}_L$	$\notin \mathfrak{o}_L$	$\frac{1}{\alpha-\bar{\alpha}}\left(\frac{x}{\bar{y}} - \frac{x}{y}\right) - z_2 + \left(\frac{x}{y} + z\right)\left(\frac{x}{\bar{y}} + \bar{z}\right) \in \mathfrak{o}^{\times}$
$vi)$	$\in \mathfrak{o}$	$\in \mathfrak{o}_L$	$\notin \mathfrak{o}_L$		always
$vii)$	$\in \mathfrak{o}$	$\notin \mathfrak{o}_L$	$\notin \mathfrak{o}_L$		always
$viii)$	$\notin \mathfrak{o}$	$\in \mathfrak{o}_L$	$\notin \mathfrak{o}_L$		never
$ix)$	$\notin \mathfrak{o}$	$\notin \mathfrak{o}_L$	$\notin \mathfrak{o}_L$	$\in \mathfrak{o}_L$	$\frac{y\bar{y}}{x^2} + \frac{y_2}{x} \in \mathfrak{o}^{\times}$
$x)$	$\notin \mathfrak{o}$	$\notin \mathfrak{o}_L$	$\notin \mathfrak{o}_L$	$\notin \mathfrak{o}_L$	always

According to these cases, $K(s)$ is the sum of ten integrals $I_{i)}, \ldots, I_{x)}$. By the support conditions, $I_{viii)} = 0$. We split the first case up into $i)a$, the case where $z_2 - z\bar{z} \in \mathfrak{o}^{\times}$, and $i)b$, the case where $z_2 - z\bar{z} \in \mathfrak{p}$ and $y_2 + (x + yz + \bar{y}\bar{z}) \in \mathfrak{o}^{\times}$. To evaluate the function $W^{\#}$ in I_1 and I_2, we will write the argument of $W^{\#}$ as $p\eta\kappa$ with $p \in P(F)$ and $\kappa \in K^H$. Only the p part is important for the evaluation. Once the argument of $W^{\#}$ is written as $p\eta\kappa$, it is straightforward to perform an initial evaluation of the integrals. We list only the results.

$$I_{i)a} = \left(\int_{\substack{\mathfrak{o}_L \\ z_2 - z\bar{z} \in \mathfrak{o}^{\times}}} \chi_0(z_2 - z\bar{z}) \, dz \right) W^{(0)}\left(\begin{bmatrix} & 1 \\ 1 & \end{bmatrix} \right)$$

$$I_{i)b} = \int_{\substack{\mathfrak{o}_L \\ z_2 - z\bar{z} \in \mathfrak{p}}} \int_{\mathfrak{o}^{\times}} W^{(0)}\left(\begin{bmatrix} x & \\ & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ z_2 - z\bar{z} & 1 \end{bmatrix} \right) dx \, dz$$

$$I_{ii)} = \int_{\substack{\mathfrak{o}_L \\ z_2 - z\bar{z} \in \mathfrak{o}^{\times}}} \int_{L \setminus \mathfrak{o}_L} |y|_L^{-3(s+\frac{1}{2})} \chi\left(\frac{1}{z_2 - z\bar{z}}\right) \Lambda(\bar{y}) W^{(0)}\left(\begin{bmatrix} y\bar{y} & y_2 + yz + \bar{y}\bar{z} \\ & 1 \end{bmatrix} \begin{bmatrix} & -1 \\ 1 & \end{bmatrix} \right) dy \, dz$$

$$I_{iii)} = \int_{\mathfrak{o}_L} \int_{\mathfrak{o}_L} \int_{F \setminus \mathfrak{o}} |x|^{-6(s+\frac{1}{2})} \chi(x^{-1}) W^{(0)}\left(\begin{bmatrix} \frac{y_2+x+yz+\bar{y}\bar{z}}{x} & \\ & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ \frac{z_2-z\bar{z}}{x} & 1 \end{bmatrix} \right) dx \, dy \, dz$$

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$$I_{iv)} = \int\limits_{\substack{\mathfrak{o}_L \\ u\in\mathfrak{o}^\times}} \int\limits_{\substack{\mathfrak{o}_L \\ yx\notin\mathfrak{o}_L}} \int\limits_{F\backslash\mathfrak{o}} |x|^{-6s-1}\chi(x^{-1})$$

$$W^{(0)}\left(\begin{bmatrix} 1 & y\bar{y}x \\ & 1 \end{bmatrix}\begin{bmatrix} y_2 - y\bar{y}z_2 + (1+yz)(1+\bar{y}\bar{z}) & \\ & 1 \end{bmatrix}\begin{bmatrix} 1 & \\ \frac{z_2-z\bar{z}}{x} & 1 \end{bmatrix}\right) dx\, dy\, dz$$

$$I_{v)} = \int\limits_{\substack{\mathfrak{o}_L \\ z_2-z\bar{z}\in\mathfrak{o}^\times}} \int\limits_{L\backslash\mathfrak{o}_L} \int\limits_{F\backslash\mathfrak{o}} |x|^{-6s-1}\chi(x^{-1})\,|y|_L^{-3(s+\frac{1}{2})}\Lambda(\bar{y})$$

$$W^{(0)}\left(\begin{bmatrix} 1 & y\bar{y}x \\ & 1 \end{bmatrix}\begin{bmatrix} y_2 - y\bar{y}z_2 + (1+yz)(1+\bar{y}\bar{z}) & \\ & 1 \end{bmatrix}\begin{bmatrix} 1 & \\ \frac{z_2-z\bar{z}}{x} & 1 \end{bmatrix}\right) dx\, dy\, dz$$

$$I_{vi)} = \int\limits_{L\backslash\mathfrak{o}_L} |z|_L^{-3s-\frac{1}{2}}\Lambda(z)W^{(0)}\left(\begin{bmatrix} 1 & \\ z\bar{z}(1-\frac{z_2}{z\bar{z}}) & \end{bmatrix}\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}\right) dz$$

$$I_{vii)} = \int\limits_{L\backslash\mathfrak{o}_L} \int\limits_{L\backslash\mathfrak{o}_L} |y|_L^{-3(s+\frac{1}{2})}|z|_L^{-3s-\frac{1}{2}}\Lambda(\bar{y}z)$$

$$\psi^{-\mathbf{c}}\left(-\frac{y}{z}-\frac{\bar{y}}{\bar{z}}+\frac{y_2}{z_2-z\bar{z}}\right)W^{(0)}\left(\begin{bmatrix} y\bar{y} & \\ & z\bar{z} \end{bmatrix}\begin{bmatrix} 1 & \\ \frac{z_2-z\bar{z}}{z\bar{z}} & 1 \end{bmatrix}\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}\right) dy\, dz$$

$$I_{ix)} = \int\limits_{L\backslash\mathfrak{o}_L} \int\limits_{\substack{\mathfrak{o}_L^\times \\ y_2-y\bar{y}\in\mathfrak{o}^\times}} \int\limits_{F\backslash\mathfrak{o}} |z|_L^{-3s-\frac{1}{2}}\Lambda(z)|x|^{-6s-1}\chi(-x^{-1})\psi^{-\mathbf{c}}\left(\frac{x(y\bar{y}+z\bar{z}+\bar{y}z+y\bar{z})}{z\bar{z}}\right)$$

$$W^{(0)}\left(\begin{bmatrix} 1 & \\ & z\bar{z} \end{bmatrix}\begin{bmatrix} y_2 - y\bar{y} + \frac{y\bar{y}z_2}{z\bar{z}} & \\ & 1 \end{bmatrix}\begin{bmatrix} 1 & \\ -\frac{z_2-z\bar{z}}{xz\bar{z}} & 1 \end{bmatrix}\right) dx\, dy\, dz$$

$$I_{x)} = \int\limits_{L\backslash\mathfrak{o}_L} \int\limits_{L\backslash\mathfrak{o}_L} \int\limits_{F\backslash\mathfrak{o}} |y|_L^{-3(s+\frac{1}{2})}|z|_L^{-3s-\frac{1}{2}}\Lambda(\bar{y}z)|x|^{-6s-1}\chi_0(x)$$

$$W^{(0)}\left(\begin{bmatrix} 1 & \\ & z\bar{z} \end{bmatrix}\begin{bmatrix} 1 & x(y\bar{y}+z\bar{z}-\bar{y}z-y\bar{z}) \\ & 1 \end{bmatrix}\begin{bmatrix} y\bar{y}(1+\frac{y_2}{y\bar{y}}-\frac{z_2}{z\bar{z}}) & \\ & 1 \end{bmatrix}\begin{bmatrix} 1 & \\ \frac{z_2-z\bar{z}}{xz\bar{z}} & 1 \end{bmatrix}\right) dx\, dy\, dz.$$

These integrals can be calculated further, using standard p-adic techniques and known properties of the GL_2 Whittaker function $W^{(0)}$. We will omit the details of the calculation for reasons of brevity.

The calculations for the split case (when $L = F \oplus F$) are similar. In this case the explicit formula (43) gives us an integral over five F-variables (coming from the two L-variables and one F-variable). Also, note that in the split case, we have the isomorphism

$$\mathrm{GU}(2,2;F\oplus F) \cong \mathrm{GL}_4(F) \times \mathrm{GL}_1(F)$$
$$g := (g_1, g_2) \rightarrow (g_1, \mu(g)).$$

Using this, we can break up the integral (43) into several smaller integrals, which we evaluate in a manner similar to the inert case. After all the integrals are computed and combined, one obtains the following result, which is true in the inert as well as the split case.

1.4.2 Theorem. *Let (τ, V_τ) be an irreducible, admissible, generic representation of $\mathrm{GL}_2(F)$. Assume that L/F is either an unramified field extension or $L = F \oplus F$. Assume also that the*

conductor $\mathfrak{p}^n$ of τ satisfies $n \geq 1$. Let the character χ_0 of $L^\times$ be such that $\chi_0\big|_{F^\times} = \omega_\tau$ and $\chi_0((1 + \mathfrak{P}^n) \cap \mathfrak{o}_L^\times) = 1$. Let Λ be an unramified character of $L^\times$ such that $\Lambda\big|_{F^\times} = 1$. Let the character χ of $L^\times$ be defined by (44). Let $W^\#(\,\cdot\,, s)$ be the distinguished function in $I(s, \chi, \chi_0, \tau)$ from Corollary 1.2.4, normalized such that $W^\#(\eta_0, s) = 1$. Then the function $K(s)$ defined by (41) is given by

$$K(s) = \chi_{L/F}(\varpi)^n \omega_\tau(\mathbf{c}^2/\mathbf{d}) \frac{\varepsilon(3s+1, \tilde{\tau}, \psi^{-\mathbf{c}})^2}{\varepsilon(6s, \omega_\tau^{-1}, \psi^{-\mathbf{c}})} \frac{L(6s, \chi\big|_{F^\times}) L(3s, \tau \times \mathcal{AI}(\Lambda) \times \chi\big|_{F^\times})}{L(1-6s, \chi^{-1}\big|_{F^\times}) L(3s+1, \tau \times \mathcal{AI}(\Lambda) \times \chi\big|_{F^\times})}.$$

1.5 Intertwining operator: archimedean case

In this section let $F = \mathbb{R}$. We use the notation and setup from Sect. 1.3. Hence, (τ, V_τ) is an irreducible, admissible, generic representation of $\mathrm{GL}_2(\mathbb{R})$, and $l_2 \in \mathbb{Z}$ has the same parity as the weights of τ. The character χ_0 of $\mathbb{C}^\times$ is such that $\chi_0\big|_{\mathbb{R}^\times} = \omega_\tau$ and $\chi_0(\zeta) = \zeta^{l_2}$ for $\zeta \in \mathbb{C}^\times$, $|\zeta| = 1$, and $\chi(\zeta) = \chi_0(\bar{\zeta})^{-1}$. We realize τ as a subrepresentation of some $\beta_1 \times \beta_2$, and the quantities p, q are defined by (57). Let $W^\#$ be the distinguished function in $I_W(s, \chi, \chi_0, \tau)$ defined in table (82). In this section we calculate the function $K(s)$ defined by (41). It is easily checked that the operator $M(s)$, defined by the same integral formula (39), defines an intertwining map from $I_\Phi(s, \chi, \chi_0, \tau)$ to $I_\Phi(-s, \bar{\chi}^{-1}, \chi\bar{\chi}\chi_0, \chi\tau)$. In fact, there is a commutative diagram

$$\begin{array}{ccc} I_\Phi(s, \chi, \chi_0, \tau) & \xrightarrow{\;M(s)\;} & I_\Phi(-s, \bar{\chi}^{-1}, \chi\bar{\chi}\chi_0, \chi\tau) \\ \downarrow & & \downarrow \\ I_W(s, \chi, \chi_0, \tau) & \xrightarrow[\;M(s)\;]{} & I_W(-s, \bar{\chi}^{-1}, \chi\bar{\chi}\chi_0, \chi\tau) \end{array}$$

in which the vertical maps are the intertwining operators $\Phi \mapsto W_\Phi$ given, in the region of convergence, by formula (79). The commutativity follows from a straightforward calculation in the region of convergence, and by analytic continuation outside this region. It follows that the function $K(s)$, instead of (41), can also be determined from the equation

$$M(s)\Phi^\#(\,\cdot\,, s, \chi, \chi_0, \tau) = K(s)\,\Phi^\#(\,\cdot\,, -s, \bar{\chi}^{-1}, \chi\bar{\chi}\chi_0, \chi\tau). \tag{98}$$

Here, $\Phi^\# \in I_\Phi(s, \chi, \chi_0, \tau)$ is defined in table (82). At this point, we do not yet know in all cases that a function $K(s)$ with the property (98) actually exists. We *do* know that it exists in the Cases A and B defined in (81); since $M(s)$ preserves right transformation properties, this follows from the uniqueness statement in Theorem 1.3.6 ii) and iii). In view of the normalization (74), we have the formula

$$K(s) = \int\limits_{N(\mathbb{R})} \Phi^\#(w_1 n \eta_0, s, \chi, \chi_0, \tau)\, dn \tag{99}$$

in Cases A and B. In Case C, part iv) of Theorem 1.3.6 assures that the left side of (98) is a linear combination of $\Phi^\#_{m,l,l_2,l+1}$ and $\Phi^\#_{m,l,l_2,l-1}$. It would be more precise to write these functions as

$$\Phi^\#_{m,l,l_2,l\pm1}(\,\cdot\,, -s, \bar{\chi}^{-1}, \chi\bar{\chi}\chi_0, \chi\tau) \qquad \text{or} \qquad \Phi^\#_{m,l,l_2,l\pm1}(\,\cdot\,, -s, \beta_2^{-1} \times \beta_1^{-1})$$

since they are defined with respect to the data $(-s, \bar{\chi}^{-1}, \chi\bar{\chi}\chi_0, \chi\tau)$, and $\chi\tau$ is a subrepresentation of $\beta_2^{-1} \times \beta_1^{-1}$. The calculation will show that this linear combination is precisely a function $K(s)$ times the distinguished vector $\Phi^\#(\,\cdot\,, -s, \bar{\chi}^{-1}, \chi\bar{\chi}\chi_0, \chi\tau)$ for the data $(-s, \bar{\chi}^{-1}, \chi\bar{\chi}\chi_0, \chi\tau)$. This will establish the existence of $K(s)$ with the property (98) in all cases.

Concerning the measure on $N(\mathbb{R})$, similar remarks as in the p-adic case apply. As a measure space, $N(\mathbb{R}) \cong \mathbb{R} \times \mathbb{C} \times \mathbb{C}$. The measure on $\mathbb{R}$ is the usual Lebesgue measure, but the measure on $\mathbb{C}$ is *twice* the usual Lebesgue measure; see Sect. 2.2 of [86].

Remark: The reason we are calculating $K(s)$ from equation (98) and not from equation (41) is that the relevant archimedean integrals are much easier to handle in the induced model than in the Whittaker model. The price one has to pay for this procedure are the non-explicit constants $\kappa_{l_1,p,q}$ defined in (80). They will not appear any further in this section, but later in Sect. 2.2 when we calculate local zeta integrals; see Corollary 2.2.3. In our application to the functional equation in Sect. 2.4, the unknown constants $\kappa_{l_1,p,q}$ will cancel out with the constants $a_{l_1,p,q}$ defined in (77), via the identity given in Lemma 1.3.8.

1.5.1 Theorem. *Let (τ, V_τ) be a generic, irreducible, admissible representation of $\mathrm{GL}_2(\mathbb{R})$ with central character ω_τ. We assume that τ is isomorphic to a subrepresentation of $\beta_1 \times \beta_2$ with characters β_1, β_2 of $\mathbb{R}^\times$. Let the complex numbers p and q be as defined in (57). Let l be a fixed positive integer. Let $l_2 = -l_1$ in Case A, $l_2 = -l$ in Case B, and $l_2 = 1-l$ in Case C. Let χ_0 be the character of $\mathbb{C}^\times$ such that $\chi_0\big|_{\mathbb{R}^\times} = \omega_\tau$ and $\chi_0(\zeta) = \zeta^{l_2}$ for $\zeta \in \mathbb{C}^\times$, $|\zeta| = 1$. Let χ be the character of $\mathbb{C}^\times$ given by (56). Let $W^\# \in I_W(s, \chi, \chi_0, \tau)$ be the distinguished function defined in table (82). Then the identity (41) holds with the function $K(s)$ given as follows.*

i) *In Case A,*

$$K(s) = 4\pi^{5/2}\, i^{2l - l_1} \frac{\Gamma(3s - \frac{q}{2} + \frac{1}{2})\Gamma(3s - \frac{q}{2})}{(3s - \frac{q}{2} + \frac{l_1}{2} - \frac{1}{2})^2 \Gamma(3s - \frac{q}{2} + \frac{l_1}{2} + \frac{3}{2} - l)\Gamma(3s - \frac{q}{2} + l - \frac{l_1}{2} - \frac{1}{2})}. \quad (100)$$

ii) *In Case B,*

$$K(s) = 4\pi^{5/2}\, i^l \frac{\Gamma(3s - \frac{q}{2} + \frac{1}{2})\Gamma(3s - \frac{q}{2})}{(3s - \frac{q}{2} + \frac{p}{2})(3s - \frac{q}{2} - \frac{p}{2})\Gamma(3s - \frac{q}{2} - \frac{l}{2} + \frac{1}{2})\Gamma(3s - \frac{q}{2} + \frac{l}{2} + \frac{1}{2})}. \quad (101)$$

iii) *In Case C,*

$$K(s) = -4\pi^{5/2} i^{l+1} \frac{(3s - \frac{q}{2} - \frac{l}{2})(3s - \frac{q}{2} - 1 - \frac{p}{2})}{(3s - \frac{q}{2} + 1 + \frac{p}{2})(3s - \frac{q}{2} + \frac{p}{2})(3s - \frac{q}{2} - \frac{p}{2})}$$
$$\times \frac{\Gamma(3s - \frac{q}{2} + \frac{1}{2})\Gamma(3s - \frac{q}{2})}{\Gamma(3s - \frac{q}{2} + 1 - \frac{l}{2})\Gamma(3s - \frac{q}{2} + 1 + \frac{l}{2})}. \quad (102)$$

Proof. i) In Case A, by (82), we have $\Phi^\# = \Phi^\#_{m,l,l_2,l_1}$, where $m = l_1 - l$ and l_1 is the lowest weight of the representation τ. The function $\Phi^\#_{m,l,l_2,l_1}$ is given in Proposition 1.3.4. By (99), we have to calculate

$$K(s) = \int_{N(\mathbb{R})} \Phi^\#_{m,l,-l_1,l_1}(w_1 n \eta_0, s, \chi, \chi_0, \tau)\, dn. \quad (103)$$

We abbreviate

$$u = \sqrt{1 + x^2}, \qquad v = \sqrt{1 + y\bar{y}}, \qquad w = \sqrt{1 + z\bar{z}},$$

and

$$r_1 = \begin{bmatrix} w^{-1} & zw^{-1} & & \\ -\bar{z}w^{-1} & w^{-1} & & \\ & & w^{-1} & zw^{-1} \\ & & -\bar{z}w^{-1} & w^{-1} \end{bmatrix}, \qquad r_2 = \begin{bmatrix} u^{-1} & & xu^{-1} & \\ & 1 & & \\ -xu^{-1} & & u^{-1} & \\ & & & 1 \end{bmatrix},$$

$$r_3 = \begin{bmatrix} v^{-1} & & & yv^{-1} \\ & v^{-1} & \bar{y}v^{-1} & \\ & -yv^{-1} & v^{-1} & \\ -\bar{y}v^{-1} & & & v^{-1} \end{bmatrix}.$$

The elements r_1, r_2, r_3 lie in $K_\infty^{G_2}$. Starting from (103), it is not difficult to show that

$$K(s) = \int\limits_{\mathbb{C}} \int\limits_{\mathbb{C}} \int\limits_{\mathbb{R}} u(uvw)^{q-6s-2}(vw^{-1})^p \, \Phi^{\#}_{m,l,-l_1,l_1}(w_1 r_3 r_2 r_1 \eta_0) \, dx \, dy \, dz. \tag{104}$$

A calculation verifies that, with $k = w_1 r_3 r_2 r_1 \eta_0$,

$$\det(J(k, i_2)) = -i\frac{1-ix}{u}, \qquad \det(J({}^t k, i_2)) = i\frac{1+ix}{u},$$

and

$$\hat{b}(k) = (1 - z\bar{z})\Big(\frac{v^2 - ix(1 - y\bar{y})}{uv^2 w^2}\Big) + 2i\frac{(y\bar{z} + \bar{y}z)}{v^2 w^2}.$$

Hence

$$\Phi^{\#}_{m,l,-l_1,l_1}(k) = (-i)^m \, \hat{b}(k)^{l_1 - l} \det(J({}^t k, i_2))^{l - l_1} \det(J(k, i_2))^{-l_1}$$

$$= i^{2l - l_1} \Big((1 - z\bar{z})\Big(\frac{v^2 - ix(1 - y\bar{y})}{uv^2 w^2}\Big) + 2i\frac{(y\bar{z} + \bar{y}z)}{v^2 w^2}\Big)^{l_1 - l}\Big(\frac{1+ix}{u}\Big)^l,$$

so that

$$K(s) = i^{2l - l_1} \int\limits_{\mathbb{C}} \int\limits_{\mathbb{C}} \int\limits_{\mathbb{R}} u(uvw)^{q-6s-2}(vw^{-1})^p$$

$$\Big((1 - z\bar{z})\Big(\frac{v^2 - ix(1 - y\bar{y})}{uv^2 w^2}\Big) + 2i\frac{(y\bar{z} + \bar{y}z)}{v^2 w^2}\Big)^{l_1 - l}\Big(\frac{1+ix}{u}\Big)^l dx \, dy \, dz.$$

We now introduce polar coordinates for y and z. More precisely, put $y = \sqrt{f}e^{i\theta_1}$, $z = \sqrt{g}e^{i\theta_2}$, and let $\theta = \theta_1 - \theta_2$. Also, put $s_0 = 3s + \frac{1}{2} + \frac{l_1 - l}{2} - \frac{q}{2}$. With these substitutions, and using the fact that $p = l_1 - 1$, the intertwining integral becomes

$$K(s) = i^{2l - l_1} 2\pi \int\limits_0^{2\pi} \int\limits_{\mathbb{R}} \int\limits_0^\infty \int\limits_0^\infty (1 + x^2)^{-s_0}(1 + f)^{-s_0 - 1 + \frac{l}{2}}(1 + g)^{-s_0 + \frac{l}{2} - l_1}\Big(\frac{1+ix}{u}\Big)^l$$

$$\Big((1 - g)(2 - (1 - f)(1 + ix)) + 4iu\sqrt{fg}\,\cos\theta\Big)^{l_1 - l} df \, dg \, dx \, d\theta.$$

Note here that the measure on $\mathbb{C}$ is twice the usual Lebesgue measure. By Lemma 1.5.2 further below, we get the result.

ii) Next we evaluate the intertwining integral in the case where τ contains the weight l. As in the previous case, $K(s)$ is given by formula (99). The same calculation that led to (104) now shows that

$$K(s) = \int\limits_{\mathbb{C}} \int\limits_{\mathbb{C}} \int\limits_{\mathbb{R}} u(uvw)^{q-6s-2}(vw^{-1})^p \, \Phi^{\#}_{m,l,-l,l}(w_1 r_3 r_2 r_1 \eta_0) \, dx \, dy \, dz. \tag{105}$$

This time $\Phi^{\#}_{m,l,-l,l} = i^l \left(\frac{1+ix}{u}\right)^l$, so that

$$K(s) = i^l \int\limits_{\mathbb{C}} \int\limits_{\mathbb{C}} \int\limits_{\mathbb{R}} u(uvw)^{q-6s-2}(vw^{-1})^p \left(\frac{1+ix}{u}\right)^l dx \, dy \, dz.$$

This integral can be calculated as before by using polar coordinates. The result follows.

iii) This case is the most complicated one, since we do not yet know that a function $K(s)$ with the property (98) exists. We do know, however, by part iv) of Theorem 1.3.6, that there exist functions $K_1(s)$ and $K_2(s)$ such that

$$M(s)\Phi^{\#}(\,\cdot\,, s, \chi, \chi_0, \tau) = K_1(s)\, \Phi^{\#}_{1,l,l_2,l+1}(\,\cdot\,, -s, \bar\chi^{-1}, \chi\bar\chi\chi_0, \chi\tau)$$

$$+ K_2(s)\, \Phi^{\#}_{1,l,l_2,l-1}(\,\cdot\,, -s, \bar\chi^{-1}, \chi\bar\chi\chi_0, \chi\tau). \tag{106}$$

The calculation of $K_1(s)$ and $K_2(s)$ is in the same spirit as in Cases A and B, and we omit the details. Eventually it turns out that (98) holds with $K(s)$ as in (102). ∎

We would like to thank Paul-Olivier Dehaye for his help with the proof of the following lemma, which was used in the above calculations.

1.5.2 Lemma. *For non-negative integers l and t, and for all $s \in \mathbb{C}$ with $\mathrm{Re}(s)$ large enough,*

$$\int\limits_0^{2\pi} \int\limits_{\mathbb{R}} \int\limits_0^{\infty} \int\limits_0^{\infty} (1+x^2)^{-s}(1+f)^{-s-1+\frac{l}{2}}(1+g)^{-s-\frac{l}{2}-t} \left(\frac{1+ix}{u}\right)^l$$

$$\left((1-g)(2-(1-f)(1+ix)) + 4iu\sqrt{fg}\,\cos\theta\right)^t df \, dg \, dx \, d\theta$$

$$= \frac{2\pi^{3/2}\Gamma(s-\frac{t}{2})\Gamma(s-\frac{t}{2}-\frac{1}{2})}{(s+\frac{l}{2}-1)^2\Gamma(s-\frac{l}{2}+1)\Gamma(s+\frac{l}{2}-t-1)}.$$

Proof. Let LHS denote the quantity on the left hand side of the asserted formula. We start off by completely expanding $\left((1-g)(2-(1-f)(1+ix))+4iu\sqrt{fg}\,\cos\theta\right)^t$ using the bimomial theorem. Then, using (6.16), (6.17) of [7] and the following well-known formulas,

$$\int\limits_0^{2\pi} \cos(\theta)^k \, d\theta = \begin{cases} 0 & \text{if } k \text{ is odd,} \\ 2\sqrt{\pi}\, \dfrac{\Gamma(\frac{k+1}{2})}{\Gamma(\frac{k+2}{2})} & \text{if } k \text{ is even,} \end{cases}$$

$$\int\limits_0^{\infty} r^{t_1}(1+r)^{-t_2} \, dr = \frac{\Gamma(1+t_1)\Gamma(-1+t_2-t_1)}{\Gamma(t_2)},$$

we arrive at (using the multinomial symbol)

$$LHS = \sum_{k,j,r,v} \left((-1)^{k+r+v+j} 2^{2k+t-j+1} \binom{t}{2k,j,t-2k-j} \binom{t-2k}{v} \binom{j}{v} \frac{\Gamma(k+\frac{1}{2})\Gamma(1+k+r)}{\Gamma(k+1)\Gamma(s+\frac{l}{2}+t)} \pi \right.$$

$$\left. \frac{\Gamma(-1+s+t-k-r+\frac{l}{2})\Gamma(s-v-k-\frac{l}{2})\Gamma(1+k+v)\Gamma(s-k-\frac{j}{2})\Gamma(s-k-\frac{j}{2}-\frac{1}{2})}{\Gamma(s-\frac{l}{2}-k-j)\Gamma(s+\frac{l}{2}-k)\Gamma(s-\frac{l}{2}+1)} \right),$$

where the sum is taken over non-negative integers k, j, r, v satisfying $2k + j \leq t$, $r + 2k \leq t$ and $v \leq j$. Next, using well-known summation formulas for the gamma functions and algebraic manipulations (we omit the details of this step, which were performed with the aid of Mathematica), it turns out that the above expression simplifies significantly. As a result, the lemma reduces to proving a certain algebraic identity. Let $x^{(n)} = x(x+1)\ldots(x+n-1)$ denote the Pochhammer symbol. Then the identity we are reduced to proving is

$$\sum_{k=0}^{T} \sum_{\substack{v \geq 0,\, n \geq 0 \\ v+n \leq 4T+1}} 4^k \frac{(-1)^{k+v}}{n!v!(2T-2k)!} \cdot \frac{(x+k-n)^{(n)}(x-k+1)^{(v)}}{(x-2T+v)^{(k+1)}(x+2T-k-n)^{(k+1)}} = -\frac{1}{x^2(2T)!},$$

where T is any non-negative integer, and x is an indeterminate. To prove this identity, observe that each summand above can be written using the partial fraction decomposition as a sum of rational functions, where each numerator is a rational number and the denominators are terms of the form $(x-a)^b$ with b equal to 1 or 2, and $-3T-1 \leq a \leq 3T+1$. So to prove the identity, it is enough to show that the sum of the numerators coincide on both sides for each such denominator. This is straightforward combinatorics, and we omit the details. ∎

2 Global L-functions for $\mathrm{GSp}_4 \times \mathrm{GL}_2$

In this section, we will use the integral defined by Furusawa in [19] to obtain an integral representation of the L-function $L(s, \pi \times \tau)$, where π is a cuspidal, automorphic representation of $\mathrm{GSp}_4(\mathbb{A})$ of the type corresponding to full level Siegel cusp forms, and where τ is an arbitrary cuspidal, automorphic representation of $\mathrm{GL}_2(\mathbb{A})$. We will use this to obtain the functional equation of the L-function, with some restriction on the GL_2 representation. We will first do the non-archimedean calculation, followed by the archimedean calculation and put it all together to get the global result.

2.1 Bessel models for GSp_4

Let F be an algebraic number field and $\mathbb{A}_F$ its ring of adeles. We fix three elements $\mathbf{a}, \mathbf{b}, \mathbf{c} \in F$ such that $\mathbf{d} = \mathbf{b}^2 - 4\mathbf{ac}$ is a non-square in $F^\times$. Then $L = F(\sqrt{\mathbf{d}})$ is a quadratic field extension of F. Let

$$S = \begin{bmatrix} \mathbf{a} & \frac{\mathbf{b}}{2} \\ \frac{\mathbf{b}}{2} & \mathbf{c} \end{bmatrix}, \qquad \xi = \begin{bmatrix} \frac{\mathbf{b}}{2} & \mathbf{c} \\ -\mathbf{a} & \frac{-\mathbf{b}}{2} \end{bmatrix}.$$

Then $F(\xi) = F + F\xi$ is a two-dimensional F-algebra isomorphic to L, an isomorphism being given by $x + y\xi \mapsto x + y\frac{\sqrt{\mathbf{d}}}{2}$. The determinant map on $F(\xi)$ corresponds to the norm map on L. Let

$$T = \{g \in \mathrm{GL}_2 \mid {}^t g S g = \det(g) S\}. \tag{107}$$

This is an algebraic F-group with $T(F) = F(\xi)^\times \cong L^\times$ and $T(\mathbb{A}_F) \cong \mathbb{A}_L^\times$. We consider T a subgroup of $H = \mathrm{GSp}_4$ via

$$T \ni g \longmapsto \begin{bmatrix} g & \\ & \det(g)\,{}^t g^{-1} \end{bmatrix} \in H.$$

Let

$$U = \{ \begin{bmatrix} 1_2 & X \\ & 1_2 \end{bmatrix} \in H \mid {}^t X = X \}$$

and $R = TU$. We call R the *Bessel subgroup* of H (with respect to the given data $\mathbf{a}, \mathbf{b}, \mathbf{c}$). Let ψ be a non-trivial character $F \backslash \mathbb{A}_F \to \mathbb{C}^\times$, chosen once and for all. Let $\theta : U(\mathbb{A}_F) \to \mathbb{C}^\times$ be the character given by

$$\theta(\begin{bmatrix} 1 & X \\ & 1 \end{bmatrix}) = \psi(\mathrm{tr}(SX)). \tag{108}$$

We have $\theta(t^{-1}ut) = \theta(u)$ for all $u \in U(\mathbb{A}_F)$ and $t \in T(\mathbb{A}_F)$. Hence, if Λ is any character of $T(\mathbb{A}_F) \cong \mathbb{A}_L^\times$, then the map $tu \mapsto \Lambda(t)\theta(u)$ defines a character of $R(\mathbb{A}_F)$. We denote this character by $\Lambda \otimes \theta$.

Analogous definitions can be made over any local field F. In this case, let π be an irreducible, admissible representation of $H(F)$. Let Λ be a character of $T(F) \cong L^\times$ such that the restriction of Λ to $F^\times$ coincides with the central character of π. Let $\Lambda \otimes \theta$ be the character of $R(F)$ defined above. We say that π has a *Bessel model* of type (S, Λ, ψ) if π is isomorphic to a space of functions $B : H(F) \to \mathbb{C}$ with the transformation property

$$B(tuh) = \Lambda(t)\theta(u)B(h) \qquad \text{for all } t \in T(F),\ u \in U(F),\ h \in H(F),$$

with the action of $H(F)$ on this space given by right translation. Such a model, if it exists, is known to be unique; we denote it by $\mathcal{B}_{S,\Lambda,\psi}(\pi)$.

Now let F be global, and let $\pi = \otimes \pi_v$ be a cuspidal, automorphic representation of $H(\mathbb{A}_F)$. Let V_π be the space of automorphic forms realizing π. Assume that a Hecke character Λ as above is chosen such that the restriction of Λ to $\mathbb{A}_F^\times$ coincides with ω_π, the central character of π. For each $\phi \in V_\pi$ consider the corresponding Bessel function

$$B_\phi(g) = \int\limits_{Z_H(\mathbb{A}_F)R(F)\backslash R(\mathbb{A}_F)} (\Lambda \otimes \theta)(r)^{-1}\phi(rg)\,dr, \tag{109}$$

where Z_H is the center of H. If one of these integrals is non-zero, then all are non-zero, and we obtain a model $\mathcal{B}_{S,\Lambda,\psi}(\pi)$ of π consisting of functions on $H(\mathbb{A}_F)$ with the obvious transformation property on the left with respect to $R(\mathbb{A}_F)$. In this case, we say that π has a *global Bessel model* of type (S, Λ, ψ). It implies that the *local* Bessel model $\mathcal{B}_{S,\Lambda_v,\psi_v}(\pi_v)$ exists for every place v of F. In fact, there is a canonical isomorphism

$$\bigotimes_v \mathcal{B}_{S,\Lambda_v,\psi_v}(\pi_v) \cong \mathcal{B}_{S,\Lambda,\psi}(\pi).$$

If $(B_v)_v$ is a collection of local Bessel functions $B_v \in \mathcal{B}_{S,\Lambda_v,\psi_v}(\pi_v)$ such that $B_v\big|_{H(\mathfrak{o}_v)} = 1$ for almost all v, then this isomorphism is such that $\otimes_v B_v$ corresponds to the global function

$$B(g) = \prod_v B_v(g_v), \qquad g = (g_v)_v \in H(\mathbb{A}_F). \tag{110}$$

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Explicit formulas: the spherical Bessel function

Explicit formulas for local Bessel functions are only known in a few cases. One of these is the p-adic unramified case, which we review next. Hence, let F be a non-archimedean local field of characteristic zero. Let the character ψ of F have conductor $\mathfrak{o}$, the ring of integers. Let (π, V_π) be an unramified, irreducible, admissible representation of $H(F)$. Let Λ be an unramified character of $T(F) \cong L^\times$. We assume that $V_\pi = \mathcal{B}_{S,\Lambda,\psi}(\pi)$ is the Bessel model with respect to the character $\Lambda \otimes \theta$ of $R(F)$. Let $B \in V_\pi$ be a spherical vector. By [83], Proposition 2-5, we have $B(1) \neq 0$. For $l, m \in \mathbb{Z}$ let

$$
h(l,m) = \begin{bmatrix} \varpi^{2m+l} & & & \\ & \varpi^{m+l} & & \\ & & 1 & \\ & & & \varpi^m \end{bmatrix}.
\tag{111}
$$

Then, as in (3.4.2) of [19],

$$
H(F) = \bigsqcup_{l \in \mathbb{Z}} \bigsqcup_{m \geq 0} R(F) h(l,m) K^H, \qquad K^H = H(\mathfrak{o}).
\tag{112}
$$

By Lemma (3.4.4) of [19] we have $B(h(l,m)) = 0$ for $l < 0$, so that B is determined by the values $B(h(l,m))$ for $l, m \geq 0$. In [83], 2-4, Sugano has given a formula for $B(h(l,m))$ in terms of a generating function. The full formula is required only in the case where the GL_2 representation τ is unramified; this case has been treated in [19]. For other cases we only require the values $B(h(l,0))$, which are given by

$$
\sum_{l \geq 0} B(h(l,0)) y^l = \frac{H(y)}{Q(y)},
\tag{113}
$$

where

$$
Q(y) = \prod_{i=1}^{4} \left(1 - \gamma^{(i)}(\varpi) q^{-3/2} y\right)
\tag{114}
$$

and

$$
H(y) = \begin{cases} 1 - q^{-4} \Lambda(\varpi) y^2 & \text{if } \left(\frac{L}{\mathfrak{p}}\right) = -1, \\ 1 - q^{-2} \Lambda(\varpi_L) y & \text{if } \left(\frac{L}{\mathfrak{p}}\right) = 0, \\ 1 - q^{-2} \left(\Lambda(\varpi_L) + \Lambda(\varpi \varpi_L^{-1})\right) y + q^{-4} \Lambda(\varpi) y^2 & \text{if } \left(\frac{L}{\mathfrak{p}}\right) = 1. \end{cases}
\tag{115}
$$

The $\gamma^{(i)}$ are the Satake parameters of π, as in Sect. (3.6) of [19].

Explicit formulas: the highest weight case

Another situation where an explicit formula for a Bessel function is known is the archimedean lowest weight case. Hence, let $F = \mathbb{R}$. Let l be an integer such that $l \geq 2$. Let π be the discrete series representation (or limit of such if $l = 2$) of $\mathrm{PGSp}_4(\mathbb{R})$ with minimal K-type (l,l); here, we write elements of the weight lattice as pairs of integers, precisely as in [65], Sect. 2.1. Such representations π appear as the archimedean components of the automorphic representations of $H(\mathbb{A})$ attached to (scalar valued) Siegel modular forms of weight l. Recall that S is a positive definite matrix. Let the function $B : H(\mathbb{R}) \to \mathbb{C}$ be defined by

$$
B(h) := \begin{cases} \mu_2(h)^l \, \overline{\det(J(h,i_2))^{-l}} \, e^{-2\pi i \operatorname{tr}(S h \langle i_2 \rangle)} & \text{if } h \in H^+(\mathbb{R}), \\ 0 & \text{if } h \notin H^+(\mathbb{R}), \end{cases}
\tag{116}
$$

where $i_2 = \begin{bmatrix} i \\ & i \end{bmatrix}$. One can check that B satisfies the Bessel transformation property with the character $\Lambda \otimes \theta$ of $R(\mathbb{R})$, where Λ is trivial. Also

$$B(hk) = \det(J(k, i_2))^l B(h) \qquad \text{for } h \in H(\mathbb{R}),\ k \in K_\infty^H. \tag{117}$$

In fact, by the considerations in [83] 1-3, or by [65] Theorem 3.4, B is the highest weight vector (weight $(-l, -l)$) in $\mathcal{B}_{S,\Lambda,\psi}(\pi)$. Note that the function B is determined by its values on a set of representatives for $R(\mathbb{R})\backslash H(\mathbb{R})/K_\infty^H$. Such a set can be obtained as follows. Let $T^1(\mathbb{R}) = T(\mathbb{R}) \cap \mathrm{SL}(2, \mathbb{R})$. Then $T(\mathbb{R}) = T^1(\mathbb{R}) \cdot \{ \begin{bmatrix} \zeta \\ & \zeta \end{bmatrix} \mid \zeta > 0 \}$. As in [19], p. 211, let $t_0 \in \mathrm{GL}_2(\mathbb{R})^+$ be such that $T^1(\mathbb{R}) = t_0 \mathrm{SO}(2) t_0^{-1}$. (We will make a specific choice of t_0 when we choose the matrix S below.) It is not hard to see that

$$H(\mathbb{R}) = R(\mathbb{R}) \cdot \left\{ \begin{bmatrix} \lambda t_0 \begin{bmatrix} \zeta \\ & \zeta^{-1} \end{bmatrix} \\ & {}^t t_0^{-1} \begin{bmatrix} \zeta^{-1} \\ & \zeta \end{bmatrix} \end{bmatrix} \mid \lambda \in \mathbb{R}^\times,\ \zeta \geq 1 \right\} \cdot K_\infty^H. \tag{118}$$

One can check that the double cosets in (118) are pairwise disjoint.

2.2 Local zeta integrals

Let F be a non-archimedean local field of characteristic zero, or $F = \mathbb{R}$. Let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in F$ and L, α, η be according to our conventions; see (14), (15), (16). Let τ, χ_0, χ be as in Corollary 1.2.4 (non-archimedean case) resp. Corollary 1.3.6 (archimedean case). Let $W^\#(\,\cdot\,, s)$ be the unique vector in $I(s, \chi, \chi_0, \tau)$ exhibited in these corollaries. The calculation in the proof of Theorem (2.4) of [19] shows that

$$W^\#(\eta tuh, s) = \Lambda(t)^{-1}\theta(u)^{-1}W^\#(\eta h, s) \qquad \text{for all } t \in T(F),\ u \in U(F),\ h \in H(F). \tag{119}$$

Here, Λ is an unramified character of $L^\times$ in the non-archimedean case, and $\Lambda = 1$ in the archimedean case; we always have $\chi(\zeta) = \Lambda(\bar\zeta)^{-1}\chi_0(\bar\zeta)^{-1}$. Let π be an irreducible, admissible representation of $H(F)$ which has a Bessel model of type (S, Λ, ψ). Then, for any $B \in \mathcal{B}_{S,\Lambda,\psi}(\pi)$, equation (119) shows that the integral

$$Z(s, W^\#, B) = \int_{R(F)\backslash H(F)} W^\#(\eta h, s)B(h)\, dh \tag{120}$$

makes sense. We shall now explicitly calculate these integrals in the case of B being the spherical vector in an unramified p-adic representation π, and B being the highest weight vector in an archimedean (limit of) discrete series representation with scalar minimal K-type.

The non-archimedean case

Assume that F is non-archimedean. Recall the explicit formula for the distinguished function $W^\#(\,\cdot\,, s)$ given in Corollary 1.2.4. It involves $W^{(0)}$, the normalized local newform in the Whittaker model of τ with respect to the character $\psi^{-\mathbf{c}}(x) = \psi(-\mathbf{c}x)$. Since this character has conductor $\mathfrak{o}$, the values $W^{(0)}(\begin{bmatrix} \varpi^l \\ & 1 \end{bmatrix})$ are zero for negative l. For non-negative l, one can use formulas for

the local newform with respect to the congruence subgroup $\mathrm{GL}_2(\mathfrak{o}) \cap \begin{bmatrix} \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^n & 1+\mathfrak{p}^n \end{bmatrix}$ (given, amongst other places, in [77]), together with the local functional equation, to obtain the following.

τ	$W^{(0)}(\begin{bmatrix} \varpi^l & \\ & 1 \end{bmatrix})\ (l \geq 0)$	
$\beta_1 \times \beta_2$ with β_1, β_2 unramified, $\beta_1\beta_2^{-1} \neq \lvert\lvert^{\pm 1}$	$q^{-l/2} \sum_{k=0}^{l} \beta_1(\varpi)^k \beta_2(\varpi)^{l-k}$	
$\beta_1 \times \beta_2$ with β_1 unramified, β_2 ramified	$\beta_2(\varpi^l)q^{-\frac{l}{2}}$	(121)
$\Omega\,\mathrm{St}_{\mathrm{GL}_2}$ with Ω unramified	$\Omega(\varpi^l)q^{-l}$	
supercuspidal OR ramified twist of Steinberg OR $\beta_1 \times \beta_2$ with β_1, β_2 ramified, $\beta_1\beta_2^{-1} \neq \lvert\lvert^{\pm 1}$	$1 \quad$ if $l = 0$ $0 \quad$ if $l > 0$	

Assume that π is an unramified representation and that $B \in \mathcal{B}_{S,\Lambda,\psi}(\pi)$ is the spherical Bessel function as in (113). In the following we shall assume that the conductor $\mathfrak{p}^n$ of τ satisfies $n > 0$, since for unramified τ the local integral has been computed by Furusawa; see Theorem (3.7) in [19]. Since both functions B and $W^{\#}$ are right K^H-invariant, it follows from (112) that the integral (120) is given by

$$Z(s, W^{\#}, B) = \sum_{l,m \geq 0} B(h(l,m))W^{\#}(\eta h(l,m), s)V_m q^{3m+3l}. \tag{122}$$

Here, as in Sect. 3.5 of [19], $V_m = \mathrm{vol}\big(T(F)\backslash T(F) \begin{bmatrix} \varpi^m & \\ & 1 \end{bmatrix}\mathrm{GL}_2(\mathfrak{o})\big)$. Calculations confirm that $\eta h(l,m)$ lies in the support of $W^{\#}(\,\cdot\,,s)$ if and only if $m = 0$. It follows that the sum (122) reduces to

$$Z(s, W^{\#}, B) = \sum_{l \geq 0} B(h(l,0))W^{\#}(\eta h(l,0), s)q^{3l}. \tag{123}$$

By (53),

$$W^{\#}(\eta h(l,0), s) = q^{-3(s+1/2)l}\omega_\pi(\varpi^{-l})\omega_\tau(\varpi^{-l})W^{(0)}(\begin{bmatrix} \varpi^l & \\ & 1 \end{bmatrix}). \tag{124}$$

Substituting the values of $W^{(0)}(\begin{bmatrix} \varpi^l & \\ & 1 \end{bmatrix})$ from the table above and the values of $B(h(l,0))$ from (113), we get the following result.

2.2.1 Theorem. *Let $\tau, \chi, \chi_0, \Lambda$ and $W^{\#}(\,\cdot\,, s)$ be as in Corollary 1.2.4. Let π be an irreducible, admissible, unramified representation of $H(F)$, and let B be the unramified Bessel function given by formula (113). Then the local zeta integral $Z(s, W^{\#}, B)$ defined in (120) is given by*

$$Z(s, W^{\#}, B) = \frac{L(3s + \frac{1}{2}, \tilde\pi \times \tilde\tau)}{L(6s+1, \chi|_{F^\times})L(3s+1, \tau \times \mathcal{AI}(\Lambda) \times \chi|_{F^\times})}Y(s), \tag{125}$$

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where

$$
Y(s) = \begin{cases}
1 & \text{if } \tau = \beta_1 \times \beta_2, \ \beta_1, \beta_2 \text{ unramified,} \\
L(6s+1, \chi|_{F^\times}) & \text{if } \tau = \beta_1 \times \beta_2, \ \beta_1 \text{ unram., } \beta_2 \text{ ram., } \left(\frac{L}{\mathfrak{p}}\right) = \pm 1, \\
& \quad \text{OR } \tau = \beta_1 \times \beta_2, \ \beta_1 \text{ unram., } \beta_2 \text{ ram.,} \\
& \qquad \left(\frac{L}{\mathfrak{p}}\right) = 0 \text{ and } \beta_2 \chi_{L/F} \text{ ramified,} \\
& \quad \text{OR } \tau = \Omega \mathrm{St}_{\mathrm{GL}(2)}, \ \Omega \text{ unramified,} \\[4pt]
\dfrac{L(6s+1, \chi|_{F^\times})}{1 - \Lambda(\varpi_L)(\omega_\pi \beta_2)^{-1}(\varpi)q^{-3s-1}} & \text{if } \tau = \beta_1 \times \beta_2, \ \beta_1 \text{ unram., } \beta_2 \text{ ram., } \left(\frac{L}{\mathfrak{p}}\right) = 0, \\
& \quad \text{and } \beta_2 \chi_{L/F} \text{ unramified,} \\[4pt]
L(6s+1, \chi|_{F^\times}) L(3s+1, \tau \times \mathcal{AI}(\Lambda) \times \chi|_{F^\times}) & \text{if } \tau = \beta_1 \times \beta_2, \ \beta_1, \beta_2 \text{ ramified,} \\
& \quad \text{OR } \tau = \Omega \mathrm{St}_{\mathrm{GL}(2)}, \ \Omega \text{ ramified,} \\
& \quad \text{OR } \tau \text{ supercuspidal.}
\end{cases}
$$

In (125), $\tilde{\pi}$ and $\tilde{\tau}$ denote the contragredient of π and τ, respectively. The symbol $\mathcal{AI}(\Lambda)$ stands for the $\mathrm{GL}_2(F)$ representation attached to the character Λ of $L^\times$ via automorphic induction, and $\chi_{L/F}$ stands for the quadratic character of $F^\times$ associated with the extension L/F. The function $L(3s+1, \tau \times \mathcal{AI}(\Lambda) \times \chi|_{F^\times})$ is a standard L-factor for $\mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{GL}_1$.

Proof. If $\tau = \beta_1 \times \beta_2$ with unramified β_1 and β_2, then this is Theorem (3.7) in Furusawa's paper [19]. If $\tau = \beta_1 \times \beta_2$ with unramified β_1 and ramified β_2, then, from the local Langlands correspondence, we have the following L-functions attached to the representations $\tilde{\pi} \times \tilde{\tau}$ of $\mathrm{GSp}_4(F) \times \mathrm{GL}_2(F)$ and $\tau \times \mathcal{AI}(\Lambda) \times \chi|_{F^\times}$ of $\mathrm{GL}_2(F) \times \mathrm{GL}_2(F) \times \mathrm{GL}_1(F)$,

$$
L(s, \tilde{\pi} \times \tilde{\tau}) = \prod_{i=1}^{4} \left(1 - (\gamma^{(i)} \beta_1)^{-1}(\varpi) q^{-s}\right)^{-1}, \tag{126}
$$

where $\gamma^{(i)}$ are the Satake parameters of π, as in Sect. (3.6) of [19], and

$$
\frac{1}{L(s, \tau \times \mathcal{AI}(\Lambda) \times \chi|_{F^\times})} = \begin{cases}
1 - \left(\Lambda(\omega_\pi \beta_1)^{-2}\right)(\varpi) q^{-2s} & \text{if } \left(\frac{L}{\mathfrak{p}}\right) = -1, \\
1 - \Lambda(\varpi_L)(\omega_\pi \beta_1)^{-1}(\varpi) q^{-s} & \text{if } \left(\frac{L}{\mathfrak{p}}\right) = 0 \text{ and } \beta_2 \chi_{L/F} \text{ ram.,} \\
(1 - \Lambda(\varpi_L)(\omega_\pi \beta_1)^{-1}(\varpi) q^{-s}) & \\
\quad (1 - \Lambda(\varpi_L)(\omega_\pi \beta_2)^{-1}(\varpi) q^{-s}) & \text{if } \left(\frac{L}{\mathfrak{p}}\right) = 0 \text{ and } \beta_2 \chi_{L/F} \text{ unram.,} \\
(1 - \Lambda(\varpi_L)(\omega_\pi \beta_1)^{-1}(\varpi) q^{-s}) & \\
\quad (1 - \Lambda(\varpi \varpi_L^{-1})(\omega_\pi \beta_1)^{-1}(\varpi) q^{-s}) & \text{if } \left(\frac{L}{\mathfrak{p}}\right) = 1.
\end{cases}
$$

The desired result therefore follows from (114) and (115). If τ is an unramified twist of the Steinberg representation, then the result was proved in Theorem 3.8.1 of [66]. In all remaining cases we have $L(s, \tilde{\pi} \times \tilde{\tau}) = 1$ and $Z(s, W^\#, B) = 1$, so that the asserted formula holds. ∎

The archimedean case

Now let $F = \mathbb{R}$. We will calculate the zeta integral (120) for the distinguished function $W^\#$ given in Theorem 1.3.6. It is enough to calculate these integrals for the functions $W^\#_{m,l,l_2,l_1}$, where l_1 is one of the weights occurring in τ, and where $l_2 \in \mathbb{Z}$ has the same parity as l_1. Recall the explicit formula (78) for these functions.

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As for the Bessel function ingredient in (120), let π be a (limit of) discrete series representation of $\mathrm{PGSp}_4(\mathbb{R})$ with scalar minimal K-type (l,l), where $l \geq 2$. Let $B : H(\mathbb{R}) \to \mathbb{C}$ be the function defined in (116). Then B is a vector of weight $(-l,-l)$ in $\mathcal{B}_{S,\Lambda,\psi}(\pi)$, where $\Lambda = 1$ and $\psi(x) = e^{-2\pi i x}$. By (85) and (117), the function $W^{\#}(\eta h, s)B(h)$ is right invariant under K_{∞}^{H}. Using this fact and the disjoint double coset decomposition (118), we obtain

$$Z(s, W_{m,l,l_2,l_1}^{\#}, B) = \pi \int\limits_{\mathbb{R}^{\times}} \int\limits_{1}^{\infty} W_{m,l,l_2,l_1}^{\#}\left(\eta \begin{bmatrix} \lambda t_0 \begin{bmatrix} \zeta & \\ & \zeta^{-1} \end{bmatrix} & \\ & {}^t t_0^{-1} \begin{bmatrix} \zeta^{-1} & \\ & \zeta \end{bmatrix} \end{bmatrix}, s\right)$$

$$B\left(\begin{bmatrix} \lambda t_0 \begin{bmatrix} \zeta & \\ & \zeta^{-1} \end{bmatrix} & \\ & {}^t t_0^{-1} \begin{bmatrix} \zeta^{-1} & \\ & \zeta \end{bmatrix} \end{bmatrix}\right)(\zeta - \zeta^{-3})\lambda^{-4}\, d\zeta\, d\lambda; \qquad (127)$$

see (4.6) of [19] for the relevant integration formulas. The above calculations are valid for any choice of $\mathbf{a}, \mathbf{b}, \mathbf{c}$ as long as $S = \begin{bmatrix} \mathbf{a} & \mathbf{b}/2 \\ \mathbf{b}/2 & \mathbf{c} \end{bmatrix}$ is positive definite. We will compute (127), in two special cases, namely when S is of the form $S = \begin{bmatrix} D/4 & \\ & 1 \end{bmatrix}$ or $S = \begin{bmatrix} (1+D)/4 & 1/2 \\ 1/2 & 1 \end{bmatrix}$ with a positive number D. By the argument in Sect. 4.4 of [66], we may assume that S is of the first kind. Then

$$\eta = \begin{bmatrix} 1 & & & \\ \frac{\sqrt{-D}}{2} & 1 & & \\ & & 1 & \frac{\sqrt{-D}}{2} \\ & & & 1 \end{bmatrix}, \text{ and we can choose } t_0 = \begin{bmatrix} 2^{1/2}D^{-1/4} & \\ & 2^{-1/2}D^{1/4} \end{bmatrix}. \text{ From formula (116),}$$

$$B\left(\begin{bmatrix} \lambda t_0 \begin{bmatrix} \zeta & \\ & \zeta^{-1} \end{bmatrix} & \\ & {}^t t_0^{-1} \begin{bmatrix} \zeta^{-1} & \\ & \zeta \end{bmatrix} \end{bmatrix}\right) = \begin{cases} \lambda^l e^{-2\pi\lambda D^{1/2}\frac{\zeta^2+\zeta^{-2}}{2}} & \text{if } \lambda > 0, \\ 0 & \text{if } \lambda < 0. \end{cases} \qquad (128)$$

Next, the argument of $W_{m,l,l_2,l_1}^{\#}$ can be rewritten as an element of $MNK_{\infty}^{G_2}$ as

$$\begin{bmatrix} \lambda D^{-\frac{1}{4}}\left(\frac{\zeta^2+\zeta^{-2}}{2}\right)^{-\frac{1}{2}} & & & \\ & \lambda D^{\frac{1}{4}}\left(\frac{\zeta^2+\zeta^{-2}}{2}\right)^{\frac{1}{2}} & & \\ & & D^{\frac{1}{4}}\left(\frac{\zeta^2+\zeta^{-2}}{2}\right)^{\frac{1}{2}} & \\ & & & D^{-\frac{1}{4}}\left(\frac{\zeta^2+\zeta^{-2}}{2}\right)^{-\frac{1}{2}} \end{bmatrix}\begin{bmatrix} 1 & -i\zeta^2 & & \\ 0 & 1 & & \\ & & 1 & 0 \\ & & -i\zeta^2 & 1 \end{bmatrix} k,$$

where $k = \begin{bmatrix} k_0 & \\ & k_0 \end{bmatrix}$ with $k_0 = (\zeta^2 + \zeta^{-2})^{-1/2}\begin{bmatrix} \zeta^{-1} & i\zeta \\ i\zeta & \zeta^{-1} \end{bmatrix} \in SU(2)$. From now on assume that $m = |l - l_1|$. We have

$$\Phi_{m,l,l_2,l_1}^{\#}\left(\begin{bmatrix} k_0 & \\ & k_0 \end{bmatrix}\right) = \left(\frac{\zeta^2 + \zeta^{-2}}{2}\right)^{-m},$$

and hence

$$W^{\#}_{m,l,l_2,l_1}\left(\eta \begin{bmatrix} \lambda t_0 \begin{bmatrix} \zeta & \\ & \zeta^{-1} \end{bmatrix} & \\ & t_0^{-1}\begin{bmatrix} \zeta^{-1} & \\ & \zeta \end{bmatrix} \end{bmatrix}, s\right)$$

$$= \left(\frac{\zeta^2 + \zeta^{-2}}{2}\right)^{-|l-l_1|} \left|\lambda D^{-\frac{1}{2}}\left(\frac{\zeta^2 + \zeta^{-2}}{2}\right)^{-1}\right|^{3(s+\frac{1}{2})} \omega_\tau(\lambda)^{-1} W_{l_1}\left(\begin{bmatrix} \lambda D^{\frac{1}{2}}\left(\frac{\zeta^2 + \zeta^{-2}}{2}\right) & \\ & 1 \end{bmatrix}\right). \quad (129)$$

If $q \in \mathbb{C}$ is as in (57), then $\omega_\tau(y) = y^q$ for $y > 0$. It follows from (76), (128) and (129) that

$$Z(s, W^{\#}_{m,l,l_2,l_1}, B) = a^+ \pi D^{-\frac{3s}{2} - \frac{3}{4} + \frac{q}{4}} (4\pi)^{\frac{q}{2}} \int\limits_0^\infty \int\limits_1^\infty \lambda^{3s + \frac{3}{2} + l - \frac{q}{2}} \left(\frac{\zeta^2 + \zeta^{-2}}{2}\right)^{-3s - \frac{3}{2} + \frac{q}{2} - |l-l_1|}$$

$$W_{\frac{l_1}{2}, \frac{p}{2}}\left(4\pi \lambda D^{1/2} \frac{\zeta^2 + \zeta^{-2}}{2}\right) e^{-2\pi \lambda D^{1/2} \frac{\zeta^2 + \zeta^{-2}}{2}} (\zeta - \zeta^{-3}) \lambda^{-4} \, d\zeta \, d\lambda. \quad (130)$$

Using the substitutions $u = (\zeta^2 + \zeta^{-2})/2$ and $x = 4\pi \lambda D^{1/2} u$, together with the integral formula for the Whittaker function from [58, p. 316], we get

$$Z(s, W^{\#}_{m,l,l_2,l_1}, B) = a^+ \pi \frac{D^{-3s - \frac{l}{2} + \frac{q}{2}} (4\pi)^{-3s + \frac{3}{2} - l + q}}{6s + l + |l - l_1| - q - 1} \frac{\Gamma(3s + l - 1 + \frac{p}{2} - \frac{q}{2})\Gamma(3s + l - 1 - \frac{p}{2} - \frac{q}{2})}{\Gamma(3s + l - \frac{l_1}{2} - \frac{1}{2} - \frac{q}{2})}.$$

Here, for the calculation of the u-integral, we have assumed that $\mathrm{Re}(6s + l + |l - l_1| - q - 1) > 0$. We summarize our result in the following theorem. We will use the notation

$$\Gamma_{\mathbb{R}}(s) = \pi^{-s/2} \Gamma\left(\frac{s}{2}\right), \qquad \Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s} \Gamma(s). \quad (131)$$

The proof of (134) below follows from the tables at the end of this section.

2.2.2 Theorem. *Assume that the matrix S is of the form*

$$S = \begin{bmatrix} D/4 & \\ & 1 \end{bmatrix} \qquad \text{or} \qquad S = \begin{bmatrix} (1+D)/4 & 1/2 \\ 1/2 & 1 \end{bmatrix} \quad (132)$$

with a positive number D. Let $l \geq 2$ be an integer, and let π be the (limit of) discrete series representation of $\mathrm{PGSp}_4(\mathbb{R})$ with scalar minimal K-type (l, l). Let $l_2 \in \mathbb{Z}$ and τ, χ_0, χ be as in Corollary 1.3.7. Let l_1 be one of the weights occurring in τ, and let $W^{\#}_{|l-l_1|,l,l_2,l_1}$ be the function defined in (78). Let $B : H(\mathbb{R}) \to \mathbb{C}$ be the function defined in (116). Then, for $\mathrm{Re}(6s + l + |l - l_1| - q - 1) > 0$, with the local archimedean integral as in (120),

$$Z(s, W^{\#}_{|l-l_1|,l,l_2,l_1}, B) = a^+_{l_1,p,q} D^{-3s - \frac{l}{2} + \frac{q}{2}} 2^{-3s + \frac{1}{2} - l + \frac{3q}{2} + \frac{l_1}{2}} \pi^{1 + \frac{q}{2} + \frac{l_1}{2}}$$

$$\times \frac{1}{6s + l + |l - l_1| - q - 1} \frac{\Gamma_{\mathbb{C}}(3s + l - 1 + \frac{p}{2} - \frac{q}{2})\Gamma_{\mathbb{C}}(3s + l - 1 - \frac{p}{2} - \frac{q}{2})}{\Gamma_{\mathbb{C}}(3s + l - \frac{l_1}{2} - \frac{1}{2} - \frac{q}{2})}. \quad (133)$$

Here, $a^+_{l_1,p,q}$ is as defined in (77). The numbers $p, q \in \mathbb{C}$ are defined in (57). With Λ being the trivial character, we can rewrite formula (133) as

$$Z(s, W^{\#}_{|l-l_1|,l,l_2,l_1}, B) = \frac{L(3s + \frac{1}{2}, \tilde{\pi} \times \tilde{\tau})}{L(6s + 1, \chi|_{\mathbb{R}^\times})L(3s + 1, \tau \times \mathcal{AI}(\Lambda) \times \chi|_{\mathbb{R}^\times})} Y_{l,l_1,p,q}(s), \quad (134)$$

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where, with $u = 0$ if l_1 is even and $u = 1/2$ if l_1 is odd,

$$Y_{l,l_1,p,q}(s) = \begin{cases} \dfrac{a^+_{l_1,p,q} D^{-3s-\frac{l}{2}+\frac{q}{2}} 2^{q-l+\frac{l_1}{2}+u} \pi^{1+\frac{q}{2}+\frac{l_1}{2}} \left(3s - \frac{q}{2} + \frac{p}{2}\right)}{3s + \frac{l+|l-l_1|}{2} - \frac{1}{2} - \frac{q}{2}} \\ \quad \times \dfrac{\Gamma_{\mathbb{C}}\left(3s+l-1-\frac{p}{2}-\frac{q}{2}\right)\Gamma_{\mathbb{C}}\left(3s+\frac{1}{2}-\frac{q}{2}+u\right)}{\Gamma_{\mathbb{C}}\left(3s+\frac{1}{2}-\frac{q}{2}+|l-\frac{3}{2}-\frac{p}{2}|\right)\Gamma_{\mathbb{C}}\left(3s+l-\frac{l_1}{2}-\frac{1}{2}-\frac{q}{2}\right)} \qquad \text{if } \tau = \mathcal{D}_{p,\frac{q}{2}},\ p \geq 1, \\[2em] \dfrac{a^+_{l_1,p,q} D^{-3s-\frac{l}{2}+\frac{q}{2}} 2^{q-1-l+\frac{l_1}{2}+u} \pi^{1+\frac{q}{2}+\frac{l_1}{2}} \Gamma_{\mathbb{C}}\left(3s+\frac{1}{2}-\frac{q}{2}+u\right)}{\left(3s+\frac{l+|l-l_1|}{2}-\frac{1}{2}-\frac{q}{2}\right)\Gamma_{\mathbb{C}}\left(3s+l-\frac{l_1}{2}-\frac{1}{2}-\frac{q}{2}\right)} \qquad \text{if } \tau = \beta_1 \times \beta_2. \end{cases}$$

Remarks: a) The factor $Y_{l,l_1,p,q}(s)$ is of the form D^{-3s} times a rational function in s.

b) For $l = l_1$ we recover Theorem 4.4.1 of [66]. We point out that in our present approach the number l_1 (the GL_2 weight) can be chosen independently of l (the GSp_4 weight), including the case of different parity.

c) In one of our later applications, the number D will be a fundamental discriminant satisfying $D \equiv 0 \bmod 4$ or $D \equiv 3 \bmod 4$. Having the above theorem available for the two cases of S in (132) assures that S can be chosen to be a half-integral matrix.

2.2.3 Corollary. *Let all hypotheses be as in Theorem 2.2.2. Let $W^\# \in I_W(s, \chi, \chi_0, \tau)$ be the distinguished function defined in table (82). Then*

$$Z(s, W^\#, B) = \frac{L(3s+\frac{1}{2}, \tilde{\pi} \times \tilde{\tau})}{L(6s+1, \chi|_{\mathbb{R}^\times})L(3s+1, \tau \times \mathcal{AI}(\Lambda) \times \chi|_{\mathbb{R}^\times})} Y(s), \tag{135}$$

with

$$Y(s) = \begin{cases} \kappa_{p+1,p}\, Y_{l,p+1,p,q}(s) & \text{in Case A,} \\ \kappa_{l,p}\, Y_{l,l,p,q}(s) & \text{in Case B,} \\ \kappa_{l+1,p}\, Y_{l,l+1,p,q}(s) + \left(3s - \dfrac{p+q}{2}\right)\kappa_{l-1,p}\, Y_{l,l-1,p,q}(s) & \text{in Case C.} \end{cases} \tag{136}$$

Here, the constants $\kappa_{*,p}$ are defined in (80), and the factors $Y_{l,*,p,q}(s)$ are defined in Theorem 2.2.2.

Tables for archimedean factors

The archimedean Euler factors appearing in (134) can be easily calculated via the archimedean local Langlands correspondence. We omit the details and simply show the results in the following tables. For the principal series case $\beta_1 \times \beta_2$, the numbers $p, q \in \mathbb{C}$ are such that $\beta_1(a) = a^{\frac{q+p}{2}}$ and $\beta_2(a) = a^{\frac{q-p}{2}}$.

τ	$L(s, \pi \times \tilde{\tau})$	$\varepsilon(s, \pi \times \tilde{\tau}, \psi^{-1})$
$\mathcal{D}_{p,\mu},\ p \geq 1,\ \mu \in \mathbb{C}$	$\Gamma_{\mathbb{C}}\left(s-\mu+\frac{p}{2}+\frac{1}{2}\right)\Gamma_{\mathbb{C}}\left(s-\mu+\frac{p}{2}-\frac{1}{2}\right)$ $\Gamma_{\mathbb{C}}\left(s-\mu+l-\frac{3}{2}+\frac{p}{2}\right)\Gamma_{\mathbb{C}}\left(s-\mu+|l-\frac{3}{2}-\frac{p}{2}|\right)$	$i^{2l+3p-3+|2l-3-p|}$
$\beta_1 \times \beta_2$	$\Gamma_{\mathbb{C}}\left(s+\frac{1-q-p}{2}\right)\Gamma_{\mathbb{C}}\left(s+\frac{1-q+p}{2}\right)$ $\Gamma_{\mathbb{C}}\left(s+l-\frac{q+p+3}{2}\right)\Gamma_{\mathbb{C}}\left(s+l-\frac{q-p+3}{2}\right)$	1

The next table shows L- and ε-factors for $\tau \times \mathcal{AI}(\Lambda) \times \chi|_{\mathbb{R}^\times}$.

τ	$L(s, \tau \times \mathcal{AI}(\Lambda) \times \chi|_{\mathbb{R}^\times})$	$\varepsilon(s, \tau \times \mathcal{AI}(\Lambda) \times \chi|_{\mathbb{R}^\times}, \psi^{-1})$
$\mathcal{D}_{p,\mu}, \ p \geq 1, \ \mu \in \mathbb{C}$	$\Gamma_{\mathbb{C}}\big(s - \mu + \frac{p}{2}\big)^2$	$(-1)^{p+1}$
$\beta_1 \times \beta_2$	$\Gamma_{\mathbb{C}}\big(s - \frac{q+p}{2}\big)\Gamma_{\mathbb{C}}\big(s - \frac{q-p}{2}\big)$	-1

2.3 The global integral representation

Let F be an algebraic number field and $\mathbb{A}_F$ its ring of adeles. Let L be a quadratic field extension of F; the extension L/F defines the unitary group G_2. The Eisenstein series $E(h, s; f)$ entering into the global integral (142) below will be defined from a section f in a global induced representation of $G_2(\mathbb{A}_F)$. We therefore start by discussing various models of such induced representations.

Global induced representations

Let (τ, V_τ) be a cuspidal, automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$. Let χ_0 be a character of $L^\times \backslash \mathbb{A}_L^\times$ such that the restriction of χ_0 to $\mathbb{A}_F^\times$ concides with ω_τ, the central character of τ. Then, as in (32) in the local case, χ_0 can be used to extend τ to a representation of $M^{(2)}(\mathbb{A}_F)$, denoted by $\chi_0 \times \tau$. Let χ be another character of $L^\times \backslash \mathbb{A}_L^\times$, considered as a character of $M^{(1)}(\mathbb{A}_F)$. This data defines a family of induced representations $I(s, \chi, \chi_0, \tau)$ of $G_2(\mathbb{A}_F)$ depending on a complex parameter s. The space of $I(s, \chi, \chi_0, \tau)$ consists of functions $\varphi : G_2(\mathbb{A}_F) \to V_\tau$ with the transformation property

$$\varphi(m_1 m_2 n g) = \delta_P(m_1 m_2)^{s+1/2} \chi(m_1)(\chi_0 \times \tau)(m_2)\varphi(g)$$

for all $m_1 \in M^{(1)}(\mathbb{A}_F)$, $m_2 \in M^{(2)}(\mathbb{A}_F)$ and $n \in N(\mathbb{A}_F)$. Since the representation τ is given as a space of automorphic forms, we may realize $I(s, \chi, \chi_0, \tau)$ as a space of $\mathbb{C}$-valued functions on $G_2(\mathbb{A}_F)$. More precisely, to each φ as above we may attach the function f_φ on $G_2(\mathbb{A}_F)$ given by $f_\varphi(g) = (\varphi(g))(1)$. Each function f_φ has the property that $\mathrm{GL}_2(\mathbb{A}_F) \ni h \mapsto f_\varphi(hg)$ is an element of V_τ, for each $g \in G_2(\mathbb{A}_F)$. Let $I_{\mathbb{C}}(s, \chi, \chi_0, \tau)$ be the model of $I(s, \chi, \chi_0, \tau)$ thus obtained. A third model of the same representation is obtained by attaching to $f \in I_{\mathbb{C}}(s, \chi, \chi_0, \tau)$ the function

$$W_f(g) = \int_{F \backslash \mathbb{A}_F} f\left(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & x \\ & & & 1 \end{bmatrix} g\right) \psi(\mathbf{c}x)dx, \qquad g \in G_2(\mathbb{A}_F). \tag{137}$$

Here, $\mathbf{c} \in F^\times$ is a fixed element. The map $f \mapsto W_f$ is injective since τ is cuspidal. Let $I_W(s, \chi, \chi_0, \tau)$ be the space of all functions W_f. Now write $\tau \cong \otimes \tau_v$ with local representations τ_v of $\mathrm{GL}_2(F_v)$. We also factor $\chi = \otimes \chi_v$ and $\chi_0 = \otimes \chi_{0,v}$, where χ_v and $\chi_{0,v}$ are characters of $\prod_{w|v} L_w^\times$. Then there are isomorphisms

$$
\begin{array}{ccc}
I(s, \chi, \chi_0, \tau) & \xrightarrow{\ \sim\ } & \otimes_v I(s, \chi_v, \chi_{0,v}, \tau_v) \\
\sim \downarrow & & \downarrow = \\
I_{\mathbb{C}}(s, \chi, \chi_0, \tau) & \xrightarrow{\ \sim\ } & \otimes_v I(s, \chi_v, \chi_{0,v}, \tau_v) \\
\sim \downarrow & & \downarrow \sim \\
I_W(s, \chi, \chi_0, \tau) & \xrightarrow{\ \sim\ } & \otimes_v I_W(s, \chi_v, \chi_{0,v}, \tau_v)
\end{array}
\tag{138}
$$

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Here, the local induced representation $I(s, \chi_v, \chi_{0,v}, \tau_v)$ consists of functions taking values in a model V_{τ_v} of τ_v; see Sect. 1.1 for the precise definition. Assume that $V_{\tau_v} = \mathcal{W}(\tau_v, \psi_v^{-\mathbf{c}})$ is the Whittaker model of V_{τ_v} with respect to the additive character $\psi_v^{-\mathbf{c}}$. If we attach to each $f_v \in I(s, \chi_v, \chi_{0,v}, \tau_v)$ the function $W_{f_v}(g) = f_v(g)(1)$, then we obtain the model $I_W(s, \chi_v, \chi_{0,v}, \tau_v)$ of the same induced representation. The bottom isomorphism in diagram (138) is such that if $W_v \in I_W(s, \chi_v, \chi_{0,v}, \tau_v)$ are given, with the property that $W_v\big|_{G_2(\mathfrak{o}_v)} = 1$ for almost all v, then the corresponding element of $I_W(s, \chi, \chi_0, \tau)$ is the function

$$W(g) = \prod_{v \leq \infty} W_v(g_v), \qquad g = (g_v)_v \in G_2(\mathbb{A}_F). \tag{139}$$

The global integral and the basic identity

Now let $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, S, L, \Lambda$ be as in Sect. 2.1. Let (π, V_π) be a cuspidal, automorphic representation of $H(\mathbb{A}_F)$ which has a global Bessel model of type (S, Λ, ψ). Let further (τ, V_τ) be a cuspidal, automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$, extended to a representation of $M^{(2)}(\mathbb{A}_F)$ via a character χ_0 of $L^\times \backslash \mathbb{A}_L^\times$. Define the character χ of $L^\times \backslash \mathbb{A}_L^\times$ by

$$\chi(y) = \Lambda(\bar{y})^{-1} \chi_0(\bar{y})^{-1}, \qquad y \in \mathbb{A}_L^\times. \tag{140}$$

Let $f(g, s)$ be an analytic family in $I_{\mathbb{C}}(s, \chi, \chi_0, \tau)$. For $\mathrm{Re}(s)$ large enough we can form the Eisenstein series

$$E(g, s; f) = \sum_{\gamma \in P(F) \backslash G_2(F)} f(\gamma g, s). \tag{141}$$

In fact, $E(g, s; f)$ has a meromorphic continuation to the entire complex plane. In [19] Furusawa studied integrals of the form

$$Z(s, f, \phi) = \int_{H(F) Z_H(\mathbb{A}_F) \backslash H(\mathbb{A}_F)} E(h, s; f) \phi(h) \, dh, \tag{142}$$

where $\phi \in V_\pi$. Theorem (2.4) of [19], the "Basic Identity", states that

$$Z(s, f, \phi) = Z(s, W_f, B_\phi) := \int_{R(\mathbb{A}_F) \backslash H(\mathbb{A}_F)} W_f(\eta h, s) B_\phi(h) \, dh, \qquad \eta \text{ as in (16)}, \tag{143}$$

where $R(\mathbb{A}_F)$ is the Bessel subgroup determined by (S, Λ, ψ), and B_ϕ is the Bessel function corresponding to ϕ; see (109). The function $W_f(\,\cdot\,, s)$ appearing in (143) is the element of $I_W(s, \chi, \chi_0, \tau)$ corresponding to $f(\,\cdot\,, s) \in I_{\mathbb{C}}(s, \chi, \chi_0, \tau)$; see (137) for the formula relating f and W_f.

The importance of the basic identity lies in the fact that the integral on the right side of (143) is Eulerian. Namely, assume that $f(\,\cdot\,, s)$ corresponds to a pure tensor $\otimes f_v$ via the middle isomorphism in (138). Assume that $W_v \in I_W(s, \chi_v, \chi_{0,v}, \tau_v)$ corresponds to $f_v \in I(s, \chi_v, \chi_{0,v}, \tau_v)$. Then

$$W_f(g, s) = \prod_{v \leq \infty} W_v(g_v, s), \qquad g = (g_v)_v \in G_2(\mathbb{A}_F),$$

see (139). Assume further that the global Bessel function B_ϕ factorizes as in (110). Then it follows from (143) that

$$Z(s, f, \phi) = \prod_{v \leq \infty} Z_v(s, W_v, B_v), \tag{144}$$

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with the local zeta integrals

$$Z_v(s, W_v, B_v) = \int\limits_{R(F_v)\backslash H(F_v)} W_v(\eta h, s) B_v(h)\, dh. \tag{145}$$

Furusawa has calculated the local integrals (145) in the case where all the data is unramified. In our non-archimedean Theorem 2.2.1 we calculated these integrals in the case where the GSp_4 data is still unramified, but the GL_2 data is arbitrary. Here, we took for W_v the distinguished vector $W^\#$ from Corollary 1.2.4. In our archimedean Corollary 2.2.3 we calculated these integrals in the case where the GSp_4 data is a scalar minimal K-type lowest weight representation, and the GL_2 data is arbitrary. Here, we took for W_v the distinguished vector $W^\#$ defined in table (82).

The global integral representation over $\mathbb{Q}$

The important fact in the theory outlined above is that the local functions W_v can be chosen such that the integrals (145) are all non-zero. We have to make sure, however, that the data entering the local theorems, in particular the characters χ, χ_0 and Λ, fit into a global situation. For simplicity, we assume from now on that the number field is $F = \mathbb{Q}$ (this, however, is not essential).

2.3.1 Lemma. *Let L be an imaginary quadratic field extension of $\mathbb{Q}$. Let $\omega = \otimes\omega_p$ be a character of $\mathbb{Q}^\times\backslash\mathbb{A}^\times$. Let l_2 be an integer such that $(-1)^{l_2} = \omega_\infty(-1)$. Then there exists a character $\chi_0 = \otimes\chi_{0,v}$ of $L^\times\backslash\mathbb{A}_L^\times$ such that*

i) the restriction of χ_0 to $\mathbb{A}^\times$ coincides with ω, and

ii) $\chi_{0,\infty}(\zeta) = \zeta^{l_2}$ for all $\zeta \in S^1$.

Proof. Since ω is trivial on $L^\times \cap \mathbb{A}^\times = \mathbb{Q}^\times$, we can extend ω to a character of $L^\times\mathbb{A}^\times$ in such a way that $\omega\big|_{L^\times} = 1$. Since $S^1 \cap (L^\times\mathbb{A}^\times) = \{\pm 1\}$, we can further extend ω to a character of $S^1 L^\times\mathbb{A}^\times$ in such a way that $\omega(\zeta) = \zeta^{l_2}$ for all $\zeta \in S^1$. For each finite place v of L we will choose a compact subgroup U_v of $\mathfrak{o}_{L,v}^\times$ such that ω can be extended to $S^1 L^\times\mathbb{A}^\times\big(\prod_{v<\infty} U_v\big)$, with ω trivial on $\prod_{v<\infty} U_v$ and $U_v = \mathfrak{o}_{L,v}^\times$ for almost all v. Hence, the U_v should be chosen such that ω is trivial on $\big(\prod_{v<\infty} U_v\big) \cap S^1 L^\times\mathbb{A}^\times$. We consider the intersection

$$\big(\prod_{v<\infty} U_v\big) \cap S^1 L^\times\mathbb{A}^\times = \big(\prod_{v<\infty} U_v\big) \cap \mathbb{C}^\times L^\times\big(\prod_{p<\infty} \mathbb{Z}_p^\times\big). \tag{146}$$

Let zax be an element of this intersection, where $z \in \mathbb{C}^\times$, $a \in L^\times$ and $x \in \prod_{p<\infty} \mathbb{Z}_p^\times$. We have $a \in L^\times \cap \prod_{v<\infty} \mathfrak{o}_{L,v}^\times = \mathfrak{o}_L^\times$, which is a finite set, say $\{a_1, \ldots, a_m\}$. For i such that $a_i \notin \mathbb{Q}$, choose a prime p such that $a_i \notin \mathbb{Z}_p^\times$. Then choose a place v lying above p, and choose U_v so small that $a_i \notin U_v \mathbb{Z}_p^\times$. Then the intersection (146) equals

$$\big(\prod_{v<\infty} U_v\big) \cap \mathbb{C}^\times\mathbb{Q}^\times\big(\prod_{p<\infty} \mathbb{Z}_p^\times\big). \tag{147}$$

We can choose U_v even smaller, so that ω is trivial on this intersection. We can therefore extend ω to a character of

$$S^1 L^\times\mathbb{A}^\times\big(\prod_{v<\infty} U_v\big) = \mathbb{C}^\times L^\times\big(\prod_{v<\infty} U_v\big)\big(\prod_{p<\infty} \mathbb{Z}_p^\times\big). \tag{148}$$

in such a way that ω is trivial on $\prod_{v<\infty} U_v$. The group (148) is of finite index in $\mathbb{C}^\times L^\times \left(\prod_{v<\infty} \mathfrak{o}_{L,v}^\times \right)$, and therefore of finite index in $\mathbb{A}_L^\times$ (using the finiteness of the class number). By Pontrjagin duality, we can now extend ω to a character χ_0 of $\mathbb{A}_L^\times$ with the desired properties. ∎

We now explain the setup for the global integral representation. For simplicity we will work over the rational numbers. We require the following ingredients.

- $\psi = \prod_v \psi_v$ is a character of $\mathbb{Q}\backslash\mathbb{A}$ such that $\psi_\infty(x) = e^{-2\pi i x}$. Also, we require that ψ_p has conductor $\mathbb{Z}_p$ for all finite p. There is exactly one such character ψ.

- Let $D > 0$ be such that $-D$ is a fundamental discriminant, and define $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{Q}$ and the matrix S by

$$
S = S(-D) := \begin{bmatrix} \mathbf{a} & \mathbf{b}/2 \\ \mathbf{b}/2 & \mathbf{c} \end{bmatrix} = \begin{cases} \begin{bmatrix} D/4 & \\ & 1 \end{bmatrix} & \text{if } D \equiv 0 \mod 4, \\ \begin{bmatrix} (1+D)/4 & 1/2 \\ 1/2 & 1 \end{bmatrix} & \text{if } D \equiv 3 \mod 4. \end{cases} \tag{149}
$$

- Let L be the imaginary quadratic field $\mathbb{Q}(\sqrt{-D})$. The unitary groups G_i are defined with respect to the extension $L/\mathbb{Q}$.

- Let $\pi = \otimes\pi_v$ be a cuspidal, automorphic representation of $H(\mathbb{A})$ with the following properties. The archimedean component π_∞ is a (limit of) discrete series representation with minimal K-type (l,l), where $l \geq 2$, and trivial central character. If v is a non-archimedean place, then π_v is unramified and has trivial central character.

- Let $\tau = \otimes\tau_v$ be a cuspidal, automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ with central character ω_τ.

- Let χ_0 be a character of $L^\times\backslash\mathbb{A}_L^\times$ such that $\chi_0\big|_{\mathbb{A}^\times} = \omega_\tau$ and $\chi_{0,\infty}(\zeta) = \zeta^{l_2}$ for $\zeta \in S^1$. Here, l_2 is any integer of the same parity as the weights of τ. Such a character exists by Lemma 2.3.1.

- Let $\Lambda = \otimes\Lambda_v$ be a character of $L^\times\backslash\mathbb{A}_L^\times$ such that $\Lambda_\infty = 1$ and such that Λ_v is unramified for all finite v. Hence, Λ is a character of the ideal class group

$$
\left(L^\times \mathbb{C}^\times \left(\prod_{v\nmid\infty} \mathfrak{o}_v^\times \right) \right) \big\backslash \mathbb{A}_L^\times. \tag{150}
$$

- Let χ be the character of $\mathbb{A}_L^\times$ defined by (140).

Let l_1 be any weight occurring in τ_∞. Let Ψ be the unique cusp form in the space of τ that is a newform at all non-archimedean places and corresponds to a vector of weight l_1 at the archimedean place. We normalize Ψ such that the corresponding Whittaker function

$$
W_\Psi(g) = \int_{\mathbb{Q}\backslash\mathbb{A}} \Psi\left(\begin{bmatrix} 1 & x \\ & 1 \end{bmatrix} g \right) \psi(x)\,dx \tag{151}
$$

satisfies $W_\Psi\left(\begin{bmatrix} t^+ & \\ & 1 \end{bmatrix} \right) = 1$, where t^+ is the positive real number chosen in (75), considered as an idele with trivial non-archimedean components. Let Ψ be extended to a function on $G_1(\mathbb{A})$ via

$\Psi(ag) = \chi_0(a)\Psi(g)$ for $a \in \mathbb{A}_L^\times$, $g \in \mathrm{GL}_2(\mathbb{A})$. Let us explicitly describe a section $f_{|l_1-l|,l,l_2,l_1}(g,s) \in I_{\mathbb{C}}(s,\chi,\chi_0,\tau)$. For a non-archimedean place v, let τ_v have conductor $\mathfrak{p}^n$ and let J_v be the function on $K_v^{G_2} = G_2(\mathfrak{o}_v)$ defined by

$$J_v(k) = \begin{cases} 1 & \text{if } k \in P(\mathfrak{o})\eta_0 K^H \Gamma(\mathfrak{P}^n), \\ 0 & \text{otherwise} \end{cases} \tag{152}$$

(see (45) for the definition of η_0). For $n = 0$ this is the characteristic function of $K_v^{G_2}$. Define

$$J_{|l_1-l|,l,l_2,l_1}(k,s) = \Phi_{|l_1-l|,l,l_2,l_1}^{\#}(k_\infty, s) \cdot \prod_{v<\infty} J_v(k_v), \qquad \text{where } k = (k_v)_v \in \prod_v K_v^{G_2};$$

see (73). Finally, let

$$f_{|l_1-l|,l,l_2,l_1}(g,s) = \delta_P(m_1 m_2)^{s+\frac{1}{2}} \chi(m_1)\Psi(m_2) J_{|l_1-l|,l,l_2,l_1}(k,s) \tag{153}$$

for $g = m_1 m_2 nk$ with $m_1 \in M^{(1)}(\mathbb{A})$, $m_2 \in M^{(2)}(\mathbb{A})$, $n \in N(\mathbb{A})$, $k \in \prod_v K_v^{G_2}$. It is easy to see that $f = f_{|l_1-l|,l,l_2,l_1}$ belongs to $I_{\mathbb{C}}(s,\chi,\chi_0,\tau)$ and that $(W_f)_v$ corresponds to the vector in Corollary 1.2.4 if v is non-archimedean, and to the vector $W_{|l_1-l|,l,l_2,l_1}^{\#}$ given by (78) if $v = \infty$. In view of Theorem 2.2.1 and Corollary 2.2.3, the following important result is now immediate.

2.3.2 Theorem. (Global Integral Representation) *Let ψ, D, S and $\pi, \tau, \chi_0, \chi, \Lambda$ be as above. Let $f = f_{|l_1-l|,l,l_2,l_1}$ be the section in $I_{\mathbb{C}}(s,\chi,\chi_0,\tau)$ defined above, and $\phi = \otimes\phi_v$ be a vector in the space of π such that ϕ_v is unramified for all finite v and such that ϕ_∞ is a vector of weight $(-l,-l)$ in π_∞. Then the global zeta integral $Z(s,f,\phi)$ defined in (142) is given by*

$$Z(s,f,\phi) = \frac{L(3s+\frac{1}{2}, \tilde{\pi} \times \tilde{\tau})}{L(6s+1, \chi|_{\mathbb{A}^\times})L(3s+1, \tau \times \mathcal{AI}(\Lambda) \times \chi|_{\mathbb{A}^\times})} \cdot B_\phi(1) \cdot Y_{l,l_1,p,q}(s) \cdot \prod_{v<\infty} Y_v(s), \quad (154)$$

with B_ϕ as in (109), with the factors $Y_v(s)$ for non-archimedean v given by Theorem 2.2.1, and with the archimedean factor $Y_{l,l_1,p,q}(s)$ given by Theorem 2.2.2. In (154), $\tilde{\pi}$ and $\tilde{\tau}$ denote the contragredient of π and τ, respectively. The symbol $\mathcal{AI}(\Lambda)$ stands for the $\mathrm{GL}_2(\mathbb{A})$ representation attached to the character Λ of $\mathbb{A}_L^\times$ via automorphic induction, and $L(3s+1, \tau \times \mathcal{AI}(\Lambda) \times \chi|_{\mathbb{A}^\times})$ is a standard L-factor for $\mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{GL}_1$.

Next, we state a second version of the above theorem where we choose the distinguished vector at all places, including the archimedean ones. Recall the Cases A,B,C defined in (81). Let l_2 be as in Theorem 1.5.1. The following result is also an immediate consequence of Theorem 2.2.1 and Corollary 2.2.3 and will be key for the functional equation.

2.3.3 Theorem. *Let ψ, D, S and $\pi, \tau, \chi_0, \chi, \Lambda$ be as above. Let B_v be the unramified Bessel function given by formula (113) if v is non-archimedean, and let B_v be the function defined in (116) if v is archimedean. Let $W_v^{\#}(\cdot, s)$ be as in Corollary 1.2.4 if v is non-archimedean, and as in table (82) if v is archimedean. Let*

$$W^{\#}(g,s) = \prod_v W_v^{\#}(g_v,s), \qquad B(h) = \prod_v B_v(h_v),$$

for $g = (g_v)_v \in G_2(\mathbb{A})$ and $h = (h_v)_v \in H(\mathbb{A})$. Then the global zeta integral $Z(s,W^{\#},B)$ defined in (143) is given by

$$Z(s,W^{\#},B) = \frac{L(3s+\frac{1}{2}, \tilde{\pi} \times \tilde{\tau})}{L(6s+1, \chi|_{\mathbb{A}^\times})L(3s+1, \tau \times \mathcal{AI}(\Lambda) \times \chi|_{\mathbb{A}^\times})} Y(s), \tag{155}$$

where $Y(s) = \prod_v Y_v(s)$, a finite product, with the local factors given in Theorem 2.2.1 (non-archimedean case) and Corollary 2.2.3 (archimedean case).

2.4 The functional equation

In this section we prove that, in the setting of Theorem 2.3.3, the global L-function $L(s, \pi \times \tau)$ satisfies the expected functional equation. We begin with some local preparations.

The X factor

Assume that F is a non-archimedean local field of characteristic zero, or $F = \mathbb{R}$. Let $\tau, \chi, \chi_0, \Lambda$ and π be as in Theorem 2.2.1 (non-archimedean case) and Theorem 2.2.2 (archimedean case). We will calculate the function

$$X(s) = K(s) \frac{L(6s+1, \chi|_{F^\times}) L(3s+1, \tau \times \mathcal{AI}(\Lambda) \times \chi|_{F^\times})}{L(6s, \chi|_{F^\times}) L(3s, \tau \times \mathcal{AI}(\Lambda) \times \chi|_{F^\times})}$$

$$\times \varepsilon(6s, \chi|_{F^\times}, \psi^{-1}) \, \varepsilon(3s, \tau \times \mathcal{AI}(\Lambda) \times \chi|_{F^\times}, \psi^{-1}) \frac{\hat{Y}(-s)}{Y(s)}, \tag{156}$$

which will be relevant for the functional equation. Here, $K(s)$ is the factor resulting from the local intertwining operator, defined in (41) and explicitly given in Proposition 1.4.1 (non-archimedean case with $n = 0$), Theorem 1.4.2 (non-archimedean case with $n > 0$) and Theorem 1.5.1 (archimedean case). The factor $Y(s)$ results from the local zeta integral calculation and is given in Theorem 2.2.1 (non-archimedean case) and Corollary 2.2.3 (archimedean case). The factor $\hat{Y}(s)$ is similar to $Y(s)$, but with the data (χ, χ_0, τ) replaced by $(\bar{\chi}^{-1}, \chi\bar{\chi}\chi_0, \chi\tau)$.

2.4.1 Lemma. *Assume that F is p-adic. Let δ be the valuation of the discriminant of L/F if L/F is a ramified field extension, and $\delta = 0$ otherwise. Let $X(s)$ be as in (156). Let $\mathfrak{p}^n$ be the conductor of τ. Assume that the restriction of Λ to $F^\times$ is trivial[3], so that $\chi|_{F^\times} = \omega_\tau^{-1}$.*

i) If $\tau = \beta_1 \times \beta_2$ with unramified characters β_1 and β_2 of $F^\times$, then

$$X(s) = \chi(\varpi)^\delta \chi_{L/F}(-1) q^{-6\delta s}. \tag{157}$$

ii) If L/F is an unramified field extension or $L = F \oplus F$, then

$$X(s) = \omega_\tau(c^2/d) \varepsilon(1/2, \tilde{\tau}, \psi^{-1})^4 q^{-12ns}. \tag{158}$$

Proof. i) By Proposition 1.4.1,

$$X(s) = q^{-\delta} \varepsilon(6s, \chi|_{F^\times}, \psi^{-1}) \, \varepsilon(3s, \tau \times \mathcal{AI}(\Lambda) \times \chi|_{F^\times}, \psi^{-1}) \frac{\hat{Y}(-s)}{Y(s)}.$$

For unramified τ we have $Y(s) = 1$, and the character $\chi|_{F^\times}$ is unramified. Hence

$$X(s) = q^{-\delta} \varepsilon(3s, \chi\tau \times \mathcal{AI}(\Lambda), \psi^{-1}) = \chi(\varpi)^\delta \chi_{L/F}(-1) q^{-6\delta s}.$$

ii) In the case of τ being a spherical representation, (158) follows from (157). We may therefore assume that $n > 0$. Using standard properties of the ε-factors, we can check that

$$\varepsilon(6s, \chi|_{F^\times}, \psi^{-1}) \, \varepsilon(3s, \tau \times \mathcal{AI}(\Lambda) \times \chi|_{F^\times}, \psi^{-1})$$

---

[3]If the $\mathrm{GSp}_4(F)$ representation π has a (S, Λ, ψ) Bessel model, this means that the central character of π is trivial.

equals

$$\chi_{L/F}(\varpi)^n q^{-(6s-1)(n+a(\omega_\tau))-\frac{a(\omega_\tau)}{2}} \varepsilon(\tfrac{1}{2}, \chi|_{F^\times}, \psi^{-1}) \, \varepsilon(\tfrac{1}{2}, \tilde{\tau}, \psi^{-1}).$$

Now the lemma follows directly from Theorem 1.4.2 and Theorem 2.2.1. We note here that, in the case under consideration, we have $n > 0$ and $\left(\frac{L}{\mathfrak{p}}\right) = \pm 1$, so that $Y(s) = L(6s + 1, \chi|_{F^\times})$. ∎

2.4.2 Lemma. *Assume that $F = \mathbb{R}$. Let $X(s)$ be as in (156). Assume that π is the lowest weight representation of $\mathrm{PGSp}_4(\mathbb{R})$ with scalar minimal K-type (l, l), where $l \geq 2$. We assume that $\Lambda = 1$, so that $\chi\big|_{\mathbb{R}^\times} = \omega_\tau^{-1}$. Then*

$$X(s) = -\omega_\tau(-D)^{-1} \, \varepsilon(s, \tilde{\pi} \times \tilde{\tau}, \psi^{-1}) D^{6s}.$$

Proof. The ingredients in the definition (156) of the X-factor are all known; see Theorem 1.5.1 for the factor $K(s)$, Corollary 2.2.3 for the factor $Y(s)$, and the tables in Sect. 2.2 for the L- and ε-factors of $\tau \times \mathcal{AI}(\Lambda) \times \chi|_{\mathbb{R}^\times}$ and $\tilde{\pi} \times \tilde{\tau}$. The asserted formula is then obtained by going through the various possibilities for the type of representation τ and the parity of l, substituting the ingredients and simplifying. This is where Lemma 1.3.8 is used. We omit the details. ∎

The global functional equation

We can now prove the global functional equation for many of the L-functions $L(s, \pi \times \tau)$, provided that the GSp_4 representation π is of the type considered before and has an appropriate global Bessel model. Once we complete the transfer to GL_4, we will be able to remove all restrictions on the GL_2 representation τ; see Theorem 5.2.2.

2.4.3 Theorem. (Functional Equation) *Assume that the positive integer D is such that $-D$ is the discriminant of the number field $L := \mathbb{Q}(\sqrt{-D})$. Let $S(-D)$ be as in (149). Let $\Lambda = \otimes\Lambda_w$ be a character of $L^\times \backslash \mathbb{A}_L^\times$ such that $\Lambda_\infty = 1$ and such that Λ_v is unramified for all finite places v. Let $\pi = \otimes\pi_v$ be a cuspidal, automorphic representation of $\mathrm{GSp}_4(\mathbb{A})$ with the following properties.*

i) π has trivial central character;

ii) *There exists an integer $l \geq 2$ such that π_∞ is the (limit of) discrete series representation of $\mathrm{PGSp}_4(\mathbb{R})$ with scalar minimal K-type (l, l);*

iii) π_p *is unramified for all primes p;*

iv) π *has a global Bessel model of type $(S(-D), \Lambda, \psi)$ (see Sect. 2.1).*

Let $\tau = \otimes\tau_v$ be a cuspidal, automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ such that τ_p is unramified for the primes p dividing D. Then $L(s, \pi \times \tau)$ has meromorphic continuation to all of $\mathbb{C}$ and satisfies the functional equation

$$L(s, \pi \times \tau) = \varepsilon(s, \pi \times \tau) L(1 - s, \tilde{\pi} \times \tilde{\tau}). \tag{159}$$

Here, $\varepsilon(s, \pi \times \tau) = \prod_v \varepsilon(s, \pi_v \times \tau_v, \psi_v^{-1})$, and the local ε-factors are the ones attached to $\pi_v \times \tau_v$ via the local Langlands correspondence.

Remark: The hypothesis on τ will be removed later; see Theorem 5.2.2 for a statement where τ is any cuspidal representation on any GL_n.

Proof. Note that $D = -\mathbf{d}$. Let the characters χ_0, χ and Λ of $L^\times \backslash \mathbb{A}_L^\times$ be as in Theorem 2.3.3. Let $f = \otimes f_v \in I_\mathbb{C}(s, \chi, \chi_0, \tau)$ be the function corresponding to the distinguished vector $W^\# = \otimes W_v^\#$; see the diagram (138). Let $E(g, s; f)$ be the Eisenstein series defined in (141). By the general theory of Eisenstein series,

$$E(g, s; f) = E(g, -s; M(s)f), \tag{160}$$

where $M(s)$ is the global intertwining operator given by a formula similar to (39) in the local case. Note that the Eisenstein series on the right hand side of (160) is defined with respect to the data $(\bar{\chi}^{-1}, \chi\bar{\chi}\chi_0, \chi\tau)$ instead of (χ, χ_0, τ); see (40). By our uniqueness results Corollary 1.2.4 and Corollary 1.3.7, and the explicit archimedean calculations resulting in Theorem 1.5.1,

$$M(s)f(\,\cdot\,, s, \chi, \chi_0, \tau) = K(s)f(\,\cdot\,, -s, \bar{\chi}^{-1}, \chi\bar{\chi}\chi_0, \chi\tau), \tag{161}$$

where $K(s) = \prod_v K_v(s)$, and the local functions $K_v(s)$ are the same as in (41). Hence

$$E(g, s; f) = K(s)E(g, -s; \hat{f}), \tag{162}$$

where $\hat{f}$ abbreviates $f(\,\cdot\,, -s, \bar{\chi}^{-1}, \chi\bar{\chi}\chi_0, \chi\tau)$. For the global zeta integrals defined in (142) it follows that

$$Z(s, f, \phi) = K(s)Z(-s, \hat{f}, \phi). \tag{163}$$

By the basic identity (143),

$$Z(s, W^\#, B_\phi) = K(s)Z(-s, \hat{W}^\#, B_\phi), \tag{164}$$

where $\hat{W}^\#$ abbreviates $W^\#(\,\cdot\,, -s, \bar{\chi}^{-1}, \chi\bar{\chi}\chi_0, \chi\tau)$. Now we let B_ϕ be the distinguished Bessel vector as in Theorem 2.3.3, and apply this theorem to both sides of (164). The result is

$$\frac{L(3s + \frac{1}{2}, \tilde{\pi} \times \tilde{\tau})}{L(6s + 1, \chi|_{\mathbb{A}^\times})L(3s + 1, \tau \times \mathcal{AI}(\Lambda) \times \chi|_{\mathbb{A}^\times})}Y(s)$$

$$= K(s)\frac{L(-3s + \frac{1}{2}, \tilde{\pi} \times \widetilde{\chi\tau})}{L(-6s + 1, \chi^{-1}|_{\mathbb{A}^\times})L(-3s + 1, \chi\tau \times \mathcal{AI}(\Lambda) \times \chi^{-1}|_{\mathbb{A}^\times})}\hat{Y}(-s). \tag{165}$$

Note that $\Lambda(\zeta) = \chi_0(\zeta)^{-1}\chi(\bar{\zeta})^{-1}$, and this character does not change under $(\chi, \chi_0) \mapsto (\bar{\chi}^{-1}, \chi\bar{\chi}\chi_0)$. However, since $\Lambda^{-1} = \bar{\Lambda}$, we have $\mathcal{AI}(\Lambda) = \mathcal{AI}(\bar{\Lambda}) = \mathcal{AI}(\Lambda^{-1})$. Using $\chi\tau \cong \tilde{\tau}$ and the global functional equations for characters and for representations of $\mathrm{GL}_2 \times \mathrm{GL}_2$ (see [38]), we can rewrite (165) as

$$\frac{L(3s + \frac{1}{2}, \tilde{\pi} \times \tilde{\tau})}{L(-3s + \frac{1}{2}, \tilde{\pi} \times \tau)} = K(s)\frac{L(6s + 1, \chi|_{\mathbb{A}^\times})L(3s + 1, \tau \times \mathcal{AI}(\Lambda) \times \chi|_{\mathbb{A}^\times})}{L(6s, \chi|_{\mathbb{A}^\times})L(3s, \tau \times \mathcal{AI}(\Lambda) \times \chi|_{\mathbb{A}^\times})}$$

$$\times \varepsilon(6s, \chi|_{\mathbb{A}^\times})\varepsilon(3s, \tau \times \mathcal{AI}(\Lambda) \times \chi|_{\mathbb{A}^\times})\frac{\hat{Y}(-s)}{Y(s)}$$

$$= \prod_v X_v(s), \tag{166}$$

with local quantities $X_v(s)$ as in (156). These quantities were calculated in Lemmas 2.4.1 and 2.4.2. For a prime p let δ_p be the p-valuation of D, so that $D = \prod_p p^{\delta_p}$. Let n_p be the conductor of τ_p.

Let S be the finite set of primes p such that τ_p is not unramified. By hypothesis, if $p \in S$, then $p \nmid D$, i.e., $L_p/\mathbb{Q}_p$ is not a ramified field extension. Using Lemmas 2.4.1 and 2.4.2, and the fact that $\varepsilon(s, \tilde{\pi}_p \times \tilde{\tau}_p, \psi_p^{-1}) = \varepsilon(s, \tilde{\tau}_p, \psi_p^{-1})^4$ for all finite places p, a straightforward calculation shows that

$$\prod_v X_v(s) = \varepsilon(3s + 1/2, \tilde{\pi} \times \tilde{\tau}).$$

Replacing s by $\frac{1}{3}s - \frac{1}{6}$ and τ by $\tilde{\tau}$, and observing that π is self-contragredient, we obtain the claim of the theorem. ∎

3 The pullback formula

In this section, we prove a second integral representation for our L-function. This is achieved via the "pullback formula", which expresses the (relatively complicated) Eisenstein series $E(g, s; f)$, defined in (141), as the inner product of an automorphic form in the space of τ with the pullback of a simple Siegel-type Eisenstein series on G_3.

We will first prove a local version of the pullback formula. This is the key technical ingredient behind the (global) pullback formula, which, when coupled with the results of the previous sections, will lead to the second integral representation. This will be crucial for proving the entireness of the $\mathrm{GSp}_4 \times \mathrm{GL}_2$ L-function $L(s, \pi \times \tau)$.

3.1 Local sections: non-archimedean case

Let F be p-adic. We use the notation from Theorem 1.2.3. In addition, we will assume that $\Lambda\big|_{F^\times} = 1$. We define the principal congruence subgroup

$$\Gamma^{(3)}(\mathfrak{P}^n) = \{g \in G_3(\mathfrak{o}) \mid g \equiv 1 \bmod \mathfrak{P}^n\}, \tag{167}$$

and consider the subgroup

$$N_1(\mathfrak{o}) = \iota(\begin{bmatrix} 1 & \mathfrak{o} \\ & 1 \end{bmatrix}, 1) \tag{168}$$

(see (27) for the definition of the embedding ι). The group $N_1(\mathfrak{o})$ is normalized by the group

$$\widetilde{R}(\mathfrak{o}) = \left\{ \iota(\begin{bmatrix} 1 & \\ & \lambda \end{bmatrix}, h) \mid h \in H(\mathfrak{o}), \lambda = \mu_2(h) \right\}.$$

As before, let n be such that $\mathfrak{p}^n$ is the conductor of τ. Define the congruence subgroup $C(\mathfrak{P}^n)$ of $G_3(\mathfrak{o})$ by

$$C(\mathfrak{P}^n) := \widetilde{R}(\mathfrak{o})N_1(\mathfrak{o})\Gamma^{(3)}(\mathfrak{P}^n). \tag{169}$$

Note that this is really a group, since $\Gamma^{(3)}(\mathfrak{P}^n)$ is normal in the maximal compact subgroup $G_3(\mathfrak{o})$. We note here an alternate description of $C(\mathfrak{P}^n)$ that will be useful: It consists of precisely the matrices $g \in G_3(\mathfrak{o})$ that satisfy

$$g \equiv \begin{bmatrix} \mathfrak{o} & \mathfrak{o} & & \mathfrak{o} & \mathfrak{o} & \\ \mathfrak{o} & \mathfrak{o} & & \mathfrak{o} & \mathfrak{o} & \\ & & 1 & & & \mathfrak{o} \\ \mathfrak{o} & \mathfrak{o} & & \mathfrak{o} & \mathfrak{o} & \\ \mathfrak{o} & \mathfrak{o} & & \mathfrak{o} & \mathfrak{o} & \\ & & & & & \mathfrak{o}^\times \end{bmatrix} \pmod{\mathfrak{P}^n}. \tag{170}$$

We define $\widetilde{\chi}$ to be the character on P_{12} (see (25)) given by

$$\widetilde{\chi}(m(A,v)n) = \chi(v^{-1}\det(A)). \tag{171}$$

For $s \in \mathbb{C}$, we form the induced representation

$$I(\widetilde{\chi},s) = \mathrm{Ind}_{P_{12}(F)}^{G_3(F)}\left(\widetilde{\chi}\,\delta_{12}^s\right) \tag{172}$$

(see (26)), consisting of smooth functions Ξ on $G_3(F)$ such that

$$\Xi(n_0 m(A,v)g,s) = |v|^{-9(s+\frac{1}{2})}|N(\det A)|^{3(s+\frac{1}{2})}\chi(v^{-1}\det A)\,\Xi(g,s) \tag{173}$$

for $n_0 \in N_{12}(F)$, $m(A,v) \in M_{12}(F)$, $g \in G_3(F)$. For any $t \in L$, set

$$\Omega(t) := \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & \alpha & t & 1 & \\ \bar{\alpha} & & & & 1 & \\ \bar{t} & & & & & 1 \end{bmatrix},$$

where α is the element defined in (15). We define

$$I_L = \begin{cases} \{\varpi^r \mid 0 \le r \le n\} & \text{if } \left(\frac{L}{\mathfrak{p}}\right) = -1, \\ \{(\varpi^{r_1},\varpi^{r_2}) \mid 0 \le r_1, r_2 \le n\} & \text{if } \left(\frac{L}{\mathfrak{p}}\right) = 1, \\ \{\varpi_L^r \mid 0 \le r \le 2n\} & \text{if } \left(\frac{L}{\mathfrak{p}}\right) = 0. \end{cases}$$

From Lemma 3.1.1 below it follows that there exists, for each $t \in I_L$, a unique well-defined section $\Upsilon_t \in I(\widetilde{\chi},s)$ satisfying all of the following,

i) $\Upsilon_t(\Omega(t),s) = 1$,

ii) $\Upsilon_t(gk,s) = \Upsilon_t(g,s)$ for all $g \in G_3(F)$, $k \in C(\mathfrak{P}^n)$,

iii) $\Upsilon_t(g,s) = 0$ if $g \notin P_{12}(F)\Omega(t)C(\mathfrak{P}^n)$.

We define $\Upsilon \in I(\widetilde{\chi},s)$ by

$$\Upsilon = \sum_{t \in I_L} \Upsilon_t. \tag{174}$$

3.1.1 Lemma. *Let $A \in \mathrm{GL}_3(F)$, $v \in F^{\times}$, $n_0 \in N_{12}(F)$ and $t \in \mathfrak{o}_L$ be such that*

$$\Omega(t)^{-1}n_0 m(A,v)\Omega(t) \in C(\mathfrak{P}^n).$$

Then

$$v^{-1}\det(A) \in (1+\mathfrak{P}^n) \cap \mathfrak{o}_L^{\times}.$$

Proof. Since the statement is trivial for $n = 0$, we will assume $n > 0$. Let $P = n_0 m(A, v) =$
$\begin{bmatrix} A & B \\ 0 & v\, {}^t\bar{A}^{-1} \end{bmatrix}$ with $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}$ and $B = \begin{bmatrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \\ b_7 & b_8 & b_9 \end{bmatrix}$. Note that $A^{-1}B$ is self-adjoint;
however we won't use this. Suppose $M := \Omega(t)^{-1} n_0 m(A, v) \Omega(t) \in C(\mathfrak{P}^n)$. This implies that
$A \in \mathrm{GL}_3(\mathfrak{o}_L)$, $v \in \mathfrak{o}^\times$ and $n_0 \in N_{12}(\mathfrak{o})$. Let us set $d := \det(A) \in \mathfrak{o}_L^\times$. We will use the description
given in (170) for a matrix in $C(\mathfrak{P}^n)$. Since the $(1,6), (2,6), (3,4), (3,5)$ entries of M are in $\mathfrak{P}^n$,
we obtain $b_3, b_6, b_7, b_8 \in \mathfrak{P}^n$. Looking at the $(3,2), (3,3)$ entries of M, we obtain $a_8 \in \mathfrak{P}^n$ and
$a_9 \in 1 + \mathfrak{P}^n$. Looking at the $(5,6)$ entry of M and using the fact that $v, d \in \mathfrak{o}_L^\times$ we deduce that
$a_2 a_7 \in \mathfrak{P}^n$. Calculating the determinant of A along the third row, we obtain

$$d = a_7(a_2 a_6 - a_3 a_5) - a_8(a_1 a_6 - a_3 a_4) + a_9(a_1 a_5 - a_2 a_4) \equiv a_1 a_5 - a_2 a_4 - a_7 a_3 a_5 \pmod{\mathfrak{P}^n}.$$

Since $d \in \mathfrak{o}_L^\times$, it follows that either a_2 or a_5 is a unit. Set

$$g_2 := \begin{bmatrix} a_2 + \alpha b_1 & b_1 \\ -\bar{\alpha}(a_2 + \alpha b_1) - \alpha \bar{a}_2 \frac{v}{d} & -\bar{\alpha} b_1 - \bar{a}_2 \frac{v}{d} \end{bmatrix}, \quad g_5 := \begin{bmatrix} a_5 + \alpha b_4 & b_4 \\ \alpha(-a_5 - \alpha b_4 + \bar{a}_5 \frac{v}{d}) & -\alpha b_4 + \bar{a}_5 \frac{v}{d} \end{bmatrix}.$$

Since g_2 and g_5 are submatrices of M mod $\mathfrak{P}^n$, they have entries in $\mathfrak{o} + \mathfrak{P}^n$. The following simple
fact,

$$\text{If } x \in \mathfrak{o} + \mathfrak{P}^n, \text{ then } x \equiv \bar{x} \pmod{(\alpha - \bar{\alpha})\mathfrak{P}^n}, \tag{175}$$

applied to the entries of g_2 resp. g_5, leads to the desired conclusion. ∎

3.2 The local pullback formula: non-archimedean case

In this subsection, we will prove the local pullback formula in the non-archimedean case. Recall
the congruence subgroups defined in (28) – (31). We note that

$$\Upsilon(g \cdot \iota(k_1, k_2), s) = \Upsilon(g, s) \tag{176}$$

for any pair of elements $k_1 \in K^{(1)}(\mathfrak{P}^n)$, $k_2 \in K^H \Gamma(\mathfrak{P}^n)$, satisfying $\mu_1(k_1) = \mu_2(k_2)$. This follows
from the right-invariance of Υ by $C(\mathfrak{P}^n)$. Let Q be the element

$$Q = \begin{bmatrix} 0 & 1 & & & & \\ 1 & 0 & & & & \\ & & 0 & & & -1 \\ & & & 0 & 1 & -1 \\ & & & 1 & 0 & \\ & 1 & 1 & & & 0 \end{bmatrix} \in G_3(F). \tag{177}$$

For $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $m_2(g)$ as in (20),

$$Q \cdot \iota(g, m_2(g)) \cdot Q^{-1} = \begin{bmatrix} a & & -b & b & & \\ & 1 & & & & \\ -c & & d & & & c \\ & & & d & & c \\ & & & & \mu_1(g) & \\ & & b & & & a \end{bmatrix}, \tag{178}$$

where the matrix on the right side lies in P_{12}. It follows that if $g \in G_1(\mathfrak{o})$, then for any $h \in G_3(F)$,

$$\Upsilon(Q \cdot \iota(g, m_2(g))h, s) = \chi(\mu_1(g)^{-1} \det(g))\Upsilon(Qh, s). \tag{179}$$

Let $W^{(0)}$ be the local newform for τ, as in Corollary 1.2.4. For each $0 \leq m \leq n$, let the elements η_m be as in (45). The main object of study for the local pullback formula is the following local zeta integral,

$$Z(g, s; g_2) = q(n) \int_{\mathrm{U}(1,1)(F)} \Upsilon(Q \cdot \iota(h, g_2), s)W^{(0)}(gh) \chi^{-1}(\det(h)) \, dh, \tag{180}$$

where $g \in G_1(F)$, $g_2 \in \mathrm{U}(2,2)(F)$ and $q(n)$ is a normalizing factor equal to $[G_1(\mathfrak{o}) : K^{(0)}(\mathfrak{P}^n)]^{-1}$. The above integral converges absolutely for $\mathrm{Re}(s)$ sufficiently large.

3.2.1 Theorem. (Non-archimedean Local Pullback Formula) *Let $0 \leq m \leq n$. Then, for $\mathrm{Re}(s)$ sufficiently large,*

$$Z(g, s; \eta_m) = \begin{cases} 0 & \text{if } 0 < m \leq n, \\ T(s)W^{(0)}(g) & \text{if } m = 0, \end{cases}$$

where the factor $T(s)$ satisfies

$$T(s)Z(s, W^{\#}, B) = \begin{cases} \dfrac{L(3s + \frac{1}{2}, \tilde{\pi} \times \tilde{\tau})}{L(6s + 1, \chi|_{F^{\times}})L(6s + 2, \chi_{L/F}\chi|_{F^{\times}})L(6s + 3, \chi|_{F^{\times}})} & \text{if } n = 0, \\[4mm] L(3s + \frac{1}{2}, \tilde{\pi} \times \tilde{\tau}) & \text{if } n > 0. \end{cases}$$

Here, $Z(s, W^{\#}, B)$ is the local integral computed in Theorem 2.2.1.

The proof of Theorem 3.2.1 will require the following lemmas.

3.2.2 Lemma. *As a function of h, the quantity $\Upsilon(Q \cdot \iota(h, \eta_m), s)$ depends only on the double coset $K_1^{(1)}(\mathfrak{P}^n)hK_1^{(1)}(\mathfrak{P}^n)$.*

Proof. The right invariance by $K_1^{(1)}(\mathfrak{P}^n)$ follows easily from the right invariance of Υ by $C(\mathfrak{P}^n)$. On the other hand, given $k \in K_1^{(1)}(\mathfrak{P}^n)$, we have

$$\begin{aligned} \Upsilon(Q \cdot \iota(kh, \eta_m), s) &= \Upsilon(Q \cdot \iota(kh, m_2(k)m_2(k)^{-1}\eta_m), s) \\ &= \Upsilon(Q \cdot \iota(h, m_2(k)^{-1}\eta_m), s) \\ &= \Upsilon(Q \cdot \iota(h, \eta_m\eta_m^{-1}m_2(k)^{-1}\eta_m), s) \\ &= \Upsilon(Q \cdot \iota(h, \eta_m), s) \end{aligned}$$

Note that we have used (176), (179) and the fact that $\eta_m^{-1}m_2(k)\eta_m \in K^H\Gamma(\mathfrak{P}^n)$. ∎

Next, we note down the Cartan decompositions for $\mathrm{U}(1,1)(F)$. These follow directly from the Cartan decomposition for $\mathrm{GL}_2(F)$. Suppose $\left(\frac{L}{\mathfrak{p}}\right) = -1$. Then

$$\mathrm{U}(1,1; L)(F) = \bigsqcup_{t \geq 0} K_1^{(1)}(1)A_t K_1^{(1)}(1), \qquad \text{where } A_t = \begin{bmatrix} \varpi^t & 0 \\ 0 & \varpi^{-t} \end{bmatrix}. \tag{181}$$

Suppose $\left(\frac{L}{\mathfrak{p}}\right) = 1$. Then

$$U(1, 1; L)(F) = \bigsqcup_{t_1 \geq t_2} K_1^{(1)}(1) A_{t_1, t_2} K_1^{(1)}(1), \qquad \text{where } A_{t_1, t_2} = \begin{bmatrix} \varpi_L^{t_1} \bar{\varpi}_L^{-t_2} & \\ & \varpi_L^{t_2} \bar{\varpi}_L^{-t_1} \end{bmatrix}. \qquad (182)$$

Suppose $\left(\frac{L}{\mathfrak{p}}\right) = 0$. Then

$$U(1, 1; L)(F) = \bigsqcup_{t \geq 0} K_1^{(1)}(1) A_t K_1^{(1)}(1), \qquad \text{where } A_t = \begin{bmatrix} \varpi_L^t & \\ & \bar{\varpi}_L^{-t} \end{bmatrix}. \qquad (183)$$

3.2.3 Lemma. *For each $0 \leq m \leq n$, there exists a function $L_m(s)$, depending on the local data $(F, L, \chi_0, \chi, \tau)$ but independent of g, such that, for $\mathrm{Re}(s)$ sufficiently large,*

$$Z(g, s; \eta_m) = L_m(s) W^{(0)}(g)$$

for all $g \in G_1(F)$.

Proof. We will only give the proof for the cases $\left(\frac{L}{\mathfrak{p}}\right) = -1$ or 0; the proof for the split case $\left(\frac{L}{\mathfrak{p}}\right) = 1$ is obtained by replacing A_t by A_{t_1, t_2} everywhere below. Recall that V_τ is the space of Whittaker functions on $\mathrm{GL}_2(F)$ realizing the representation τ with respect to the character $\psi^{-\mathbf{c}}$. $W^{(0)}(g)$ is (up to a constant) the unique function in V_τ that is right-invariant by $K^{(1)}(\mathfrak{p}^n)$. Observe that, by (181) resp. (183), we can write

$$q(n)^{-1} Z(g, s; \eta_m) = \sum_{t \geq 0} \int_{K_1^{(1)}(1) A_t K_1^{(1)}(1)} \Upsilon(Q \cdot \iota(h, \eta_m), s) W^{(0)}(gh) \chi^{-1}(\det h) \, dh. \qquad (184)$$

For $g \in G_1(F)$, denote

$$I_t(g; s) = \int_{K_1^{(1)}(1) A_t K_1^{(1)}(1)} \Upsilon(Q \cdot \iota(h, \eta_m), s) W^{(0)}(gh) \chi^{-1}(\det h) \, dh. \qquad (185)$$

By writing $K_1^{(1)}(1) A_t K_1^{(1)}(1)$ as a finite disjoint union $\bigsqcup_\gamma \gamma K_1^{(1)}(\mathfrak{P}^n)$ and using Lemma 3.2.2, we see that I_t is a finite sum of right translates of $W^{(0)}$. Thus, I_t lies in V_τ for each t. In fact, we will show that I_t is a multiple of $W^{(0)}$. Let $k \in K_1^{(1)}(\mathfrak{P}^n)$. By a change of variables, and using Lemma 3.2.2, we see that

$$I_t(gk, s) = I_t(g, s). \qquad (186)$$

Next, let $l \in \mathfrak{o}^\times$ and put $k_l = \begin{bmatrix} 1 & \\ & l \end{bmatrix}$. Then

$$I_t(gk_l, s) = \int_{K_1^{(1)}(1) A_t K_1^{(1)}(1)} \Upsilon(Q \cdot \iota(h, \eta_m), s) W^{(0)}(gk_l h) \chi^{-1}(\det h) \, dh$$

$$= \int_{K_1^{(1)}(1) A_t K_1^{(1)}(1)} \Upsilon(Q \cdot \iota(k_l^{-1} h k_l, m_2(k_l)^{-1}(m_2(k_l) \eta_m m_2(k_l)^{-1}) m_2(k_l)), s)$$

$$\times W^{(0)}(gh) \chi^{-1}(\det h) \, dh$$

$$= \int_{K_1^{(1)}(1)A_tK_1^{(1)}(1)} \Upsilon(Q \cdot \iota(h, \eta_m), s)W^{(0)}(gh)\chi^{-1}(\det h)\, dh. \tag{187}$$

In the last step above we used (179) and the fact that $m_2(k_l)\eta_m m_2(k_l)^{-1} = \eta_m$. The above calculations show that

$$I_t(gk_l, s) = I_t(g, s) \tag{188}$$

for all $l \in \mathfrak{o}^\times$. From this and (186), we conclude that $I_t(gk, s) = I_t(g, s)$ for all $k \in K^{(1)}(\mathfrak{P}^n)$. The fact that the conductor of τ equals $\mathfrak{p}^n$ implies that, for each s, the function $I_t(\cdot, s)$ is a multiple of $W^{(0)}$. Now the assertion follows immediately from (184) and (185). ∎

Proof of Theorem 3.2.1. Let us first prove that $Z(g, s; \eta_m) = 0$ for $0 < m \le n$. We assume $n > 0$ as otherwise the assertion is vacuous. Recall from Lemma 3.2.3 that, for each s, the function $Z(g, s; \eta_m)$ restricted to $\mathrm{GL}_2(F)$ lies in V_τ. Using $\eta_m^{-1}m_2(k)\eta_m \in K^H\Gamma(\mathfrak{P}^n)$ for $k \in K^{(1)}(\mathfrak{p}^{n-m})$, and a similar calculation as in (187), we get

$$Z(gk, s; \eta_m) = Z(g, s; \eta_m)$$

for any $k \in K^{(1)}(\mathfrak{p}^{n-m}) \cap \mathrm{SL}_2(\mathfrak{o})$. Together with (188) it follows that $Z(g, s; \eta_m)$ is right invariant under $K^{(1)}(\mathfrak{p}^{n-m})$. However, because the conductor of τ is n, V_τ does not contain any non-zero function that is right invariant under $K^{(1)}(\mathfrak{p}^{n-m})$ for $m > 0$. This proves that $Z(g, s; \eta_m) = 0$ whenever $m > 0$.

For the rest of this proof, we assume that $m = 0$, so $\eta_m = \eta$. Our task is to evaluate $Z(g, s; \eta)$. We first consider the case $\left(\frac{L}{\mathfrak{p}}\right) = -1$. For $l \in L$, we use $\widetilde{l}$ to denote the element $\begin{bmatrix} l \\ & \bar{l}^{-1} \end{bmatrix}$. It is not hard to prove that the following decomposition holds,

$$\begin{aligned}
\mathrm{U}(1,1)(F) = & \bigsqcup_{l \in \mathfrak{o}_L^\times/(1+\mathfrak{P})} K_1^{(1)}(\mathfrak{P})\widetilde{l}K_1^{(1)}(\mathfrak{P}) \quad \sqcup \bigsqcup_{l \in \mathfrak{o}_L^\times/(1+\mathfrak{P})} K_1^{(1)}(\mathfrak{P})w\widetilde{l}K_1^{(1)}(\mathfrak{P}) \\
& \sqcup \bigsqcup_{\substack{t>0 \\ l \in \mathfrak{o}_L^\times/(1+\mathfrak{P})}} K_1^{(1)}(\mathfrak{P})A_t\widetilde{l}K_1^{(1)}(\mathfrak{P}) \quad \sqcup \bigsqcup_{\substack{t>0 \\ l \in \mathfrak{o}_L^\times/(1+\mathfrak{P})}} K_1^{(1)}(\mathfrak{P})A_tw\widetilde{l}K_1^{(1)}(\mathfrak{P}) \\
& \sqcup \bigsqcup_{\substack{t>0 \\ l \in \mathfrak{o}_L^\times/(1+\mathfrak{P})}} K_1^{(1)}(\mathfrak{P})wA_t\widetilde{l}K_1^{(1)}(\mathfrak{P}) \quad \sqcup \bigsqcup_{\substack{t>0 \\ l \in \mathfrak{o}_L^\times/(1+\mathfrak{P})}} K_1^{(1)}(\mathfrak{P})wA_tw\widetilde{l}K_1^{(1)}(\mathfrak{P}),
\end{aligned} \tag{189}$$

where $w = \begin{bmatrix} & 1 \\ -1 & \end{bmatrix}$. We have the following facts about the support of Υ,

$$Q \cdot \iota(W, \eta) \notin P_{12}(F)\Omega(u)C(\mathfrak{P}), \qquad \text{for } W \in \{A_tw\widetilde{l},\, wA_t\widetilde{l},\, wA_tw\widetilde{l} \mid t \in \mathbb{Z}_{>0}\},\ u \in I_L, \tag{190}$$

$$Q \cdot \iota(w\widetilde{l}, \eta) \notin P_{12}(F)\Omega(u)C(\mathfrak{P}), \qquad \text{for } u \in I_L. \tag{191}$$

The statements (190) and (191) are proved by direct computations involving the relevant 6×6 matrices; we omit the details. From the above statements, Lemma 3.2.3, and (189), we see that

$$Z(g, s; \eta) = q(n)\frac{W^{(0)}(g)}{W^{(0)}(1)} \sum_{t \ge 0} \sum_{l \in \mathfrak{o}_L^\times/(1+\mathfrak{P})} \int_{K_1^{(1)}(\mathfrak{P})A_t\widetilde{l}K_1^{(1)}(\mathfrak{P})} \Upsilon(Q \cdot \iota(A_t\widetilde{l}, \eta), s)W^{(0)}(h)\chi^{-1}(\det h)\, dh.$$

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Now, we have the non-disjoint double coset decomposition

$$\bigsqcup_{l\in\mathfrak{o}_L^\times/(1+\mathfrak{P})} K_1^{(1)}(\mathfrak{P})A_t\tilde{l}K_1^{(1)}(\mathfrak{P}) = \bigsqcup_{k=1}^{n} \bigsqcup_{l\in\mathfrak{o}_L^\times/(1+\mathfrak{P}^n)} \bigcup_{\substack{y\in\mathfrak{p}/\mathfrak{p}^n \\ v(y)=k}} K_1^{(1)}(\mathfrak{P}^n)A_t\begin{bmatrix}1 & \\ y & 1\end{bmatrix}\tilde{l}K_1^{(1)}(\mathfrak{P}^n). \quad (192)$$

Again, by explicit calculation, one verifies that $Q\cdot\iota(A_t\begin{bmatrix}1 & \\ y & 1\end{bmatrix}\tilde{l},\eta)$ does not belong to any of the sets $P_{12}(F)\Omega_r C(\mathfrak{P}^n)$ if $v(y)<n$. It follows that

$$Z(g,s;\eta) = q(n)\frac{W^{(0)}(g)}{W^{(0)}(1)} \sum_{t\geq 0} \sum_{l\in\mathfrak{o}_L^\times/(1+\mathfrak{P}^n)} \Upsilon(Q\cdot\iota(A_t\tilde{l},\eta),s) \int_{K_1^{(1)}(\mathfrak{P}^n)A_t\tilde{l}K_1^{(1)}(\mathfrak{P}^n)} W^{(0)}(h)\chi^{-1}(\det h)\, dh$$

$$= q(n)\frac{W^{(0)}(g)}{W^{(0)}(1)} \sum_{t\geq 0} \sum_{l\in\mathfrak{o}_L^\times/(1+\mathfrak{P}^n)} \Upsilon(Q\cdot\iota(A_t\tilde{l},\eta),s)\chi(l^{-1}) \int_{K_1^{(1)}(\mathfrak{P}^n)A_tK_1^{(1)}(\mathfrak{P}^n)} W^{(0)}(h)\, dh$$

$$= T(s)W^{(0)}(g),$$

where

$$T(s) = \frac{q(n)}{W^{(0)}(1)} \sum_{t\geq 0} \sum_{l\in\mathfrak{o}_L^\times/(1+\mathfrak{P}^n)} \Upsilon(Q\cdot\iota(A_t\tilde{l},\eta),s)\chi(l^{-1}) \int_{K_1^{(1)}(\mathfrak{P}^n)A_tK_1^{(1)}(\mathfrak{P}^n)} W^{(0)}(h)\, dh. \quad (193)$$

To evaluate $T(s)$, we note from the theory of Hecke operators on $\mathrm{GL}_2(F)$ that

$$\int_{K_1^{(1)}(\mathfrak{P}^n)A_tK_1^{(1)}(\mathfrak{P}^n)} \tau(h)W^{(0)}\, dh = \mathrm{vol}(K_1^{(1)}(\mathfrak{P}^n))\lambda_t W^{(0)}, \quad (194)$$

where λ_t depends on t and τ. Using familiar double coset decompositions, the eigenvalues λ_t can easily be calculated. The result is as follows.

- If $\tau = \beta_1 \times \beta_2$ with unramified characters β_1, β_2, then $\lambda_t = \gamma_t - \gamma_{t-1}$ where

$$\gamma_t = q^t\omega_\tau(\varpi)^{-t}\frac{\beta_1(\varpi)^{2t+1} - \beta_2(\varpi)^{2t+1}}{\beta_1(\varpi) - \beta_2(\varpi)}$$

  for $t\geq 0$, and $\gamma_t = 0$ for $t < 0$ (for $\beta_1 = \beta_2$, the fraction is to be interpreted as $(2t+1)\beta_1(\varpi)^{2t}$).

- If τ is an unramified twist of the Steinberg representation, then $\lambda_t = 1$ for all $t\geq 0$.

- If $\tau = \beta_1 \times \beta_2$ is a principal series representation with an unramified character β_1 and a ramified character β_2, then $\lambda_t = q^t\beta_1(\varpi)^{-t}\beta_2(\varpi)^t$ for all $t\geq 0$.

- If τ is supercuspidal, or a ramified twist of the Steinberg representation, or an irreducible principal series representation induced from two ramified characters β_1,β_2, then $\lambda_0 = 1$ and $\lambda_t = 0$ for $t > 0$.

We substitute the above formulas for λ_t in the integral inside (193). Then, we use the definition of Υ to compute the term $\Upsilon(Q\cdot\iota(A_t\tilde{l},\eta),s)$; it turns out that

$$\Upsilon(Q\cdot\iota(A_t\tilde{l},\eta),s) = q^{-6t(s+\frac{1}{2})}\chi(l)\chi(\varpi^t). \quad (195)$$

After making these substitutions, it is easy to evaluate $T(s)$ for the possible types of τ listed above simply by summing the geometric series. This proves Theorem 3.2.1 in the inert case. The proofs for the cases $\left(\frac{L}{\mathfrak{p}}\right) = 0$ or 1 are very similar to the above. The details are left to the reader. ∎

3.3 Local sections: archimedean case

In this subsection, $F = \mathbb{R}$ and $L = \mathbb{C}$. Let τ be as in Sect. 1.3, and let l_1 be any of the weights occurring in τ. Let χ_0 be the character of $\mathbb{C}^\times$ such that $\chi_0|_{\mathbb{R}^\times} = \omega_\tau$ and $\chi_0(\zeta) = \zeta^{-l_1}$ for $\zeta \in \mathbb{C}^\times$, $|\zeta| = 1$. Let χ be the character of $\mathbb{C}^\times$ given by $\chi(\zeta) = \chi_0(\bar{\zeta})^{-1}$.

We define $I(\widetilde{\chi}, s)$ in the present (archimedean) case in exactly the same manner as it was defined in the non-archimedean case (see (172), (173)). In this subsection, we will construct a special element of $I(\widetilde{\chi}, s)$. Let η_0 be the matrix defined in (74). For $\theta \in \mathbb{R}$, let

$$r(\theta) = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \in \mathrm{SO}(2),$$

and

$$r_\times(\theta) = \begin{bmatrix} \cos(\theta) & & & & & \sin(\theta) \\ & 1 & & & & \\ & & \cos(\theta) & \sin(\theta) & & \\ & & -\sin(\theta) & \cos(\theta) & & \\ & & & & 1 & \\ -\sin(\theta) & & & & & \cos(\theta) \end{bmatrix} \in K_\infty^{G_3}, \tag{196}$$

where $K_\infty^{G_3}$ is the maximal compact subgroup of $G_3^+(\mathbb{R}) = \{g \in G_3(\mathbb{R}) \mid \mu_3(g) > 0\}$. Explicitly,

$$K_\infty^{G_3} = \left\{ \begin{bmatrix} A & B \\ -B & A \end{bmatrix} \mid A, B \in \mathrm{Mat}_{3,3}(\mathbb{C}),\ {}^t\bar{A}B = {}^t\bar{B}A,\ {}^t\bar{A}A + {}^t\bar{B}B = 1 \right\}.$$

Also, we let

$$w_Q = \begin{bmatrix} & 1 & & & & \\ 1 & & & & & \\ & & & & & -1 \\ & & & & 1 & \\ & & & 1 & & \\ & & 1 & & & \end{bmatrix}, \qquad s_1 = \begin{bmatrix} & 1 & & & \\ 1 & & & & \\ & & 1 & & \\ & & & & 1 \\ & & & 1 & \\ & & & & & 1 \end{bmatrix}, \tag{197}$$

and

$$t_\infty = w_Q \cdot \iota(1, \eta_0) = \iota(r(\pi/2), s_1\eta_0). \tag{198}$$

Let l be a positive integer (in our application we will consider a discrete series representation of $\mathrm{PGSp}_4(\mathbb{R})$ with scalar minimal K-type (l, l)). To ease notation, we will denote the function $\Phi_{|l-l_1|,l,-l_1,l_1}^\#$ defined in (73) by J_∞. Explicitly,

$$J_\infty = \begin{cases} i^{l-l_1}\hat{b}^{l_1-l}(\hat{a}\hat{d} - \hat{b}\hat{c})^{l-l_1}D^{-l} & \text{if } l \le l_1, \\ i^{l_1-l}\hat{c}^{l-l_1}D^{-l} & \text{if } l \ge l_1, \end{cases} \tag{199}$$

where $D(g) = \det(J(g, i_2))$, and the functions $\hat{a}, \hat{b}, \hat{c}, \hat{d}$ are defined before Lemma 1.3.3. Note that $J_\infty(\eta_0) = 1$.

By the Iwasawa decomposition, $G_3(\mathbb{R}) = P_{12}(\mathbb{R})K_\infty^{G_3}$. The following lemma provides a criterion for when functions on $K_\infty^{G_3}$ can be extented to nice sections in $I(\widetilde{\chi}, s)$.

3.3.1 Lemma. *Suppose Υ_∞ is an analytic function on $K_\infty^{G_3}$ that satisfies the following conditions.*

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i) For all $A \in \mathrm{U}(3)$ and all $g \in K_\infty^{G_3}$,

$$\Upsilon_\infty(\begin{bmatrix} A \\ & A \end{bmatrix} g) = \det(A)^{-l_1} \Upsilon_\infty(g). \tag{200}$$

ii) For all $\theta \in \mathbb{R}$ and all $k \in K_\infty^{G_2}$,

$$\Upsilon_\infty(r_\times(\theta) \, t_\infty \, \iota(1, k)) = \Upsilon_\infty(r_\times(\theta) \, t_\infty) J_\infty(\eta_0 k). \tag{201}$$

iii) For all $\varphi \in \mathbb{R}$ and all $g \in K_\infty^{G_3}$,

$$\Upsilon_\infty(g \, \iota(r(\varphi), 1)) = e^{-i l_1 \varphi} \, \Upsilon_\infty(g). \tag{202}$$

Then Υ_∞ can be extended in a unique way to an analytic function on $G_3(\mathbb{R})$ satisfying the following conditions.

i)

$$\Upsilon_\infty \in I(\tilde{\chi}, s). \tag{203}$$

ii) For all $\zeta \in S^1$ and all $h \in \mathrm{U}(1,1)(\mathbb{R})$

$$\Upsilon_\infty(Q \cdot \iota(\begin{bmatrix} \zeta \\ & \zeta \end{bmatrix} h, \eta_0), s) = \zeta^{-l_1} \, \Upsilon_\infty(Q \cdot \iota(h, \eta_0), s). \tag{204}$$

iii) We have the following equation for any $k \in K_\infty^{G_2}$ and $h \in \mathrm{U}(1,1)(\mathbb{R})$:

$$\Upsilon_\infty(Q \cdot \iota(h, \eta_0 k), s) = \Upsilon_\infty(Q \cdot \iota(h, \eta_0), s) J_\infty(\eta_0 k). \tag{205}$$

iv) For all $\varphi \in \mathbb{R}$ and all $g \in G_3(\mathbb{R})$,

$$\Upsilon_\infty(g \, \iota(r(\varphi), 1), s) = e^{-i l_1 \varphi} \, \Upsilon_\infty(g, s). \tag{206}$$

v) For all $\varphi \in \mathbb{R}$ and all $h \in \mathrm{U}(1,1)(\mathbb{R})$,

$$\Upsilon_\infty(Q \cdot \iota(r(\varphi)h, \eta_0), s) = e^{-i l_1 \varphi} \, \Upsilon_\infty(Q \cdot \iota(h, \eta_0), s). \tag{207}$$

Proof. Using the Iwasawa decomposition, it is easy to see that Υ_∞ can be extended in a unique way to an analytic function on $G_3(\mathbb{R})$ satisfying condition (203). Note that condition (200) is tailored so that the extension is well-defined. Next, another appeal to the Iwasawa decomposition and the fact that $\Upsilon_\infty \in I(\tilde{\chi}, s)$ shows that (202) implies (206). We now prove (205). We have the identity

$$Q \cdot \iota(\begin{bmatrix} \sqrt{a} \\ & \sqrt{a}^{-1} \end{bmatrix}, 1) = p_a k_a w_Q \tag{208}$$

with

$$p_a = \begin{bmatrix} 1 & & & & \frac{1}{1+a} \\ & 1 & & & \\ & & 1 & \frac{1}{1+a} & \\ & & & 1 & \\ & & & & 1 \\ & & & & & 1 \end{bmatrix} \begin{bmatrix} \sqrt{\frac{a}{1+a}} & & & & \\ & 1 & & & \\ & & \sqrt{\frac{1}{1+a}} & & \\ & & & \sqrt{\frac{1+a}{a}} & \\ & & & & 1 \\ & & & & & \sqrt{1+a} \end{bmatrix} \in P_{12}(\mathbb{R}) \tag{209}$$

and

$$k_a = \begin{bmatrix} \sqrt{\frac{a}{1+a}} & & & & \frac{-1}{\sqrt{1+a}} \\ & 1 & & 0 & \\ & & \sqrt{\frac{a}{1+a}} & \frac{-1}{\sqrt{1+a}} & \\ & & \frac{1}{\sqrt{1+a}} & \sqrt{\frac{a}{1+a}} & \\ & 0 & & 1 & \\ \frac{1}{\sqrt{1+a}} & & & & \sqrt{\frac{a}{1+a}} \end{bmatrix} \in K_\infty^{G_3}. \tag{210}$$

On the other hand, observe that $k_a = r_\times(\theta)$ for θ ranging in an open subset of $\mathbb{R}/2\pi\mathbb{Z}$; so condition (201) is equivalent to

$$\Upsilon_\infty(k_a w_Q \cdot \iota(1, \eta_0 k), s) = \Upsilon_\infty(k_a w_Q \cdot \iota(1, \eta_0), s) J_\infty(\eta_0 k). \tag{211}$$

Using (178) and (208), properties of J_∞ imply that condition (205) holds for all h of the form $r(\varphi) \begin{bmatrix} \sqrt{a} & \\ & \sqrt{a}^{-1} \end{bmatrix}$. A similar calculation shows that (205) holds for all elements h of the form $\begin{bmatrix} \zeta & \\ & \zeta \end{bmatrix} r(\varphi) \begin{bmatrix} \sqrt{a} & \\ & \sqrt{a}^{-1} \end{bmatrix}$. In combination with (206) it follows that (205) holds for all $h \in \mathrm{U}(1,1)(\mathbb{R})$. Finally, (204) and (207) can be verified using (179), (205), and the properties of J_∞. ∎

We define the functions x_{ij} on $K_\infty^{G_3}$ by

$$x_{ij}(g) = ij\text{-coefficient of } J({}^t\bar{g}\tilde{g}, I), \qquad g \in K_\infty^{G_3}, \tag{212}$$

where

$$\tilde{g} = \begin{bmatrix} A & -B \\ B & A \end{bmatrix} \text{ for } g = \begin{bmatrix} A & B \\ -B & A \end{bmatrix}.$$

Any polynomial expression in the functions x_{ij} and their complex conjugates is $K_\infty^{G_3}$-finite. We further define

$$X_1 := \big((1 - |x_{33}|^2)x_{11} + x_{13}x_{31}\overline{x_{33}}\big)\overline{x_{13}} + \big((1 - |x_{33}|^2)x_{12} + x_{13}x_{32}\overline{x_{33}}\big)\overline{x_{23}},$$
$$X_2 := \big((1 - |x_{33}|^2)x_{21} + x_{23}x_{31}\overline{x_{33}}\big)\overline{x_{13}} + \big((1 - |x_{33}|^2)x_{22} + x_{23}x_{32}\overline{x_{33}}\big)\overline{x_{23}},$$
$$Y_1 := \big((1 - |x_{33}|^2)\overline{x_{11}} + \overline{x_{13}x_{31}}x_{33}\big)x_{31} + \big((1 - |x_{33}|^2)\overline{x_{21}} + \overline{x_{23}x_{31}}x_{33}\big)x_{32},$$
$$Y_2 := \big((1 - |x_{33}|^2)\overline{x_{12}} + \overline{x_{13}x_{32}}x_{33}\big)x_{31} + \big((1 - |x_{33}|^2)\overline{x_{22}} + \overline{x_{23}x_{32}}x_{33}\big)x_{32}.$$

Let Υ_0 be the function on $K_\infty^{G_3}$ given by

$$\Upsilon_0 = \begin{cases} \big(\overline{x_{31}}Y_2 - \overline{x_{32}}Y_1\big)^{l_1 - l} & \text{if } l \le l_1, \\ \big(x_{13}X_2 - x_{23}X_1\big)^{l - l_1} & \text{if } l \ge l_1. \end{cases} \tag{213}$$

By explicit calculation, one verifies that

$$\Upsilon_0(r_\times(\theta)\, \iota(1, s_1\eta_0)) = (-1)^{l_1 - l}\, \sin(2\theta)^{4|l - l_1|}. \tag{214}$$

3.3.2 Lemma. Let Υ_0 be as in (213). Then the function $\Upsilon_\infty(g) := \Upsilon_0(g)\det(J(g, i_2))^{-l_1}$ is $K_\infty^{G_3}$-finite and satisfies the conditions from Lemma 3.3.1. Moreover,

$$\Upsilon_\infty(r_\times(\theta)\, t_\infty) = (-i)^{l_1}\, (-1)^l\, \sin(2\theta)^{4|l - l_1|} \tag{215}$$

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for all $\theta \in \mathbb{R}$. If $\Upsilon_\infty(\,\cdot\,, s)$ denotes the extension of Υ_∞ to a function on all of $G_3(\mathbb{R})$, then

$$\Upsilon_\infty(Q\,\iota(\begin{bmatrix} \sqrt{a} & \\ & \sqrt{a}^{-1} \end{bmatrix}, \eta_0), s) = 2^{4|l-l_1|}(-i)^{l_1}\,(-1)^l\,(\sqrt{a} + \sqrt{a}^{-1})^{q-6(s+\frac{1}{2})-4|l-l_1|} \tag{216}$$

for all $a > 0$. Here, $q \in \mathbb{C}$ is such that $\omega_\tau(a) = a^q$ for $a > 0$.

Proof. From the construction, it is a routine calculation to verify that Υ_∞ satisfies the conditions (200) and (201). Property (202) follows from the right transformation properties of the functions x_{ij}. Property (215) follows easily from (214). To prove (216), note that, by (208),

$$Q\,\iota(\begin{bmatrix} \sqrt{a} & \\ & \sqrt{a}^{-1} \end{bmatrix}, 1) = p_a r_\times(\theta)\,\iota(\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}, s_1) = p_a r_\times(\theta)\,t_\infty\,\iota(1, \eta_0^{-1})$$

with p_a as in (209) and $\theta \in \mathbb{R}$ such that $\cos(\theta) = \sqrt{\frac{a}{1+a}}$ and $\sin(\theta) = \frac{-1}{\sqrt{1+a}}$. This leads to the claimed result in a straightforward manner. ∎

3.4 The local pullback formula: archimedean case

In this section, we will prove the local pullback formula in the archimedean case. Let $\Upsilon_\infty(\,\cdot\,, s)$ be the element of $I(\widetilde{\chi}, s)$ constructed in Lemma 3.3.2. For any $g_2 \in \mathrm{U}(2,2)(F)$, $g \in G_1(F)$ and $s \in \mathbb{C}$ let

$$Z_\infty(g, s; g_2) = \int\limits_{\mathrm{U}(1,1)(\mathbb{R})} \Upsilon_\infty(Q\cdot\iota(h, g_2), s)\,W_{l_1}(gh)\,\chi(\det(h))^{-1}\,dh, \tag{217}$$

which converges absolutely for $\mathrm{Re}(s)$ sufficiently large. Here, W_{l_1} is as in (76). For simplicity, we will assume that $\mathbf{c} = 1$.

3.4.1 Theorem. (Archimedean Local Pullback Formula) *Let l be a positive integer, and let l_1 be any of the weights occurring in τ. Then, for $\mathrm{Re}(s)$ sufficiently large,*

$$Z_\infty(g, s; \eta_0) = T_\infty(s)\,W_{l_1}(g), \tag{218}$$

where, up to a non-zero constant (depending on τ_∞ and l and l_1, but not on s),

$$T_\infty(s) = 2^{-6s}\,\frac{\Gamma(3s + 1 - \frac{q}{2} + 2|l_1 - l| - \frac{p}{2})\,\Gamma(3s + 1 - \frac{q}{2} + 2|l_1 - l| + \frac{p}{2})}{\Gamma(3s + \frac{3}{2} - \frac{q}{2} + 2|l_1 - l| - \frac{l_1}{2})\,\Gamma(3s + \frac{3}{2} - \frac{q}{2} + 2|l_1 - l| + \frac{l_1}{2})}.$$

Here, p and q are as in (57).

Proof. Recall that we have chosen $t^+ \in \mathbb{R}_{>0}$ and normalized the function W_{l_1} such that $W_{l_1}(\begin{bmatrix} t^+ & \\ & 1 \end{bmatrix}) = 1$; see (77). By changing the value of p slightly and using the holomorphy of both sides of (218) in p, we may work under the additional assumption that $W_{l_1}(1) \neq 0$. We have

$$Z_\infty(g, s; \eta_0) = \int\limits_{\mathrm{U}(1,1)(\mathbb{R})} \Upsilon_\infty(Q\cdot\iota(h, \eta_0), s)\,W_{l_1}(gh)\,\chi^{-1}(\det(h))\,dh$$

$$\overset{(204)}{=} \int\limits_{\mathrm{U}(1,1)(\mathbb{R})/Z} \Upsilon_\infty(Q\cdot\iota(h, \eta_0), s)\,W_{l_1}(gh)\,\chi^{-1}(\det(h))\,dh$$

$$= \int\limits_{\mathrm{SL}(2,\mathbb{R})} F_1(h)\, W_{l_1}(gh)\, dh,$$

where the function F_1 on $\mathrm{SL}_2(\mathbb{R})$ is defined by $F_1(h) := \Upsilon_\infty(Q\,\iota(h,\eta_0),s)$. Hence $Z_\infty(g,s;\eta_0)$ is in the space of τ. It follows from (207) that $Z_\infty(g,s;\eta_0)$ is a vector of weight l_1. By irreducibility, there is (up to multiples) only one vector of weight l_1 in the space of τ, namely W_{l_1}. It follows that $Z_\infty(g,s;\eta_0)$ is a multiple of $W_{l_1}(g)$. By an easy calculation, in terms of the Iwasawa decomposition,

$$F_1\!\left(\begin{bmatrix}1 & b \\ & 1\end{bmatrix}\begin{bmatrix}\sqrt{a} & \\ & \sqrt{a}^{-1}\end{bmatrix} r(\theta)\right) = (-i)^{l_1} e^{-il_1\theta}\left(\frac{(1+a)^2+b^2}{a}\right)^{\frac{q}{2}-3(s+\frac{1}{2})-2|l_1-l|}\left(\frac{b-i(a+1)}{|b-i(a+1)|}\right)^{-l_1}.$$

So we get that $Z_\infty(g,s;\eta_0) = T_\infty(s)W_{l_1}(g)$, where

$$T_\infty(s)W_{l_1}(1) = Z_\infty(1,s;\eta_0)$$

$$= (-i)^{l_1}\int\limits_{-\infty}^{\infty}\int\limits_{0}^{\infty} a^{-1}\left(\frac{(1+a)^2+b^2}{a}\right)^{\frac{q}{2}-3(s+\frac{1}{2})-2|l_1-l|}\left(\frac{b-i(a+1)}{|b-i(a+1)|}\right)^{-l_1}$$

$$e^{2\pi i b}\, W_{l_1}\!\left(\begin{bmatrix}\sqrt{a} & \\ & \sqrt{a}^{-1}\end{bmatrix}\right) d^\times a\, db.$$

For brevity, we make the following substitutions,

$$s' = \frac{q}{2} - 3\left(s+\frac{1}{2}\right) - 2|l_1 - l|,$$

$$s_1 = -s' - \frac{l_1}{2}, \qquad s_2 = -s' - \frac{1}{2} - \frac{p}{2}, \qquad s_3 = -s' - \frac{1}{2} + \frac{p}{2}, \qquad s_4 = -s' + \frac{l_1}{2}.$$

By the integral formula from [31, (6.11)]), the first formula in Sect. 7.5.2 of [58], and Lemma 3.4.2 below,

$$Z_\infty(1,s;\eta_0) = \frac{(2\pi)^{-2s'}}{\Gamma(s_1)\Gamma(s_4)}\int\limits_{0}^{\infty}\int\limits_{0}^{\infty} a^{-s'-1}t^{s_1-1}(t+1)^{s_4-1}e^{-2\pi(a+1)(1+2t)}\, W_{l_1}\!\left(\begin{bmatrix}\sqrt{a} & \\ & \sqrt{a}^{-1}\end{bmatrix}\right) d^\times a\, dt$$

$$= a^+\, 2^{q+2}\pi^{-s'+1+\frac{q}{2}}\frac{\Gamma(s_2)\Gamma(s_3)}{\Gamma(s_1)^2\Gamma(s_4)}\int\limits_{0}^{\infty} t^{s_1-1}(t+1)^{s_4-1}e^{-2\pi(1+2t)}\, {}_2F_1\!\left(s_2,s_3,s_1;-t\right) dt$$

$$= a^+\, 2^{q+1-p-2s_2}\pi^{1+\frac{q}{2}}\frac{\Gamma(s_2)\Gamma(s_3)}{\Gamma(s_1)\Gamma(s_4)} W_{\frac{l_1}{2},\frac{p}{2}}(4\pi)$$

$$= 4^{s'+1}\pi\frac{\Gamma(s_2)\Gamma(s_3)}{\Gamma(s_1)\Gamma(s_4)} W_{l_1}(1),$$

and so

$$T_\infty(s) = 4^{s'+1}\pi\frac{\Gamma(s_2)\Gamma(s_3)}{\Gamma(s_1)\Gamma(s_4)}.$$

This concludes the proof. ∎

3.4.2 Lemma. *For complex numbers s_1, s_2, s_3 with $\mathrm{Re}(s_1) > 0$,*

$$\int\limits_{0}^{\infty} t^{s_1-1}(t+1)^{s_2+s_3-s_1}e^{-4\pi t}\, {}_2F_1\!\left(s_2,s_3,s_1;-t\right) dt = \Gamma(s_1)(4\pi)^{-\frac{s_2+s_3+1}{2}}e^{2\pi}\, W_{\frac{s_2+s_3+1}{2}-s_1,\frac{s_3-s_2}{2}}(4\pi).$$

Proof. This follows by first applying the third equation of [28, (9.131, 1)], followed by the integral formula [28, (7.522, 1)]. ∎

3.5 The global pullback formula

In the following we use the global set-up of Theorem 2.3.2. We set the number l_2 to be $-l_1$. We will hence work with the section $f = f_{|l_1-l|,l,-l_1,l_1}$ in $I_{\mathbb{C}}(s, \chi, \chi_0, \tau)$. It gives rise to the Eisenstein series $E(g, s; f)$ via (141). In this section we will prove the global pullback formula, which expresses the Eisenstein series $E(g, s; f)$ on $G_2(\mathbb{A})$ in terms of the pullback of a simpler Eisenstein series on $G_3(\mathbb{A})$.

Let $\widetilde{\chi}$ be the character on $P_{12}(\mathbb{A})$ defined by $\widetilde{\chi}(m(A, v)n) = \chi(v^{-1}\det(A))$; see (171) for the corresponding local definition. For $s \in \mathbb{C}$, we form the global induced representation

$$I(\widetilde{\chi}, s) = \mathrm{Ind}_{P_{12}(\mathbb{A})}^{G_3(\mathbb{A})}(\widetilde{\chi}\delta_{12}^s) \tag{219}$$

(see (26)), consisting of functions Υ on $G_3(\mathbb{A})$ such that

$$\Upsilon(m(A, v)ng, s) = |v|^{-9(s+\frac{1}{2})}|N(\det A)|^{3(s+\frac{1}{2})}\chi(v^{-1}\det(A))\,\Upsilon(g, s) \tag{220}$$

for $n \in N_{12}(\mathbb{A})$, $m(A, v) \in M_{12}(\mathbb{A})$, $g \in G_3(\mathbb{A})$. Now, let $\Upsilon = \otimes_v \Upsilon_v \in I(\widetilde{\chi}, s)$, where Υ_v is defined by (174) in the non-archimedean case and defined as in Lemma 3.3.2 in the archimedean case. We define the Eisenstein series $E_\Upsilon(g, s)$ on $G_3(\mathbb{A})$ by

$$E_\Upsilon(g, s) = \sum_{\gamma \in P_{12}(\mathbb{Q})\backslash G_3(\mathbb{Q})} \Upsilon(\gamma g, s) \tag{221}$$

for $\mathrm{Re}(s)$ sufficiently large, and by analytic continuation elsewhere. Furthermore, let

$$T(s) = \prod_v T_v(s), \tag{222}$$

where the local functions $T_v(s)$ are defined by Theorem 3.2.1 in the non-archimedean case and by Theorem 3.4.1 in the archimedean case. Note that though (222) makes sense for $\mathrm{Re}(s)$ sufficiently large; it is clear from the definitions of $T_v(s)$ that $T(s)$ can be analytically continued to a meromorphic function on the entire complex plane (it is effectively just a ratio of global L-functions).

3.5.1 Theorem. (Global Pullback Formula) *Let Ψ be the cusp form in the space of τ corresponding to a local newform at all non-archimedean places, a vector of weight l_1 at the archimedean place, and with the same normalization for the corresponding Whittaker function W_Ψ as in (151). Let Ψ be extended to a function on $G_1(\mathbb{A})$ via $\Psi(ag) = \chi_0(a)\Psi(g)$ for $a \in \mathbb{A}_L^\times$, $g \in \mathrm{GL}_2(\mathbb{A})$. For an element $g \in G_2(\mathbb{A})$, let $\mathrm{U}[g](\mathbb{A})$ denote the subset of $G_1(\mathbb{A})$ consisting of all elements h such that $\mu_2(g) = \mu_1(h)$. Then we have the following identity of meromorphic functions,*

$$\chi(\mu_2(g)) \int_{\mathrm{U}(1,1)(\mathbb{Q})\backslash \mathrm{U}[g](\mathbb{A})} E_\Upsilon(\iota(h, g), s)\,\Psi(h)\,\chi(\det(h))^{-1}\,dh = T(s)E(g, s; f), \tag{223}$$

where $f = f_{|l_1-l|,l,-l_1,l_1}$ as in Theorem 2.3.2.

Proof. Let $\mathcal{E}(g,s)$ denote the left hand side above. Note that $E_\Upsilon(\iota(g,h),s)$ is slowly increasing away from its poles, while $\Psi(h)$ is rapidly decreasing. Thus $\mathcal{E}(g,s)$ converges uniformly and absolutely for $s \in \mathbb{C}$ away from the poles of the Eisenstein series E_Υ. Hence, it is enough to prove the theorem for $\mathrm{Re}(s)$ sufficiently large. Since E_Υ is left invariant by $G_3(\mathbb{Q})$,

$$\mathcal{E}(g,s) = \chi(\mu_2(g)) \int_{\mathrm{U}(1,1)(\mathbb{Q})\backslash \mathrm{U}[g](\mathbb{A})} E_\Upsilon(Q \cdot \iota(h,g),s)\,\Psi(h)\,\chi(\det(h))^{-1}\,dh. \tag{224}$$

Let $V(\mathbb{Q})$ denote the subgroup of $G_3(\mathbb{Q})$ defined by

$$V(\mathbb{Q}) = \{Q\,\iota(g_1,g_2)\,Q^{-1} \mid g_i \in G_i(\mathbb{Q}),\ \mu_1(g_1) = \mu_2(g_2)\}.$$

Recall from [82, Prop. 2.4] that $|P_{12}(\mathbb{Q})\backslash G_3(\mathbb{Q})/V(\mathbb{Q})| = 2$. We take the identity element as one of the double coset representatives, and denote the other one by v. Thus

$$G_3(\mathbb{Q}) = P_{12}(\mathbb{Q})V(\mathbb{Q}) \sqcup P_{12}(\mathbb{Q})vV(\mathbb{Q}).$$

Let $R_1 \subset V(\mathbb{Q})$ and $R_2 \subset vV(\mathbb{Q})$ be corresponding sets of coset representatives, such that

$$P_{12}(\mathbb{Q})V(\mathbb{Q}) = \bigsqcup_{s \in R_1} P_{12}(\mathbb{Q})s, \quad P_{12}(\mathbb{Q})vV(\mathbb{Q}) = \bigsqcup_{s \in R_2} P_{12}(\mathbb{Q})s.$$

For the Eisenstein series defined in (221), we can write $E_\Upsilon(h,s) = E_\Upsilon^1(h,s) + E_\Upsilon^2(h,s)$, where

$$E_\Upsilon^1(h,s) = \sum_{\gamma \in R_1} \Upsilon(\gamma h,s), \qquad E_\Upsilon^2(h,s) = \sum_{\gamma \in R_2} \Upsilon(\gamma h,s).$$

Now, by [82, 22.9] the orbit of v is 'negligible' for our integral, that is,

$$\int_{\mathrm{U}(1,1)(\mathbb{Q})\backslash \mathrm{U}[g](\mathbb{A})} E_\Upsilon^2(Q \cdot \iota(h,g),s)\,\Psi(h)\,\chi(\det(h))^{-1}\,dh = 0$$

for all $g \in G_2(\mathbb{A})$. It follows that

$$\mathcal{E}(g,s) = \chi(\mu_2(g)) \int_{\mathrm{U}(1,1)(\mathbb{Q})\backslash \mathrm{U}[g](\mathbb{A})} E_\Upsilon^1(Q \cdot \iota(h,g),s)\,\Psi(h)\,\chi(\det(h))^{-1}\,dh. \tag{225}$$

On the other hand, by [82, Prop. 2.7], we can take R_1 to be the following set,

$$R_1 = \{Q\,\iota(1,m_2(\xi)\beta)\,Q^{-1} \mid \xi \in \mathrm{U}(1,1)(\mathbb{Q}),\ \beta \in P(\mathbb{Q})\backslash G_2(\mathbb{Q})\}, \tag{226}$$

where $m_2(\xi)$ is as in (20), and where the β are chosen to have $\mu_2(\beta) = 1$. For $\mathrm{Re}(s)$ large, we therefore have

$$E_\Upsilon^1(Q \cdot \iota(h,g),s) = \sum_{\substack{\xi \in \mathrm{U}(1,1)(\mathbb{Q}) \\ \beta \in P(\mathbb{Q})\backslash G_2(\mathbb{Q})}} \Upsilon(Q \cdot \iota(h,m_2(\xi)\beta g),s).$$

Substituting into (225) and using that $Q\,\iota(\xi,m_2(\xi))\,Q^{-1} \in P_{12}(\mathbb{Q})$ by (178), we have

$$\mathcal{E}(g,s) = \chi(\mu_2(g)) \int_{\mathrm{U}(1,1)(\mathbb{Q})\backslash \mathrm{U}[g](\mathbb{A})} \sum_{\substack{\xi \in \mathrm{U}(1,1)(\mathbb{Q}) \\ \beta \in P(\mathbb{Q})\backslash G_2(\mathbb{Q})}} \Upsilon(Q \cdot \iota(h,m_2(\xi)\beta g),s)\,\Psi(h)\,\chi(\det(h))^{-1}\,dh$$

$$= \sum_{\beta \in P(\mathbb{Q}) \backslash G_2(\mathbb{Q})} \chi(\mu_2(g)) \int_{U[g](\mathbb{A})} \Upsilon(Q \cdot \iota(h, \beta g), s) \, \Psi(h) \, \chi(\det(h))^{-1} \, dh.$$

Let

$$\Upsilon_\Psi(g, s) = \chi(\mu_2(g)) \int_{U[g](\mathbb{A})} \Upsilon(Q \cdot \iota(h, g), s) \, \Psi(h) \, \chi(\det(h))^{-1} \, dh. \tag{227}$$

If we can show that, for each $g \in G_2(\mathbb{Q})$,

$$\Upsilon_\Psi(g, s) = T(s) f(g, s), \tag{228}$$

the proof will be complete. By [82], we know that the integral above converges absolutely and uniformly on compact sets for $\mathrm{Re}(s)$ large. We are going to evaluate the above integral for such s. For a finite place p such that τ_p has conductor $\mathfrak{p}^n$, note that

$$G_2(\mathbb{Q}_p) = \bigsqcup_{m=0}^{n} P(\mathbb{Q}_p) \eta_m K^H \Gamma(\mathfrak{P}^n)$$

by Proposition 1.2.1. For $k \in K^H \Gamma(\mathfrak{P}^n)$, we may write $k = m_2(\begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix}) k'$, where $\lambda = \mu_2(k)$ and $\mu_2(k') = 1$. Using the fact that both sides of (228) are invariant under the right action on g by elements $k_p \in K^H \Gamma(\mathfrak{P}^n)$ satisfying $\mu_2(k_p) = 1$, and the above observations, it follows that in order to prove our theorem, it is enough to prove (228) for $g \in G_2(\mathbb{A})$ of the form

$$g = m_1(a) m_2(b) n \, \kappa \, k_\infty,$$

where $m_i \in M^{(i)}(\mathbb{A})$, $n \in N(\mathbb{A})$, $k_\infty \in K_\infty^{G_2}$, and $\kappa = (\kappa_v)_v \in \prod_v K_v^{G_2}$ satisfies

- $\kappa_v \in \{\eta_0, \cdots \eta_n\}$ if $v = p$ and τ_p has conductor p^n, $n > 0$,

- $\kappa_v = \eta_0$ if $v = \infty$,

- $\kappa_v = 1$ otherwise.

For such g, we calculate

$$\Upsilon_\Psi(g, s) = \chi(\mu_1(b)) \int_{U[m_2(b)](\mathbb{A})} \Upsilon(Q \cdot \iota(h, m_1(a) m_2(b) n \kappa k_\infty), s) \, \Psi(h) \, \chi(\det(h))^{-1} \, dh$$

$$\overset{(178)}{=} |\mu_1(b)|^{-3(s+\frac{1}{2})} \int_{U[m_2(b)](\mathbb{A})} \Upsilon(Q \cdot \iota(b^{-1}h, m_1(a) n \kappa k_\infty), s) \, \Psi(bb^{-1}h) \, \chi(\det(b^{-1}h))^{-1} \, dh$$

$$\overset{(205)}{=} |\mu_1(b)|^{-3(s+\frac{1}{2})} J_\infty(\eta_0 k_\infty) \int_{U(1,1)(\mathbb{A})} \Upsilon(Q \cdot \iota(h, m_1(a) n \kappa), s) \, \Psi(bh) \, \chi(\det(h))^{-1} \, dh$$

$$= \chi(a) |N(a) \mu_1(b)^{-1}|^{3(s+\frac{1}{2})} J_\infty(\eta_0 k_\infty) \int_{U(1,1)(\mathbb{A})} \Upsilon(Q \cdot \iota(h, \kappa), s) \, \Psi(bh) \, \chi(\det(h))^{-1} \, dh.$$

Using the Whittaker expansion

$$\Psi(g) = \sum_{\lambda \in \mathbb{Q}^\times} W_\Psi(\begin{bmatrix} \lambda & 0 \\ 0 & 1 \end{bmatrix} g), \tag{229}$$

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we have

$$\int_{U(1,1)(\mathbb{A})} \Upsilon(Q \cdot \iota(h,\kappa),s) \, \Psi(bh) \, \chi(\det(h))^{-1} \, dh = \sum_{\lambda \in \mathbb{Q}^\times} Z(\begin{bmatrix} \lambda & 0 \\ 0 & 1 \end{bmatrix} b, s; \kappa), \qquad (230)$$

where for $g \in G_1(\mathbb{A})$, $g_2 \in U(2,2)(\mathbb{A})$,

$$Z(g,s;g_2) = \int_{U(1,1)(\mathbb{A})} \Upsilon(Q \cdot \iota(h,g_2),s) \, W_\Psi(gh) \, \chi(\det(h))^{-1} \, dh.$$

Note that the uniqueness of the Whittaker function implies $Z(g,s;\kappa) = \prod_v Z_v(g_v, s, \kappa_v)$, where the local zeta integral $Z_v(g_v, s, \kappa_v)$ is defined by

$$Z_v(g_v, s, \kappa_v) = \int_{U(1,1)(\mathbb{Q}_v)} \Upsilon(Q \cdot \iota(h,\kappa_v),s) \, W^{(0)}(g_v h) \, \chi_v(\det(h))^{-1} \, dh;$$

at the archimedean place we understand $W^{(0)} = W_{l_1}$. Hence, by Theorems 3.2.1 and 3.4.1,

$$\Upsilon_\Psi(g,s) = \chi(a)|N(a)\mu_1(b)^{-1}|^{3(s+1/2)} T(s) \Psi(b) J_\infty(\eta_0 k_\infty, s) \prod_{\substack{p<\infty \\ \tau_p \text{ ramified}}} J_p(\kappa_p), \qquad (231)$$

where for a finite place p with τ_p of conductor p^n, $n > 0$,

$$J_p(\kappa_p) = \begin{cases} 1 & \text{if } \kappa_p = \eta_0, \\ 0 & \text{otherwise.} \end{cases} \qquad (232)$$

This proves (228) and hence completes the proof of the theorem. ∎

3.5.2 Remark. Pullback formulas in the spirit of Theorem 3.5.1 as a method to express complicated Eisenstein series on lower rank groups in terms of simpler Eisenstein series on higher rank groups have a long history. Garrett [21] used pullback formulas for Eisenstein series on symplectic groups to study the triple product L-function, as well as to establish the algebraicity of certain ratios of inner products of Siegel modular forms. Pullback formulas for Eisenstein series on unitary groups were first proved in a classical setting by Shimura [82]. Unfortunately, Shimura only considers certain special types of Eisenstein series in his work, which do not include ours except in the very specific case when τ is unramified principal series at all finite places and holomorphic discrete series at infinity.

3.6   The second global integral representation

In Theorem 2.3.2 we supplied a global integral representation for $L(s, \tilde\pi \times \tilde\tau)$. Using Theorem 3.5.1, we can modify it into a second integral representation that is more suitable for certain purposes. Let

$$R(s) = \frac{L(3s+1, \tau \times \mathcal{AI}(\Lambda) \times \chi|_{F^\times})}{L(6s+2, \chi_{L/F}\chi|_{F^\times})L(6s+3, \chi|_{F^\times}) \cdot T(s) \cdot Y_{l,l_1,p,q}(s) \cdot \prod_{v<\infty} Y_v(s)}, \qquad (233)$$

where $T(s)$ is defined by (222), the factors $Y_v(s)$ for non-archimedean v are given by Theorem 2.2.1, and the archimedean factor $Y_{l,l_1,p,q}(s)$ is given by Theorem 2.2.2. Note that $R(s)$ has an obvious Euler product $R(s) = \prod_v R_v(s)$, and that $R_v(s) = 1$ for all finite places v where τ_v is unramified.

Recall the Eisenstein series $E_\Upsilon(g, s)$ defined in (221). We define the normalized Eisenstein series

$$E_\Upsilon^*(g, s) = L(6s + 1, \chi|_{\mathbb{A}^\times})L(6s + 2, \chi_{L/F}\chi|_{\mathbb{A}^\times})L(6s + 3, \chi|_{\mathbb{A}^\times})E_\Upsilon(g, s). \tag{234}$$

Let Z_H and Z_{G_1} denote respectively the centers of $H = \mathrm{GSp}_4$ and $G_1 = \mathrm{GU}(1, 1)$. Given any $g \in G_1$ we define $H[g]$ to be the subgroup of H consisting of elements $h \in H$ with $\mu_2(h) = \mu_1(g)$. From Theorem 2.3.2 and Theorem 3.5.1 we get the following result.

3.6.1 Theorem. *Let $\phi = \otimes\phi_v$ be a vector in the space of π such that ϕ_v is unramified for all finite v and such that ϕ_∞ is a vector of weight $(-l, -l)$ in π_∞. Let Ψ be as in Theorem 3.5.1. The following meromorphic functions are all equal,*

   i) $R(s)^{-1}B_\phi(1)L(3s + \frac{1}{2}, \tilde{\pi} \times \tilde{\tau})$

   ii) $\displaystyle\int\limits_{Z_H(\mathbb{A})H(\mathbb{Q})\backslash H(\mathbb{A})} \phi(h)\chi(\mu_2(h)) \int\limits_{\mathrm{U}(1,1)(\mathbb{Q})\backslash \mathrm{U}[h](\mathbb{A})} E_\Upsilon^*(\iota(g, h), s)\Psi(g)\chi(\det(g))^{-1}\, dg\, dh,$

   iii) $\displaystyle\int\limits_{Z_{G_1}(\mathbb{A})G_1(\mathbb{Q})\backslash G_1(\mathbb{A})} \Psi(g)\chi\Big(\frac{\mu_1(g)}{\det(g)}\Big) \int\limits_{\mathrm{Sp}_4(\mathbb{Q})\backslash H[g](\mathbb{A})} E_\Upsilon^*(\iota(g, h), s))\phi(h)\, dh\, dg.$

For future reference, we record the following result about the poles of $E_\Upsilon^*(g, s)$.

3.6.2 Proposition. *Assume that the number q defined in (57) is zero. Then $E_\Upsilon^*(g, s)$ has no poles in the region $0 \leq \mathrm{Re}(s) \leq \frac{1}{4}$ except possibly a simple pole at the point $s = \frac{1}{6}$; this pole can exist only if $\omega_\tau = 1$.*

Proof. First, note that the Eisenstein series $E_\Upsilon^*(g, s)$ on $\mathrm{GU}(3, 3)$ has a pole at s_0 if and only if its restriction to $\mathrm{U}(3, 3)$, which is an Eisenstein series on $\mathrm{U}(3, 3)$, has a pole at $s = s_0$. Now the proof of the main Theorem of [85] shows exactly what we want. However, the statement there is a little ambiguous and seems to also allow for a possible simple pole at $s = 0$, in addition to the one at $s = \frac{1}{6}$. So we sketch the proof of holomorphy at $s = 0$ here for completeness. Let $I_v(\chi, 0)$ be as defined in [85]; this space is completely reducible at each non-archimedean inert place v (it is the direct sum of two irreducible representations). Now, we may choose any one of these irreducible components and work through the proof exactly as in [85]. ∎

4 Holomorphy of global L-functions for $\mathrm{GSp}_4 \times \mathrm{GL}_2$

In this section we will prove that the global L-function $L(s, \pi \times \tau)$ appearing in Theorem 2.4.3 is entire. Our main tools are the global integral representation Theorem 3.6.1 and Ichino's regularized Siegel-Weil formula for unitary groups, Theorem 4.1 of [35].

4.1 Preliminary considerations

Our goal is to prove the following theorem.

4.1.1 Theorem. (Holomorphy for $\mathrm{GSp}_4 \times \mathrm{GL}_2$) *Let $\pi = \otimes\pi_v$ be a cuspidal, automorphic representation of $\mathrm{GSp}_4(\mathbb{A})$ with the properties enumerated in Theorem 2.4.3 and such that π is not a Saito-Kurokawa lift. Let $\tau = \otimes\tau_v$ be a cuspidal, automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ such that τ_p is unramified for the primes p dividing D. Then $L(s, \pi \times \tau)$ is an entire function.*

The proof will be completed in Section 4.3 below. To begin with, note that τ may be twisted by an unramified Hecke character of the form $|\cdot|^t$ to make sure that ω_τ is of finite order. Such a twist will merely shift the argument of the L-function, because of the equation $L(s, \pi \times \tau \times |\cdot|^t) = L(s+t, \pi \times \tau)$. It is therefore sufficient to prove Theorem 4.1.1 under the following assumption, which we will make throughout this section.

$$\textit{The central character } \omega_\tau \textit{ of } \tau \textit{ is of finite order.} \qquad (235)$$

In particular, this means that the number q defined in (57) is zero. Since ε-factors never have any zeros or poles, it follows from the functional equation Theorem 2.4.3 that in order to prove Theorem 4.1.1, *it is enough to prove that $L(s, \pi \times \tau)$ has no poles in the region* $\mathrm{Re}(s) \geq \frac{1}{2}$.

Remark: Recall that the hypothesis that τ_p is unramified for the primes p dividing D was necessary for Theorem 2.4.3. This is the only reason for this hypothesis in Theorem 4.1.1; the following arguments work for general τ. The restriction on τ will be removed in Theorem 5.2.2.

Let $L_f(s, \pi \times \tau)$ be the finite part of $L(s, \pi \times \tau)$, i.e.,

$$L_f(s, \pi \times \tau) = \prod_{p < \infty} L_p(s, \pi_p \times \tau_p).$$

4.1.2 Lemma. *The Dirichlet series defining $L_f(s, \pi \times \tau)$ converges absolutely for* $\mathrm{Re}(s) > \frac{5}{4}$.

Proof. In fact, the Dirichlet series converges absolutely for $\mathrm{Re}(s) > \frac{71}{64}$. This follows directly from the global temperedness of π due to Weissauer [90] and the best known bound towards the Ramanujan conjecture for cusp forms on GL_2 due to Kim-Sarnak [42]. ∎

As a consequence, we get the following.

4.1.3 Lemma. *The completed L-function $L(s, \pi \times \tau)$ has no poles in the region* $\mathrm{Re}(s) > \frac{5}{4}$.

Proof. In view of Lemma 4.1.2, we only have to show that $L_\infty(s, \pi \times \tau)$ has no poles in that region. In fact it turns out that $L_\infty(s, \pi \times \tau)$ has no poles in the region $\mathrm{Re}(s) > 1$. To see this, first note that q, μ are equal to zero by our assumption on ω_τ. Next, by the unitarizability of τ_∞, it follows that p is a non-negative integer when τ_∞ is discrete series (or limit of discrete series) and $p \in i\mathbb{R} \cup (-1, 1)$ if τ_∞ is principal series. Also, we have $l \geq 2$. Now the holomorphy of $L_\infty(s, \pi \times \tau)$ in the desired right-half plane follows from the tables following Corollary 2.2.3. ∎

We will now use the second integral representation to reduce the possible set of poles to at most one point.

4.1.4 Proposition. *$L(s, \pi \times \tau)$ has no poles in the region $\mathrm{Re}(s) \geq \frac{1}{2}$ except possibly a simple pole at the point $s = 1$. This pole can exist only if $\omega_\tau = 1$.*

Proof. In Theorem 3.6.1, the functions Ψ, Υ, χ, $R(s)$ all depend on a choice of an integer l_1 such that τ has a vector of weight l_1. We now make such a choice. If τ_∞ is a principal series representation, then put $l_1 = 0$ or 1 (exactly one of these weights occurs in τ). If τ_∞ is a discrete series (or a limit of discrete series) representation, then put $l_1 = p+1$; hence, l_1 is the lowest weight of τ_∞.

With this choice, we can check by an explicit calculation that the function $R(s)$ defined in (233) has no poles in the region $0 \leq \mathrm{Re}(s) \leq \frac{1}{4}$. Indeed, the only possible pole in that region can come from $R_\infty(s)$, and so it boils down to checking that the function $T_\infty(s)$ defined in Theorem 3.4.1 and the function $Y_{l,l_1,p,q}(s)$ defined in Theorem 2.2.2 are non-zero when $0 \leq \mathrm{Re}(s) \leq \frac{1}{4}$. It is easy to verify that this is true with our choice of l_1.

On the other hand, by Theorem 3.6.2, the only possible pole of $E_\Upsilon^*(g,s)$ in the region $0 \leq \mathrm{Re}(s) \leq \frac{1}{4}$ is at $s = \frac{1}{6}$; this pole can occur only if $\omega_\tau = 1$. The result now follows from Theorem 3.6.1 and Lemma 4.1.3. ∎

4.2 Eisenstein series and Weil representations

In view of Proposition 4.1.4, we will now assume that $\omega_\tau = 1$, and that the integer l_1 used in the definition of Υ_∞ is equal to $p + 1$ in the discrete series case, and 0 or 1 otherwise. By abuse of notation, we continue to use $E_\Upsilon(g,s)$ to denote its restriction to $\mathrm{U}(3,3)(\mathbb{A})$. Indeed, this restricted function is an Eisenstein series on $\mathrm{U}(3,3)(\mathbb{A})$. For brevity, we will use G_3' to denote $\mathrm{U}(3,3)$. Let $K^{G_3'}$ denote the standard maximal compact subgroup of $G_3'(\mathbb{A})$. Let $I(\chi, s)$ be the set of holomorphic vectors in the global induced representation defined analogously to $I(\widetilde{\chi}, s)$ as in (219), except that we are now dealing with functions on $\mathrm{U}(3,3)$ rather than $\mathrm{GU}(3,3)$. In other words $I(\chi, s)$ consists of the sections $f^{(s)}$ on $G_3'(\mathbb{A})$ such that

$$f^{(s)}(m(A,1)ng, s) = |N(\det A)|^{3(s+\frac{1}{2})} \chi(\det(A)) f^{(s)}(g, s) \tag{236}$$

for all $g \in G_3'(\mathbb{A})$, and so that $f^{(s)}$ is holomorphic (in the sense of [35, p. 251]). In particular, any such section can be written as a finite linear combination of standard sections with holomorphic coefficients. A key example of a standard section is simply the restriction of the previously defined $\Upsilon(g, s)$ to $\mathrm{U}(3,3)$.

Recall that $\phi = \otimes \phi_v$ is a vector in the space of π such that ϕ_v is unramified for all finite v and such that ϕ_∞ is a vector of weight $(-l, -l)$ in π_∞. We have the following lemma.

4.2.1 Lemma. *Suppose that the Eisenstein series $E_\Upsilon(g, s)$ on $\mathrm{U}(3,3)(\mathbb{A})$ has the property that for all $g_1 \in \mathrm{U}(1,1)(\mathbb{A})$, we have*

$$\int_{\mathrm{Sp}_4(\mathbb{Q}) \backslash \mathrm{Sp}_4(\mathbb{A})} \mathrm{Res}_{s=\frac{1}{6}} E_\Upsilon(\iota(g_1, h_1), s)) \, \phi(h_1) \, dh_1 = 0. \tag{237}$$

Then $L(s, \pi \times \tau)$ is holomorphic at $s = 1$.

Proof. By Theorem 3.6.1, the fact that $R(s)$ has no pole at $s = \frac{1}{6}$, and the fact that

$$L(6s+1, \chi|_{\mathbb{A}^\times})L(6s+2, \chi_{L/F}\,\chi|_{\mathbb{A}^\times})L(6s+3, \chi|_{\mathbb{A}^\times})$$

is finite and non-zero at $s = \frac{1}{6}$, it follows that if

$$\int_{\mathrm{Sp}_4(\mathbb{Q}) \backslash H[g](\mathbb{A})} \mathrm{Res}_{s=\frac{1}{6}} E_\Upsilon(\iota(g, h), s)) \, \phi(h) \, dh = 0 \qquad \text{for all } g \in G_1(\mathbb{A}),$$

then $L(s, \pi \times \tau)$ is holomorphic at $s = 1$. Suppose $E_\Upsilon(g, s)$ has the property (237). If $g \in G_1(\mathbb{A})$ with $\mu_1(g) = m$, we can write $m = \lambda z k$ with $\lambda \in \mathbb{Q}^\times$, $z \in \mathbb{R}^+$, $k \in \prod_{p < \infty} \mathbb{Z}_p^\times$. It follows that we can write

$$g = \begin{bmatrix} 1 & \\ & \lambda \end{bmatrix} g_1 \begin{bmatrix} z^{1/2} & \\ & z^{1/2} \end{bmatrix} \begin{bmatrix} 1 & \\ & k \end{bmatrix}$$

with $g_1 \in \mathrm{U}(1,1)(\mathbb{A})$. A similar decomposition holds for h with $\mu_2(h) = m$. Thus

$$E_\Upsilon(\iota(g, h), s))\phi(h) = E_\Upsilon(\iota(g_1, h_1), s))\phi(h_1)$$

with g_1, h_1 belonging to $\mathrm{U}(1,1)(\mathbb{A})$, $\mathrm{Sp}_4(\mathbb{A})$ respectively. The lemma follows. ∎

We will reinterpret the condition of the lemma in terms of Weil representations and theta liftings. Let $(V, \mathcal{Q})$ be a non-degenerate Hermitian space over L of dimension 4. We identify $\mathcal{Q}$ with a Hermitian matrix of size 4. Let $\mathrm{U}(V)$ be the unitary group of V; thus

$$\mathrm{U}(V)(\mathbb{Q}) = \{g \in \mathrm{GL}_4(L) \mid {}^t\bar{g}\mathcal{Q}g = \mathcal{Q}\}.$$

Let χ be as above. Fix an additive character ψ as before. As described in [35], there is a Weil representation $\omega_\mathcal{Q} = \omega_{\mathcal{Q}, \psi, \chi}$ of $G_3'(\mathbb{A}) \times \mathrm{U}(V)(\mathbb{A})$ acting on the Schwartz space $\mathcal{S}(V^3(\mathbb{A}))$. The explicit formulas for the action can be found in [35, p. 246].

Let $s_0 = \frac{1}{6}$. Let $S(V^3(\mathbb{A}))$ denote the space of $K^{G_3'}$-finite vectors in $\mathcal{S}(V^3(\mathbb{A}))$. Write $\Pi(V)$ for the image of the $G_3'(\mathbb{A})$ intertwining map from $S(V^3(\mathbb{A}))$ to $I(\chi, s_0)$ given by

$$\varphi \mapsto f_\varphi^{(s_0)},$$

where $f_\varphi^{(s_0)}(g) = (\omega_\mathcal{Q}(g)\varphi)(0)$. We can extend $f_\varphi^{(s_0)}$ to a standard section $f_\varphi^{(s)} \in I(\chi, s)$ via

$$f_\varphi^{(s)}(g, s) = |N(\det A)|^{3(s-s_0)} f_\varphi^{(s_0)}(g),$$

where we use the Iwasawa decomposition to write $g = m(A, 1)nk$ with $A \in \mathrm{GL}_3(\mathbb{A}_L)$, n in the unipotent radical of the Siegel parabolic subgroup, and $k \in K^{G_3'}$.

Next we deal with the local picture. Suppose that $(V^{(v)}, \mathcal{Q}^{(v)})$ is a non-degenerate Hermitian space over L_v of dimension 4. Then we have the local Weil representation $\omega_{\mathcal{Q}_v} = \omega_{\mathcal{Q}_v, \psi_v, \chi_v}$ of $G_3'(\mathbb{Q}_v) \times \mathrm{U}(V^{(v)})(\mathbb{Q}_v)$ acting on the Schwartz space $\mathcal{S}((V^{(v)})^3)$. We define $R(V^{(v)})$ to be the image of the $G_3'(\mathbb{Q}_v)$ intertwining map from $S((V^{(v)})^3)$ to $I_v(\chi_v, s_0)$ given by

$$\varphi \mapsto f_\varphi^{(s_0)},$$

where $f_\varphi^{(s_0)}(g) = (\omega_{\mathcal{Q}_v}(g)\varphi)(0)$. The span of the various subspaces $R(V^{(v)})$ of $I_v(\chi_v, s_0)$ as $V^{(v)}$ ranges over the various inequivalent non-degenerate Hermitian spaces over L_v of dimension 4 is well understood. The non-archimedean case is treated in [48] while the archimedean case is treated in [54]. For instance, the following result [48, Thm. 1.2] describes the case when v is non-archimedean and L_v is a field.

4.2.2 Theorem. (Kudla–Sweet) *Suppose that v is non-archimedean and L_v is a field. Let $V_1^{(v)}$ and $V_2^{(v)}$ be the two inequivalent non-degenerate Hermitian vector spaces over L_v of dimension 4. Then $R(V_1^{(v)})$ and $R(V_2^{(v)})$ are distinct maximal submodules of $I_v(\chi_v, s_0)$, so that*

$$I_v(\chi_v, s_0) = R(V_1^{(v)}) + R(V_2^{(v)}).$$

In the case when v is non-archimedean and $L_v = F_v \oplus F_v$, a similar result is provided by [48, Thm. 1.3], while the case $v = \infty$ is dealt with in [54]. Now, let $\mathcal{C} = \{V^{(v)}\}$ be a collection, over all places v of $\mathbb{Q}$, of local non-degenerate Hermitian spaces over L_v of dimension 4. Whenever v is non-archimedean and L_v is a field, there are two inequivalent choices for $V^{(v)}$. Each of these spaces has an isotropic vector [48, Lemma 5.2]. If v is non-archimedean and $L_v = F_v \oplus F_v$, then the "Galois" automorphism is given by $(x_1, x_2) \mapsto (x_2, x_1)$. In this case the resulting "norm" map from L_v to F_v is surjective. So there is only one isometry class for $V^{(v)}$ and the "unitary" group of $V^{(v)}$ is isomorphic to $\mathrm{GL}_4(F_v)$. Indeed, up to isometry, the space $V^{(v)}$ is explicitly given by $V^{(v)} = F_v^4 \oplus F_v^4$ with $(a, b) = [{}^t a_1 \cdot b_2, {}^t a_2 \cdot b_1]$ where $a = (a_1, a_2)$, $b = (b_1, b_2)$. Finally, if $v = \infty$, there are 5 such $V^{(v)}$, corresponding to spaces of signature (p, q) with $p + q = 4$. For any such collection $\mathcal{C}$ as above, let $\Pi(\mathcal{C})$ be the representation space defined by

$$\Pi(\mathcal{C}) = \otimes_v R(V^{(v)}).$$

The upshot of the local results from [48] and [54] is that the natural map from $\oplus_{\mathcal{C}} \Pi(\mathcal{C})$ to $I(\chi, s_0)$ is surjective; here the sum ranges over all inequivalent collections $\mathcal{C}$ as above. Let $\mathcal{A}(G_3')$ denote the space of automorphic forms over $G_3'(\mathbb{A})$. Define A_{-1} to be the $G_3'(\mathbb{A})$ intertwining map from $I(\chi, s_0)$ to $\mathcal{A}(G_3')$ given by

$$f^{(s_0)} \longmapsto \mathrm{Res}_{s=s_0} E_{f^{(s)}}(g, s).$$

We note here (see [35, p. 252]) that the residue of the Eisenstein series at some point s_0 only depends on the section at s_0, so the above map is indeed well defined.

Next, for any local Hermitian space $V^{(v)}$ as above, with v non-archimedean, let $V_0^{(v)}$ denote the complementary space, which is defined to be the space of dimension 2 over L_v in the same Witt class as $V^{(v)}$. Note that such a space exists because (by our comments above) $V^{(v)}$ always has an isotropic vector if v is non-archimedean. The subspace $R(V_0^{(v)})$ of $I_v(\chi_v, -s_0)$ is defined similarly as above. It turns out (see [35], [48]) that for any non-archimedean place v, the restriction of the intertwining operator maps $R(V^{(v)})$ onto $R(V_0^{(v)})$. This identifies $R(V_0^{(v)})$ as a quotient of $R(V^{(v)})$; in fact it is the unique irreducible quotient of $R(V^{(v)})$. Moreover, if $f^{(s_0)}$ is a factorizable section, and the local section at a non-archimedean place v lies in the kernel of the above map from $R(V^{(v)})$ to $R(V_0^{(v)})$, then $A_{-1}(f^{(s_0)}) = 0$. This follows from [35, Lemma 6.1]; the lemma only states the result for the case that L_v is a field, but the same proof also works for the split case using the local results from [48, Sect. 7].

From the above discussion, we conclude that the map A_{-1} factors through the quotient

$$I_\infty(\chi_\infty, s_0) \otimes \left(\oplus \Pi(\mathcal{C}') \right),$$

where $\mathcal{C}' = \{V_0^{(v)}\}$ runs over all inequivalent collections of local Hermitian spaces $V_0^{(v)}$ of dimension 2 over L_v with v ranging over the *non-archimedean* places. (Compare [46, Prop. 4.2] for the analogous result in the symplectic case.) But we can say more. For any global Hermitian space V_0 of dimension 2 over L, let $\Pi(V_0)$ be the image of the $G_3'(\mathbb{A})$ intertwining map from $S(V_0^3(\mathbb{A}))$ to $I(\chi, -s_0)$. Note that (at each place, and hence globally) $\Pi(V_0)$ is naturally a quotient (via the intertwining operator) of $\Pi(V)$, where V is the complementary global Hermitian space of dimension 4 over L, obtained by adding a split space of dimension 2 to V_0.

4.2.3 Proposition. *The map A_{-1} from $I(\chi, s_0)$ to $\mathcal{A}(G_3')$ factors through $\oplus_{V_0} \Pi(V_0)$ where V_0 runs through all global Hermitian spaces of dimension 2 over L.*

Proof. We have already seen that the map A_{-1} factors through the quotient $I_\infty(\chi_\infty, s_0) \otimes (\oplus \Pi(\mathcal{C}'))$, where $\mathcal{C}' = \{V_0^{(v)}\}$ runs over all inequivalent collections of local Hermitian spaces $V_0^{(v)}$ of dimension 2 over L_v with v ranging over the non-archimedean places of $\mathbb{Q}$. The argument of [84, p. 363–364] takes care of the archimedean place, and we get that A_{-1} factors through $\oplus \Pi(\mathcal{C}')$, where $\mathcal{C}' = \{V_0^{(v)}\}$ runs over all inequivalent collections of local Hermitian spaces $V_0^{(v)}$ of dimension 2 over L_v; here v ranges over all the places of $\mathbb{Q}$ including ∞.

The question now is if there exists a global Hermitian space V_0 whose localizations are precisely the local spaces $\{V_0^{(v)}\}$ in the collection $\mathcal{C}'$. If such a global Hermitian space does not exist, then the collection $\mathcal{C}'$ is called incoherent, otherwise it is called coherent. From the local results quoted above, we know that each $\Pi(\mathcal{C}')$ is irreducible. Thus to complete the proof we only need to show that $\Pi(\mathcal{C}')$ cannot be embedded in $\mathcal{A}(G_3')$ if $\mathcal{C}'$ is incoherent.

The proof that such an embedding cannot exist is fairly standard. See, for instance [46, Thm. 3.1 (ii)], [47, Prop. 2.6] or [84, Cor. 4.1.12]. Thus, we will be brief. For any (global) Hermitian matrix β of size 3, let $W_\beta : \mathcal{A}(G_3') \to \mathbb{C}$ denote the β-th Fourier coefficient, defined by

$$W_\beta(f) = \int_{N_{12}(\mathbb{Q}) \backslash N_{12}(\mathbb{A})} f(nb) \psi(-\mathrm{tr}(b\beta)) \, db.$$

Let $\mathcal{D}$ be a non trivial embedding of $\Pi(\mathcal{C}')$ in $\mathcal{A}(G_3')$ where $\mathcal{C}'$ is incoherent, and put $\mathcal{D}_\beta = W_\beta \circ D$. Then there must exist some β such that $\mathcal{D}_\beta$ is non-zero. Moreover, if $\mathcal{D}_\beta = 0$ for all β of rank ≥ 2, then the argument of [47, Lemma 2.5] shows that $D = 0$. So there exists β with $\mathrm{rank}(\beta) \geq 2$ and $\mathcal{D}_\beta \neq 0$. By well-known results on the twisted Jacquet functor (see [35, Lemmas 5.1 and 5.2]), this implies that β is locally represented by $\{V_0^{(v)}\}$ at each place v, i.e., there exists $v_0^{(v)} \in (V_0^{(v)})^3$ such that $(v_0^{(v)}, v_0^{(v)}) = \beta$. Since the dimension of $V_0^{(v)}$ is 2, this implies that such a β cannot be non-singular; thus $\mathrm{rank}(\beta) = 2$. Hence β is (globally) equivalent to $\begin{bmatrix} \beta_0 & \\ & 0 \end{bmatrix}$ where β_0 is of size 2 and non-singular. Let $\epsilon_v(V_0^{(v)}) = \pm 1$ denote the Hasse invariant of the local Hermitian space $V_0^{(v)}$. Since the collection $\mathcal{C}'$ is incoherent, we have $\prod_v \epsilon_v(V_0^{(v)}) = -1$. On the other hand, because β is locally represented by $\{V_0^{(v)}\}$, and $\mathrm{rank}(\beta) = \dim(V_0^{(v)}) = 2$, it follows that the matrix for $V_0^{(v)}$ equals β_0 for some suitable basis. But this means that $\epsilon_v(V_0^{(v)}) = \epsilon_v(\beta_0)$. So $\prod_v \epsilon_v(V_0^{(v)}) = \prod_v \epsilon_v(\beta_0) = 1$, a contradiction. ∎

4.3 The Siegel-Weil formula and the proof of entireness

In the previous subsection, we proved that the map A_{-1} from $I(\chi, s_0)$ to $\mathcal{A}(G_3')$ given by

$$f^{(s_0)} \longmapsto \mathrm{Res}_{s=s_0} E_{f^{(s)}}(g, s)$$

factors through $\oplus_{V_0} \Pi(V_0)$, where V_0 runs through all global Hermitian spaces of dimension 2 over L. It turns out that the same map is also given by a regularized theta integral. This is the content of the regularized Siegel-Weil formula, which we now recall. Let $(V_0, \mathcal{Q}_0)$ be a global Hermitian space of dimension 2 over L and let $(V, \mathcal{Q})$ be the global Hermitian space of dimension 4 over L obtained by adding a split space of dimension 2 to V_0. Note that the Witt index of V is at least 1, thus V cannot be anisotropic. Given $\varphi_0 \in S(V_0^3(\mathbb{A}))$ we define the theta function

$$\Theta(g, h; \varphi_0) = \sum_{x \in V_0^3(\mathbb{Q})} \omega_{\mathcal{Q}_0}(g, h) \varphi_0(x). \tag{238}$$

This is a slowly increasing function on $(G'_3(\mathbb{Q})\backslash G'_3(\mathbb{A})) \times (\mathrm{U}(V_0)(\mathbb{Q})\backslash \mathrm{U}(V_0)(\mathbb{A}))$. If $\mathcal{Q}_0$ is anisotropic, we define

$$I_{\mathcal{Q}_0}(g, \varphi_0) = \int\limits_{\mathrm{U}(V_0)(\mathbb{Q})\backslash \mathrm{U}(V_0)(\mathbb{A})} \Theta(g, h; \varphi_0)\, dh.$$

If $\mathcal{Q}_0$ is isotropic, the above integral does not converge, so we define

$$I_{\mathcal{Q}_0}(g, \varphi_0) = c_\alpha^{-1} \int\limits_{\mathrm{U}(V_0)(\mathbb{Q})\backslash \mathrm{U}(V_0)(\mathbb{A})} \Theta(g, h; \omega_{\mathcal{Q}_0}(\alpha)\varphi_0)\, dh,$$

where α, c_α are defined as in Sect. 2 of [35]. In fact, in the convergent case, the second definition automatically equals the first, so we might as well use it in both the cases. Next, one has a map of Schwartz functions $\pi_{\mathcal{Q}}^{\mathcal{Q}_0}\pi_K$ from $S(V^3(\mathbb{A}))$ to K_0-invariant functions in $S(V_0^3(\mathbb{A}))$; here K_0 is the standard maximal compact subgroup of $\mathrm{U}(V_0(\mathbb{A}))$. We refer the reader to [35] for definitions and details. Let $\varphi \in S(V^3(\mathbb{A}))$. Let $f_\varphi^{(s)} \in I(\chi, s)$ be the standard section attached to φ via the Weil representation. Then the regularized Siegel-Weil formula [35, Thm. 4.1] in this setting says the following.

4.3.1 Theorem. (Ichino) *We have*

$$\mathrm{Res}_{s=s_0} E_{f_\varphi^{(s)}}(g, s) = c\, I_{\mathcal{Q}_0}(g, \pi_{\mathcal{Q}}^{\mathcal{Q}_0}\pi_K \varphi)$$

for an explicit constant c depending only on the normalization of Haar measures.

Theorem 4.3.1 and Proposition 4.2.3 imply the following result.

4.3.2 Proposition. *Suppose that the Eisenstein series $E_\Upsilon(g, s)$ does not satisfy the property (237). Then there exists a Hermitian space $(V_0, \mathcal{Q}_0)$ of dimension 2 over L and a K_0-invariant Schwartz function $\varphi_0 \in S(V_0^3(\mathbb{A}))$ such that, for some $g \in \mathrm{U}(1,1)(\mathbb{A})$,*

$$\int\limits_{\mathrm{Sp}_4(\mathbb{Q})\backslash \mathrm{Sp}_4(\mathbb{A})} I_{\mathcal{Q}_0}(\iota(g, h), \varphi_0)\phi(h)\, dh \neq 0.$$

We will now prove Theorem 4.1.1. In order to do so, it suffices to show that the conclusion of Proposition 4.3.2 leads to a contradiction. First note that, given Schwartz functions $\varphi_1 \in S(V_0(\mathbb{A}))$, $\varphi_2 \in S(V_0^2(\mathbb{A}))$, we may form the Schwartz function $\varphi_0 = \varphi_2 \otimes \varphi_1 \in S(V_0^3(\mathbb{A}))$ defined by $\varphi(v_1, v_2, v_3) = \varphi_1(v_3)\varphi_2(v_1, v_2)$. The space generated by linear combinations of functions of this type is the full Schwartz space $S(V_0^3(\mathbb{A}))$. Suppose that the conclusion of Proposition 4.3.2 holds. By the definition of $I_{\mathcal{Q}_0}$ and the above discussion, it follows that we can find $\varphi_1 \in S(V_0(\mathbb{A}))$, $\varphi_2 \in S(V_0^2(\mathbb{A}))$ such that for some $g \in \mathrm{U}(1,1)(\mathbb{A})$, we have

$$\int\limits_{\mathrm{Sp}_4(\mathbb{Q})\backslash \mathrm{Sp}_4(\mathbb{A})} \int\limits_{U(V_0)(\mathbb{Q})\backslash U(V_0)(\mathbb{A})} \Theta(\iota(g, h), h'; \varphi_2 \otimes \varphi_1)\, \phi(h)\, dh'\, dh \neq 0. \tag{239}$$

For $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{U}(1,1)$ let $\hat{g} = \begin{bmatrix} a & -b \\ -c & d \end{bmatrix}$. It is easy to check that

$$\omega_{\mathcal{Q}_0}(\iota(g, h))(\varphi_2 \otimes \varphi_1) = \omega_{\mathcal{Q}_0}(h)\varphi_2 \otimes \omega_{\mathcal{Q}_0}(\hat{g})\varphi_1.$$

Here, we are abusing notation and using $\omega_{\mathcal{Q}_0}$ to denote the Weil representation of $G_i(\mathbb{A})$ on $\mathcal{S}(V_0^i(\mathbb{A}))$ for various i. This gives the following factorization,

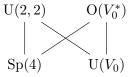$$\Theta(\iota(g,h),h';\varphi_2 \otimes \varphi_1) = \Theta(\hat{g},h';\varphi_1)\Theta(h,h';\varphi_2). \tag{240}$$

Define the automorphic form $\Theta(h';\phi,\varphi_2)$ on $\mathrm{U}(V_0)(\mathbb{Q})\backslash\mathrm{U}(V_0)(\mathbb{A})$ by

$$\Theta(h';\phi,\varphi_2) = \int\limits_{\mathrm{Sp}_4(\mathbb{Q})\backslash\mathrm{Sp}_4(\mathbb{A})} \Theta(h,h';\varphi_2)\phi(h)\,dh.$$

Equations (239) and (240) imply the following.

4.3.3 Lemma. *Suppose that the conclusion of Proposition 4.3.2 holds. Then there exists a Schwartz function $\varphi_2 \in S(V_0^2(\mathbb{A}))$ such that the automorphic form $\Theta(h';\phi,\varphi_2)$ on $\mathrm{U}(V_0)(\mathbb{A})$ is non-zero.*

We will now interpret the conclusion of this lemma in terms of theta liftings. Let V_0^* denote the 4-dimensional orthogonal space over $\mathbb{Q}$ obtained by considering V_0 as a space over $\mathbb{Q}$ and composing the hermitian form on V_0 with $\mathrm{tr}_{L/\mathbb{Q}}$. We have the following seesaw diagram (see [45, p. 252]) of dual reductive pairs.[4]

$$
\begin{array}{ccc}
\mathrm{U}(2,2) & & \mathrm{O}(V_0^*) \\
| & \times & | \\
\mathrm{Sp}(4) & & \mathrm{U}(V_0)
\end{array}
$$

Note that, at each place, V_0^* is either the unique anisotropic space of dimension four, or the split quadratic space $V_{2,2}$. Let π_1 be the representation of $\mathrm{Sp}_4(\mathbb{A})$ generated by the restriction of ϕ to $\mathrm{Sp}_4(\mathbb{A})$. By [60], we know that π_1 is an irreducible, automorphic, cuspidal representation. Moreover, π_1 is an anti-holomorphic discrete series representation at infinity. The above seesaw diagram and Lemma 4.3.3 imply that if the conclusion of Proposition 4.3.2 holds, then π_1 has a non-zero theta lift to $\mathrm{O}(V_0^*)$.

However, if π_1 has a non-zero theta lift to $\mathrm{O}(V_0^*)$, then V_0^* cannot be split at infinity. This is because there is no local archimedean theta lift of an anti-holomorphic discrete series representation from $\mathrm{Sp}_4(\mathbb{R})$ to $\mathrm{O}(2,2)(\mathbb{R})$, see [69]. This means there must be a non-archimedean place v where V_0^* is ramified. But this implies that π_1 is also ramified at v; else the local theta lift would be zero. However, we know that π_1 is unramified at all finite places because ϕ is right-invariant under $\mathrm{Sp}_4(\mathbb{Z}_p)$ at all finite places p. This contradiction shows that the conclusion of Proposition 4.3.2 cannot hold. Therefore the Eisenstein series $E_\Upsilon(g,s)$ on $\mathrm{U}(3,3)(\mathbb{A})$ has the property that, for all $g \in \mathrm{U}(1,1)(\mathbb{A})$,

$$\int\limits_{\mathrm{Sp}_4(\mathbb{Q})\backslash\mathrm{Sp}_4(\mathbb{A})} \mathrm{Res}_{s=\frac{1}{6}} E_\Upsilon(\iota(g,h),s))\phi(h)\,dh = 0,$$

and hence $L(s,\pi\times\tau)$ is holomorphic at $s=1$. This completes the proof of Theorem 4.1.1.

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[4]We would like to thank Paul Nelson for pointing this out to us.

5 Applications

As a special case of Langlands functoriality, one expects that automorphic forms on GSp_4 have a functorial transfer to automorphic forms on GL_4, coming from the natural embedding of dual groups $\mathrm{GSp}_4(\mathbb{C}) \subset \mathrm{GL}_4(\mathbb{C})$. For generic automorphic representations on GSp_4 this transfer was established in [4]. There is also a conjectured functorial transfer from automorphic forms on PGSp_4 to automorphic forms on GL_5, coming from the morphism $\rho_5 : \mathrm{Sp}_4(\mathbb{C}) \to \mathrm{GL}_5(\mathbb{C})$ of dual groups, where ρ_5 is the irreducible 5-dimensional representation of $\mathrm{Sp}_4(\mathbb{C})$. Here, we are going to show the existence of both these transfers for full level holomorphic cuspidal Siegel eigenforms. Note that the automorphic representation generated by such a Siegel modular form is not globally generic, since its archimedean component, a holomorphic discrete series representation, is non-generic.

We will use the transfer results to prove analytic properties of several L-functions related to Siegel modular forms. In the last subsection we will derive some special value results for $\mathrm{GSp}_4 \times \mathrm{GL}_1$ and $\mathrm{GSp}_4 \times \mathrm{GL}_2$ L-functions.

5.1 The transfer theorems

In the following let $\mathbb{A}$ be the ring of adeles of $\mathbb{Q}$. As before we write H for GSp_4, considered as an algebraic group over $\mathbb{Q}$. Let $\pi = \otimes \pi_v$ be a cuspidal, automorphic representation of $H(\mathbb{A})$ with the following properties.

- π has trivial central character.

- The archimedean component π_∞ is a holomorphic discrete series representation with scalar minimal K-type (l, l), where $l \geq 3$.

- For each finite place p, the local representation π_p is unramified.

It is well known that every such π gives rise to a holomorphic cuspidal Siegel eigenform of degree 2 and weight l with respect to the full modular group $\mathrm{Sp}_4(\mathbb{Z})$; see [3]. Conversely, every such eigenform generates an automorphic representation π as above (which is in fact irreducible; see [60]). Well-known facts about classical full-level Siegel modular forms show that the cuspidality condition implies $l \geq 10$. For the following lemma let ψ be the standard global additive character that was used in Theorem 2.3.2 and Theorem 3.5.1. Recall the definition of global Bessel models from Sect. 2.1.

5.1.1 Lemma. *Let π be as above, and let F be the corresponding Siegel cusp form. Assume that the Fourier expansion of F is given by*

$$F(Z) = \sum_S a(F, S) e^{2\pi i \, \mathrm{tr}(SZ)}, \tag{241}$$

where Z lies in the Siegel upper half space of degree 2, and S runs through 2×2 positive definite, semi-integral, symmetric matrices. Then, given a positive integer D such that $-D$ is a fundamental discriminant, the following are equivalent.

i) $a(F, S) \neq 0$ for some S with $D = 4 \det(S)$.

ii) π has a Bessel model of type $(S(-D), \Lambda, \psi)$, where $S(-D)$ is the matrix defined in (149), and where Λ is a character of the ideal class group (150) of $L = \mathbb{Q}(\sqrt{-D})$.

Proof: This follows from equation (4.3.4) in [19] (which is based on (1-26) of [83]). ∎

The second author has recently shown that condition i) of the lemma is always satisfied for some D; see [75]. In fact, independently of whether F is an eigenform or not, there exist infinitely many non-zero Fourier coefficients $a(F, S)$ such that $D = 4 \det(S)$ is odd and squarefree (in which case $-D$ is automatically a fundamental discriminant). The important fact for us to note is that *there always exists a positive integer D such that $-D$ is a fundamental discriminant and such that π satisfies the hypotheses of Theorem 2.4.3.*

We shall write down the explicit form of the local parameters of the representations π_v. These are admissible homomorphisms from the local Weil groups to the dual group $\mathrm{GSp}_4(\mathbb{C})$. Note that the trivial central character condition implies that the image of each local parameter lies in $\mathrm{Sp}_4(\mathbb{C})$. As in [87] (1.4.3), the real Weil group $W_\mathbb{R}$ is given by $W_\mathbb{R} = \mathbb{C}^\times \sqcup j\mathbb{C}^\times$ with the rules $j^2 = -1$ and $jzj^{-1} = \bar{z}$ for $z \in \mathbb{C}^\times$. Then the parameter of π_∞ is given by

$$\mathbb{C}^\times \ni re^{i\theta} \longmapsto \begin{bmatrix} e^{i(2l-3)\theta} & & & \\ & e^{i\theta} & & \\ & & e^{-i(2l-3)\theta} & \\ & & & e^{-i\theta} \end{bmatrix}, \qquad j \longmapsto \begin{bmatrix} & & & -1 \\ & & -1 & \\ 1 & & & \\ & 1 & & \end{bmatrix}. \tag{242}$$

For a finite place p, there exist unramified characters χ_1, χ_2 and σ of $\mathbb{Q}_p^\times$ such that π_p is the spherical component of a parabolically induced representation $\chi_1 \times \chi_2 \rtimes \sigma$ (using the notation of [76]). If we identify characters of $\mathbb{Q}_p^\times$ with characters of the local Weil group $W_{\mathbb{Q}_p}$ via local class field theory, then the L-parameter of π_p is given by

$$W_{\mathbb{Q}_p} \ni w \longmapsto \begin{bmatrix} \sigma(w)\chi_1(w) & & & \\ & \sigma(w)\chi_1(w)\chi_2(w) & & \\ & & \sigma(w)\chi_2(w) & \\ & & & \sigma(w) \end{bmatrix}. \tag{243}$$

The central character condition is $\chi_1\chi_2\sigma^2 = 1$, so that the image of this parameter lies in $\mathrm{Sp}_4(\mathbb{C})$. Now, let Π_∞ be the irreducible, admissible representation of $\mathrm{GL}_4(\mathbb{R})$ with L-parameter (242). For a prime number p, let Π_p be the irreducible, admissible representation of $\mathrm{GL}_4(\mathbb{Q}_p)$ with L-parameter (243). Then the irreducible, admissible representation

$$\Pi_4 := \otimes \Pi_v \tag{244}$$

of $\mathrm{GL}_4(\mathbb{A})$ is our candidate representation for the transfer of π to GL_4. Clearly, Π_4 is self-contragredient.

5.1.2 Theorem. *Let π be a cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbb{A})$ as above, related to a cuspidal Siegel eigenform F. We assume that F is not of Saito-Kurokawa type. Then the admissible representation Π_4 of $\mathrm{GL}_4(\mathbb{A})$ defined above is cuspidal automorphic. Hence Π_4 is a strong functorial lifting of π. This representation is symplectic, i.e., the exterior square L-function $L(s, \Pi_4, \Lambda^2)$ has a pole at $s = 1$.*

Proof. We will use the converse theorem for GL_4 from [11], and therefore have to establish the "niceness" of the L-functions of twists of Π by cusp forms on GL_1 and GL_2. As remarked above, there exists a positive integer D such that $-D$ is a fundamental discriminant and such that

π satisfies the hypotheses of Theorem 2.4.3; we will fix such a D. Let $\tau = \otimes \tau_p$ be a cuspidal, automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ such that τ_p is unramified for $p | D$. By definition of the candidate representation Π, the $\mathrm{GL}_4 \times \mathrm{GL}_2$ L-function $L(s, \Pi_4 \times \tau)$ coincides with the $\mathrm{GSp}_4 \times \mathrm{GL}_2$ L-function $L(s, \pi \times \tau)$. Therefore, by Theorem 4.1.1, the L-function $L(s, \Pi_4 \times \tau)$ has analytic continuation to an entire function. Moreover, by Theorem 2.4.3, it satisfies the functional equation

$$L(s, \Pi_4 \times \tau) = \varepsilon(s, \Pi_4 \times \tau) L(1-s, \tilde{\Pi}_4 \times \tilde{\tau}). \tag{245}$$

We will next prove that $L(s, \Pi_4 \times \tau)$ is bounded in vertical strips.[5] Consider the group GSp_8 and its Levi subgroup $\mathrm{GL}_2 \times \mathrm{GSp}_4$. One of the representations of the dual parabolic with Levi $\mathrm{GL}_2(\mathbb{C}) \times \mathrm{GSp}_4(\mathbb{C})$ on the dual unipotent radical is the tensor product representation. This means that our L-function $L(s, \pi \times \tau)$ is accessible via Langlands' method; see [52]. Now, Gelbart and Lapid proved that *any* L-function that is accessible via Langlands' method is meromorphic of finite order; this is Theorem 2 in [24]. Here, a function $f : \mathbb{C} \to \mathbb{C}$ being of finite order means that there exist positive constants r, c, C such that

$$|f(z)| \leq C\, e^{c|z|^r} \qquad \text{for all } z \in \mathbb{C}.$$

By the Phragmen-Lindelöf Theorem, if a holomorphic function of finite order is bounded on the left and right boundary of a vertical strip, then it is bounded on the entire vertical strip. For a large enough positive number M, our function $L(s, \pi \times \tau)$ is bounded on $\mathrm{Re}(s) = M$, since it is given as a product of archimedean Euler factors, which are bounded on vertical lines, times a convergent Dirichlet series. By the functional equation, $L(s, \pi \times \tau)$ is also bounded on $\mathrm{Re}(s) = -M$. It follows that $L(s, \pi \times \tau)$ is bounded on $-M \leq \mathrm{Re}(s) \leq M$. This proves that $L(s, \Pi_4 \times \tau)$ is bounded in vertical strips.

A similar argument applies to twists of Π_4 by Hecke characters χ of $\mathbb{A}^\times$. The required functional equation of $L(s, \Pi_4 \times \chi) = L(s, \pi \times \chi)$ is provided by [44]. The holomorphy follows from Theorem 2.2 of [63].

By Theorem 2 of [11], there exists an automorphic representation $\Pi' = \otimes \Pi'_v$ of $\mathrm{GL}_4(\mathbb{A})$ such that $\Pi'_\infty \cong \Pi_\infty$ and $\Pi'_p \cong \Pi_p$ for all primes $p \nmid D$. We claim that in fact $\Pi'_p \cong \Pi_p$ for *all* primes p; this will prove that the candidate representation Π_4 is automorphic (but not yet the cuspidality). To prove our claim, observe that we have the functional equations

$$L(s, \Pi_4) = \varepsilon(s, \Pi_4) L(1-s, \tilde{\Pi}_4) \tag{246}$$

and

$$L(s, \Pi') = \varepsilon(s, \Pi') L(1-s, \tilde{\Pi}'). \tag{247}$$

We have (246) because $L(s, \Pi_4) = L(s, \pi)$ and $\varepsilon(s, \Pi_4) = \varepsilon(s, \pi)$ by definition of Π_4, so that we can use Andrianov's classical theory; see [1]. We have (247) because Π' is an automorphic representation of $\mathrm{GL}_4(\mathbb{A})$. Dividing (246) by (247) and observing that the local factors outside D coincide, we obtain

$$\prod_{p|D} \frac{L(s, \Pi_p) L(1-s, \tilde{\Pi}'_p) \varepsilon(s, \Pi'_p)}{L(s, \Pi'_p) L(1-s, \tilde{\Pi}_p) \varepsilon(s, \Pi_p)} = 1. \tag{248}$$

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[5]We would like to thank Mark McKee for explaining this argument to us.

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It follows from unique prime factorization that if $p_1, \ldots, p_r$ are distinct primes, and if $R_1, \ldots, R_r \in \mathbb{C}(X)$ are such that

$$\prod_{i=1}^{r} R_i(p_i^s) = 1 \qquad \text{for all } s \in \mathbb{C}, \tag{249}$$

then the rational functions R_i are all constant. Hence, it follows from (248) that

$$\frac{L(s, \Pi_p)L(1 - s, \tilde{\Pi}'_p)\varepsilon(s, \Pi'_p)}{L(s, \Pi'_p)L(1 - s, \tilde{\Pi}_p)\varepsilon(s, \Pi_p)} \qquad \text{is constant for each } p|D. \tag{250}$$

Fix a prime $p|D$, and write (250) as

$$\frac{1}{L(s, \Pi_p)} = c_p X^m \frac{L(1 - s, \tilde{\Pi}'_p)}{L(s, \Pi'_p)L(1 - s, \tilde{\Pi}_p)}, \tag{251}$$

where c_p is a constant, $X = p^{-s}$, and m is some exponent coming from the ε-factors. Let $\alpha, \beta, \gamma, \delta$ be the Satake parameters of Π_p, so that

$$L(s, \Pi_p) = \frac{1}{(1 - \alpha p^{-s})(1 - \beta p^{-s})(1 - \gamma p^{-s})(1 - \delta p^{-s})}.$$

Substituting into (251), we obtain

$$(1 - \alpha X)(1 - \beta X)(1 - \gamma X)(1 - \delta X)$$

$$= (1 - (\alpha p)^{-1} X^{-1})(1 - (\beta p)^{-1} X^{-1})(1 - (\gamma p)^{-1} X^{-1})(1 - (\delta p)^{-1} X^{-1}) c_p X^m \frac{L(1 - s, \tilde{\Pi}'_p)}{L(s, \Pi'_p)}$$

$$= (X - (\alpha p)^{-1})(X - (\beta p)^{-1})(X - (\gamma p)^{-1})(X - (\delta p)^{-1}) c_p X^{m-4} \frac{L(1 - s, \tilde{\Pi}'_p)}{L(s, \Pi'_p)}.$$

Consider the zeros of the functions on both sides of this equation. On the left hand side, we have zeros exactly when $X = p^s$ is equal to

$$\alpha^{-1}, \quad \beta^{-1}, \quad \gamma^{-1}, \quad \delta^{-1} \tag{252}$$

(with repetitions allowed). On the right hand side, the factor $L(1 - s, \tilde{\Pi}'_p)$ does not contribute any zeros, since local L-factors are never zero. The factor X^{m-4} might contribute the zero 0, but this zero does certainly not appear amongst the numbers (252). Then there are the obvious *possible* zeros when X equals

$$(\alpha p)^{-1}, \quad (\beta p)^{-1}, \quad (\gamma p)^{-1}, \quad (\delta p)^{-1}. \tag{253}$$

Recalling that $\alpha, \beta, \gamma, \delta$ originate from the Satake parameters of a holomorphic Siegel cusp form, the Ramanujan conjecture for such modular forms, proven in [90], implies that $|\alpha| = |\beta| = |\gamma| = |\delta| = 1$. (Even without the full Ramanujan conjecture, known estimates as those in [67] would lead to the same conclusion.) Hence there is no overlap between the numbers in (252) and (253). It follows that the factor $L(s, \Pi'_p)$ must contribute the zeros (252) for the right hand side. In particular, $L(s, \Pi'_p)^{-1}$ is a polynomial in p^{-s} of degree 4, so that Π'_p is a spherical representation. And then, evidently, its Satake parameters are precisely α, β, γ and δ. This is equivalent to saying $\Pi'_p \cong \Pi_p$, proving our claim.

We now proved that the candidate representation $\Pi_4 = \otimes \Pi_v$ is automorphic, and it remains to prove it is cuspidal. Assume that Π_4 is not cuspidal; we will obtain a contradiction. Being not cuspidal, Π_4 is a constituent of a globally induced representation from a proper parabolic subgroup of GL_4. It follows that $L(s, \Pi_4)$ is, up to finitely many Euler factors, of one of the following forms.

i) $L(s, \chi_1)L(s, \chi_2)L(s, \chi_3)L(s, \chi_4)$ with Hecke characters χ_i of $\mathbb{A}^\times$.

ii) $L(s, \chi_1)L(s, \chi_2)L(s, \tau)$ with Hecke characters χ_1, χ_2 of $\mathbb{A}^\times$ and a cuspidal, automorphic representation τ of $\mathrm{GL}_2(\mathbb{A})$.

iii) $L(s, \chi_1)L(s, \tau)$ with a Hecke character χ_1 of $\mathbb{A}^\times$ and a cuspidal, automorphic representation τ of $\mathrm{GL}_3(\mathbb{A})$.

iv) $L(s, \tau_1)L(s, \tau_2)$ with cuspidal, automorphic representations τ_1, τ_2 of $\mathrm{GL}_2(\mathbb{A})$.

Note that all the characters and representations in this list must be unramified at every finite place, since the same is true for Π_4. If one of the cases i), ii) or iii) is true, then $L(s, \Pi_4 \times \chi_1^{-1})$ has a pole. Since $L(s, \Pi_4 \times \chi_1^{-1}) = L(s, \pi \times \chi_1^{-1})$ and we are assuming that F is not of Saito-Kurokawa type, this contradicts Theorem 2.2 of [63]. Hence we are in case iv). But then $L(s, \Pi_4 \times \tilde\tau_1) = L(s, \pi \times \tilde\tau_1)$ has a pole, contradicting Theorem 4.1.1. This contradiction shows that Π_4 must be cuspidal.

It remains to prove the last statement. Since Π_4 is self-dual, it is well known that exactly one of the L-functions

$$L(s, \Pi_4, \Lambda^2) \qquad \text{or} \qquad L(s, \Pi_4, \mathrm{Sym}^2)$$

has a pole at $s = 1$. If $L(s, \Pi_4, \mathrm{Sym}^2)$ would have a pole at $s = 1$, then Π_4 would be a (strong) lifting from the split orthogonal group SO_4; see the Theorem on p. 680 of [26] and the comments thereafter. By Lemma 5.1.3 below, this is impossible. It follows that $L(s, \Pi_4, \Lambda^2)$ has a pole at $s = 1$. ∎

5.1.3 Lemma. *Let F and $\pi = \otimes \pi_v$ be as in Theorem 5.1.2, and let Π_4 be the resulting lifting to $\mathrm{GL}_4(\mathbb{A})$. Then there does not exist a cuspidal, automorphic representation σ of $\mathrm{SO}_4(\mathbb{A})$ such that Π_4 is a Langlands functorial lifting of σ.*

Proof. The obstruction comes from the archimedean place. Recall that the dual group of SO_4 is $\mathrm{SO}_4(\mathbb{C})$, which we realize as

$$\mathrm{SO}_4(\mathbb{C}) = \{ g \in \mathrm{SL}_4(\mathbb{C}) \mid {}^t g \begin{bmatrix} & 1_2 \\ 1_2 & \end{bmatrix} g = \begin{bmatrix} & 1_2 \\ 1_2 & \end{bmatrix} \}.$$

Let $\varphi : W_{\mathbb{R}} \to \mathrm{GL}_4(\mathbb{C})$ be the archimedean L-parameter given explicitly in (242). If Π_4 would come from SO_4, there would exist a matrix $g \in \mathrm{GL}_4(\mathbb{C})$ such that

$$g\varphi(w)g^{-1} \in \mathrm{SO}_4(\mathbb{C}) \qquad \text{for all } w \in W_{\mathbb{R}}.$$

Then ${}^t(g\varphi(w)g^{-1}) \begin{bmatrix} & 1_2 \\ 1_2 & \end{bmatrix} (g\varphi(w)g^{-1}) = \begin{bmatrix} & 1_2 \\ 1_2 & \end{bmatrix}$ for all $w \in W_{\mathbb{R}}$, or equivalently

$$ {}^t\varphi(w)S\varphi(w) = S, \qquad \text{where } S = {}^t g \begin{bmatrix} & 1_2 \\ 1_2 & \end{bmatrix} g.$$

Letting w run through non-zero complex numbers $re^{i\theta}$ shows that S is of the form

$$S = \begin{bmatrix} & & a & \\ & & & b \\ a & & & \\ & b & & \end{bmatrix}.$$

But then letting $w = j$ yields the contradiction $-S = S$. ∎

We will next consider a backwards lifting of Π_4 in order to obtain a globally generic, cuspidal, automorphic representation on $\mathrm{GSp}_4(\mathbb{A})$ in the same L-packet as π.

5.1.4 Theorem. *Let F and $\pi = \otimes\pi_v$ be as in Theorem 5.1.2. Then there exists a globally generic, cuspidal, automorphic representation $\pi^g = \otimes\pi_v^g$ of $\mathrm{GSp}_4(\mathbb{A})$ such that $\pi_p^g \cong \pi_p$ for all primes p, and such that π_∞^g is the generic discrete series representation of $\mathrm{PGSp}_4(\mathbb{R})$ lying in the same L-packet as π_∞. Any globally generic, cuspidal automorphic representation $\sigma = \otimes\sigma_v$ of $\mathrm{GSp}_4(\mathbb{A})$ such that $\sigma_p \cong \pi_p$ for almost all p coincides with π^g.*

Proof. Let Π_4 be the lifting of π to GL_4 constructed in Theorem 5.1.2. Since Π_4 is symplectic, we can apply Theorem 4 of [27]. The conclusion is that there exists a non-zero representation $\sigma = \sigma_1 \oplus \ldots \oplus \sigma_m$ of $\mathrm{PGSp}_4(\mathbb{A})$ such that each σ_i is globally generic, cuspidal, automorphic and weakly lifts to Π_4. By Theorem 9 of [27], there can be only one σ_i, i.e., σ is itself irreducible. Note that "weak lift" in [27] includes the condition that the lift is functorial with respect to archimedean L-parameters (see [27], p. 733). In particular, the archimedean component of σ is the generic discrete series representation of $\mathrm{PGSp}_4(\mathbb{R})$ lying in the same L-packet as π_∞. Evidently, the local components σ_p and π_p are isomorphic for almost all primes p. It remains to show that this is the case for *all* primes p. This can be done by a similar argument as in the proof of Theorem 5.1.2. Dividing the functional equations for the degree 4 L-functions $L(s, \pi)$ and $L(s, \sigma)$, and comparing the resulting zeros at a particular prime p, shows first that $L(s, \sigma_p)$ is a degree 4 Euler factor. Hence σ_p is an unramified representation. The same comparison of zeros then also implies that σ_p and π_p have the same Satake parameters. The last assertion follows from the strong multiplicity one result Theorem 9 of [27]. ∎

With F and π as above, we constructed a strong functorial lifting of π to GL_4 with respect to the natural inclusion of dual groups $\mathrm{Sp}_4(\mathbb{C}) \subset \mathrm{GL}_4(\mathbb{C})$. Similarly, we will now produce a strong functorial lifting of π to GL_5 with respect to the morphism $\rho_5 : \mathrm{Sp}_4(\mathbb{C}) \to \mathrm{GL}_5(\mathbb{C})$ of dual groups, where ρ_5 is the irreducible 5-dimensional representation of $\mathrm{Sp}_4(\mathbb{C})$. Let $L(s, \pi, \rho_5)$ be the degree 5 (standard) L-function of F. If the L-parameter at a prime p is given by (243), then

$$L(s, \pi_p, \rho_5) = \frac{1}{(1 - p^{-s})(1 - \chi_1(p)p^{-s})(1 - \chi_1^{-1}(p)p^{-s})(1 - \chi_2(p)p^{-s})(1 - \chi_2^{-1}(p)p^{-s})}. \quad (254)$$

5.1.5 Theorem. *Let F and $\pi = \otimes\pi_v$ be as in Theorem 5.1.2. Then there exists a cuspidal, automorphic representation Π_5 of $\mathrm{GL}_5(\mathbb{A})$ such that*

$$L(s, \pi, \rho_5) = L(s, \Pi_5) \quad (255)$$

(equality of completed Euler products). The representation Π_5 is a strong functorial lifting of π to GL_5 with respect to the morphism $\rho_5 : \mathrm{Sp}_4(\mathbb{C}) \to \mathrm{GL}_5(\mathbb{C})$ of dual groups. Moreover, Π_5 is orthogonal, i.e., the symmetric square L-function $L(s, \Pi_5, \mathrm{Sym}^2)$ has a pole at $s = 1$.

Proof. A straightforward calculation verifies that

$$L_f(s, \Pi_4, \Lambda^2) = L_f(s, \pi, \rho_5)\zeta(s). \quad (256)$$

Here, the subscript f indicates that the Euler product defining the L-functions is taken over finite places only, and $\zeta(s)$ denotes the Riemann zeta function. By Theorem 5.1.2, the function

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$L_f(s, \Pi_4, \Lambda^2)$ has a simple pole at $s = 1$. It follows that $L_f(s, \pi, \rho_5)$ is holomorphic and non-zero at $s = 1$. Together with [29], Theorem 2, we obtain that $L_f(s, \pi, \rho_5)$ has no poles on $\mathrm{Re}(s) = 1$. Now by [42], Theorem A, $L(s, \Pi_4, \Lambda^2)$ is the L-function of an automorphic representation of $\mathrm{GSp}_6(\mathbb{A})$ of the form

$$\mathrm{Ind}(\tau_1 \otimes \ldots \otimes \tau_m) \tag{257}$$

where $\tau_1, \ldots, \tau_m$ are unitary, cuspidal, automorphic representations of $\mathrm{GL}_{n_i}(\mathbb{A})$, $n_1 + \ldots + n_m = 6$. Since $L_f(s, \Pi_4, \Lambda^2)$ has a simple pole at $s = 1$, it follows that exactly one of the τ_i, say τ_m, is the trivial representation of $\mathrm{GL}_1(\mathbb{A})$. Cancelling out one zeta factor, we see that

$$L_f(s, \pi, \rho_5) = L_f(s, \tau_1) \ldots L_f(s, \tau_{m-1}). \tag{258}$$

Observe that since π is unramified at every finite place, the same must be true for the τ_i. If we had $n_i = 1$ for some i, then $L(s, \tau_i)$, and therefore the right hand side of (257), would have a pole on $\mathrm{Re}(s) = 1$. This contradicts the observation from above that $L_f(s, \pi, \rho_5)$ has no poles on $\mathrm{Re}(s) = 1$. Hence $n_i > 1$ for all i, so that the only possibilities for the set $\{n_1, \ldots, n_{m-1}\}$ are $\{2, 3\}$ and $\{5\}$. Assume the former is the case, so that, say, τ_1 is a cuspidal representation of $\mathrm{GL}_2(\mathbb{A})$ and τ_2 is a cuspidal representation of $\mathrm{GL}_3(\mathbb{A})$. Let $\Pi_5 = \mathrm{Ind}(\tau_1 \otimes \tau_2)$. It is not hard to verify that

$$L_f(s, \Pi_5, \Lambda^2) = L_f(s, \Pi_4, \mathrm{Sym}^2), \tag{259}$$

which we know is an entire function. On the other hand,

$$L_f(s, \Pi_5, \Lambda^2) = L_f(s, \omega_{\tau_1}) L_f(s, \tau_1 \times \tau_2) L_f(s, \omega_{\tau_2} \times \tilde{\tau}_2), \tag{260}$$

where ω_{τ_i} is the central character of τ_i. Since the latter is everywhere unramified, the right hand side of (260) has a pole on $\mathrm{Re}(s) = 1$. This contradiction shows that the assumption $\{n_1, \ldots, n_{m-1}\} = \{2, 3\}$ must be wrong. Hence $\Pi_5 := \tau_1$ is a cuspidal representation of $\mathrm{GL}_5(\mathbb{A})$ such that

$$L_f(s, \pi, \rho_5) = L_f(s, \Pi_5). \tag{261}$$

This implies that Π_5 is a lifting of π (with respect to the morphism ρ_5 of dual groups) at every finite place. At the archimedean place, observe that the L-parameter of $\mathrm{Ind}(\tau_1 \otimes \tau_2)$ equals the exterior square of the L-parameter of Π_4, since the lifting of [42] is strong. On the other hand, an explicit calculation shows that the exterior square of the L-parameter of Π_4 equals the L-parameter of π composed with ρ_5, plus the trivial representation of $W_{\mathbb{R}}$ (in other words, the archimedean place behaves exactly as the finite places, so that (256) holds in fact for the completed L-functions). Cancelling out the trivial representation on both sides, one obtains an equality of the L-parameter of τ_1 with the L-parameter of π composed with ρ_5. Hence Π_5 is a functorial lifting of π also at the archimedean place.

Finally, Π_5 is orthogonal since the exterior square $L_f(s, \Pi_5, \Lambda^2)$ has no pole at $s = 1$; see (259). This concludes the proof. ∎

5.2 Analytic properties of L-functions

For $n \in \{1, 4, 5, 10, 14, 16\}$ let ρ_n be the n-dimensional irreducible representation of $\mathrm{Sp}_4(\mathbb{C})$. In the notation of [18], Sect. 16.2, we have $\rho_4 = \Gamma_{1,0}$, $\rho_5 = \Gamma_{0,1}$, $\rho_{10} = \Gamma_{2,0}$, $\rho_{14} = \Gamma_{0,2}$ and $\rho_{16} = \Gamma_{1,1}$. Of course, ρ_4 is the natural representation of $\mathrm{Sp}_4(\mathbb{C})$ on $\mathbb{C}^4$, which is also called the spin representation. An explicit formula for the representation ρ_5 as a map $\mathrm{Sp}_4(\mathbb{C}) \to \mathrm{SO}_5(\mathbb{C})$ is given in Appendix A.7

of [70]. (Somewhat confusingly, in the theory of Siegel modular forms ρ_5 is often referred to as the standard representation, even though it is ρ_4 that is the non-trivial representation of lowest dimension.) The representation ρ_{10} is the adjoint representation of $\mathrm{Sp}_4(\mathbb{C})$ on its Lie algebra. We have the following relations,

$$\Lambda^2 \rho_4 = \rho_1 + \rho_5, \tag{262}$$

$$\Lambda^2 \rho_5 = \mathrm{Sym}^2 \rho_4 = \rho_{10}, \tag{263}$$

$$\mathrm{Sym}^2 \rho_5 = \rho_1 + \rho_{14}, \tag{264}$$

$$\rho_4 \otimes \rho_5 = \rho_4 + \rho_{16}. \tag{265}$$

Let F and π be as in Theorem 5.1.2. To each ρ_n we have an associated global L-function $L(s, \pi, \rho_n)$. We will list the archimedean L- and ε-factors (the latter with respect to the character ψ^{-1}, where $\psi(x) = e^{-2\pi i x}$). Let $\Gamma_{\mathbb{R}}$ and $\Gamma_{\mathbb{C}}$ be as in (131). The archimedean factors depend only on the minimal K-type (l, l) of π_∞.

ρ	$L(s, \pi_\infty, \rho)$	$\varepsilon(s, \pi_\infty, \rho, \psi^{-1})$
ρ_1	$\Gamma_{\mathbb{R}}(s)$	1
ρ_4	$\Gamma_{\mathbb{C}}(s + \frac{1}{2})\Gamma_{\mathbb{C}}(s + l - \frac{3}{2})$	$(-1)^l$
ρ_5	$\Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{C}}(s + l - 1)\Gamma_{\mathbb{C}}(s + l - 2)$	1
ρ_{10}	$\Gamma_{\mathbb{R}}(s + 1)^2\,\Gamma_{\mathbb{C}}(s + 1)\Gamma_{\mathbb{C}}(s + l - 1)\Gamma_{\mathbb{C}}(s + l - 2)\Gamma_{\mathbb{C}}(s + 2l - 3)$	1
ρ_{14}	$\Gamma_{\mathbb{R}}(s)^2\,\Gamma_{\mathbb{C}}(s + 1)\Gamma_{\mathbb{C}}(s + l - 1)\Gamma_{\mathbb{C}}(s + l - 2)$ $\Gamma_{\mathbb{C}}(s + 2l - 2)\Gamma_{\mathbb{C}}(s + 2l - 3)\Gamma_{\mathbb{C}}(s + 2l - 4)$	1
ρ_{16}	$\Gamma_{\mathbb{C}}(s + \frac{1}{2})^2\,\Gamma_{\mathbb{C}}(s + l - \frac{1}{2})\Gamma_{\mathbb{C}}(s + l - \frac{3}{2})^2\,\Gamma_{\mathbb{C}}(s + l - \frac{5}{2})$ $\Gamma_{\mathbb{C}}(2 + 2l - \frac{5}{2})\Gamma_{\mathbb{C}}(2 + 2l - \frac{7}{2})$	-1

These factors are normalized so that they fit into a functional equation relating s and $1 - s$, and hence differ from the traditional factors used in the theory of Siegel modular forms. For example, the classical Andrianov spin L-function relates s and $2l - 2 - s$; see [1], Theorem 3.1.1. To obtain the Andrianov Γ-factors, one has to replace s by $s - l + \frac{3}{2}$ in the above factor for ρ_4.

5.2.1 Theorem. *Let F and π be as in Theorem 5.1.2. The Euler products defining the L-functions $L_f(s, \pi, \rho_n)$, for $n \in \{4, 5, 10, 14, 16\}$, are absolutely convergent for $\mathrm{Re}(s) > 1$. They have meromorphic continuation to the entire complex plane, have no zeros or poles on $\mathrm{Re}(s) \geq 1$, and the completed L-functions (using the above archimedean factors) satisfy the functional equation*

$$L(s, \pi, \rho_n) = \varepsilon(s, \pi, \rho_n)L(1 - s, \pi, \rho_n).$$

Furthermore, for $n \in \{4, 5, 10\}$, the functions $L(s, \pi, \rho_n)$ are entire and bounded in vertical strips.

Proof. By definition, $L(s, \pi, \rho_4) = L(s, \Pi_4)$ and $L(s, \pi, \rho_5) = L(s, \Pi_5)$. Hence, the analytic properties of $L(s, \pi, \rho_4)$ and $L(s, \pi, \rho_5)$ follow from the known analytic properties of L-functions of cuspidal representations on GL_n. For the absolute convergence of the Euler products in $\mathrm{Re}(s) > 1$, see [39], Theorem 5.3. As for the adjoint L-function, it follows from (263) that

$$L(s, \Pi_4, \mathrm{Sym}^2) = L(s, \pi, \rho_{10}). \tag{266}$$

Since Π_4 is symplectic by Theorem 5.1.2, this is an entire function; see [8, Thm. 7.5]. The absolute convergence in $\mathrm{Re}(s) > 1$ follows from [39, Thm. 5.3], together with the known automorphy,

hence absolute convergence, of $L(s, \Pi_4, \Lambda^2)$. Since symmetric square L-functions are accessible via the Langlands-Shahidi method, the boundedness in vertical strips follows from [25], and the functional equation follows from [80, Cor. 6.7]. The non-vanishing on $\mathrm{Re}(s) = 1$ follows also from the Langlands-Shahidi method; see Sect. 5 of [79]. From (264) we get

$$L(s, \Pi_5, \mathrm{Sym}^2) = Z(s)L(s, \pi, \rho_{14}), \tag{267}$$

where $Z(s) = \Gamma_{\mathbb{R}}(s)\zeta(s)$ is the completed Riemann zeta function. Observe that $L(s, \Pi_5, \Lambda^2)$ is absolutely convergent for $\mathrm{Re}(s) > 1$ by (259). Together with [39], Theorem 5.3, this implies the absolute convergence of $L(s, \Pi_5, \mathrm{Sym}^2)$, and hence of $L(s, \pi, \rho_{14})$, in $\mathrm{Re}(s) > 1$. The meromorphic continuation of $L(s, \pi, \rho_{14})$ is obvious from (267). Since this is an identity of complete Euler products, and since our liftings are strongly functorial, it also implies the asserted functional equation. By Theorem 5.1.5 the function $L(s, \Pi_5, \mathrm{Sym}^2)$ has a simple pole at $s = 1$, while otherwise it is holomorphic and non-vanishing on $\mathrm{Re}(s) = 1$. Since the same is true for $Z(s)$, it follows that $L(s, \pi, \rho_{14})$ is holomorphic and non-vanishing on $\mathrm{Re}(s) = 1$. Since

$$L(s, \Pi_4 \times \Pi_5) = L(s, \pi)L(s, \pi, \rho_{16}). \tag{268}$$

by (265), similar arguments apply to $L(s, \pi, \rho_{16})$. ∎

Let r be a positive integer, and τ a cuspidal, automorphic representation of $\mathrm{GL}_r(\mathbb{A})$. Let σ_r be the standard representation of the dual group $\mathrm{GL}_r(\mathbb{C})$. Then we can consider the Rankin-Selberg Euler products $L(s, \pi \times \tau, \rho_n \otimes \sigma_r)$, where ρ_n is one of the irreducible representations of $\mathrm{Sp}_4(\mathbb{C})$ considered above. For $n = 4$ or $n = 5$, since Π_4 and Π_5 are functorial liftings of π, we have

$$L(s, \pi \times \tau, \rho_n \times \sigma_r) = L(s, \Pi_n \times \tau), \tag{269}$$

where the L-function on the right is a standard Rankin-Selberg L-function for $\mathrm{GL}_n \times \mathrm{GL}_r$. From the well-known properties of these L-functions, the following result is immediate. For $\varepsilon > 0$ and a closed interval I on the real line we use the notation $T_{\varepsilon,I} = \{s \in \mathbb{C} \mid \mathrm{Re}(s) \in I, \, |\mathrm{Im}(s)| \geq \varepsilon\}$, as in [25].

5.2.2 Theorem. *Let F and π be as in Theorem 5.1.2. Let r be a positive integer, and τ a (unitary) cuspidal, automorphic representation of $\mathrm{GL}_r(\mathbb{A})$. Let $n = 4$ or $n = 5$. Then the Euler products defining the $\mathrm{GSp}_4 \times \mathrm{GL}_r$ L-functions $L(s, \pi \times \tau, \rho_n \otimes \sigma_r)$ are absolutely convergent for $\mathrm{Re}(s) > 1$. They have meromorphic continuation to the entire complex plane, and the completed L-functions satisfy the functional equation*

$$L(s, \pi \times \tau, \rho_n \otimes \sigma_r) = \varepsilon(s, \pi \times \tau, \rho_n \otimes \sigma_r)L(1 - s, \tilde{\pi} \times \tilde{\tau}, \rho_n \otimes \sigma_r). \tag{270}$$

These L-functions are entire, bounded in vertical strips, and non-vanishing on $\mathrm{Re}(s) \geq 1$, except in the cases

- *$n = r = 4$ and $\tau = |\det|^{it} \otimes \Pi_4$, where $t \in \mathbb{R}$ and Π_4 is the lifting of π from Theorem 5.1.2, or*

- *$n = r = 5$ and $\tau = |\det|^{it} \otimes \Pi_5$, where $t \in \mathbb{R}$ and Π_5 is the lifting of π from Theorem 5.1.5.*

In these cases the function $L(s, \pi \times \tau, \rho_n \otimes \sigma_r)$ is holomorphic except for simple poles at $s = -it$ and $s = 1 - it$, and is bounded on all sets of the form $T_{\varepsilon,I}$ with $\varepsilon > |t|$.

Proof. For the precise location of poles, see Theorem 2.4 of [12]. For boundedness in vertical strips, see Corollary 2 on p. 80 of [25]. ∎

5.2.3 Theorem. *Let F and F' be Siegel cusp forms with respect to $\mathrm{Sp}_4(\mathbb{Z})$. Assume that F and F' are Hecke eigenforms, that they are not Saito-Kurokawa lifts and that π resp. π' are the associated cuspidal, automorphic representations of $\mathrm{GSp}_4(\mathbb{A})$. Let $n \in \{4,5\}$ and $n' \in \{4,5\}$. Then the Euler products defining the $\mathrm{GSp}_4 \times \mathrm{GSp}_4$ L-functions $L(s, \pi \times \pi', \rho_n \otimes \rho_{n'})$ are absolutely convergent for $\mathrm{Re}(s) > 1$. They have meromorphic continuation to the entire complex plane, and the completed L-functions satisfy the expected functional equation. These functions are entire, bounded in vertical strips, and non-vanishing on $\mathrm{Re}(s) \geq 1$, except if $n = n'$ and F and F' have the same Hecke eigenvalues. In these cases the function $L(s, \pi \times \pi', \rho_n \otimes \rho_{n'})$ is holomorphic except for simple poles at $s = 0$ and $s = 1$, and is bounded on all sets of the form $T_{\varepsilon,I}$ with $\varepsilon > 0$.*

Proof. By definition,
$$L(s, \pi \times \pi', \rho_n \otimes \rho_{n'}) = L(s, \Pi_n \times \Pi'_{n'}), \tag{271}$$
where Π_n (resp. $\Pi'_{n'}$) is the lifting of π (resp. π') to GL_n (resp. $\mathrm{GL}_{n'}$). Evidently, F and F' have the same Hecke eigenvalues if and only if π and π' are nearly equivalent if and only if $\Pi_n = \Pi'_n$. Hence everything follows from the properties of L-functions for $\mathrm{GL}_n \times \mathrm{GL}_{n'}$. ∎

5.2.4 Theorem. *Let F and F' be Siegel cusp forms with respect to $\mathrm{Sp}_4(\mathbb{Z})$. Assume that F and F' are Hecke eigenforms, that they are not Saito-Kurokawa lifts and that π resp. π' are the associated cuspidal, automorphic representations of $\mathrm{GSp}_4(\mathbb{A})$. Let χ be a Hecke character of $\mathbb{A}^\times$ (possibly trivial) such that $\chi^2 = 1$, τ_2 be a unitary, cuspidal, automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ with trivial central character, and τ_3 be a unitary, self-dual, cuspidal, automorphic representation of $\mathrm{GL}_3(\mathbb{A})$. Then the central values*

$$L(1/2, \pi \otimes \chi, \rho_4), \quad L(1/2, \pi \otimes \tau_2, \rho_5 \otimes \sigma_2), \quad L(1/2, \pi \otimes \tau_3, \rho_4 \otimes \sigma_3), \quad L(1/2, \pi \times \pi', \rho_4 \otimes \rho_5),$$

are all non-negative.

Proof. Recall that the lifting Π_4 is symplectic by Theorem 5.1.2, and the lifting Π_5 is orthogonal by Theorem 5.1.5. Furthermore, τ_2 is symplectic and τ_3 is orthogonal. All the assertions now follow from Theorem 1.1 of [53]. ∎

5.3 Critical values of L-functions

If $L(s, \mathcal{M})$ is an arithmetically defined (or motivic) L-series associated to an arithmetic object $\mathcal{M}$, it is of interest to study its values at certain critical points $s = m$. For these critical points, conjectures due to Deligne predict that $L(m, \mathcal{M})$ is the product of a suitable transcendental number Ω and an algebraic number $A(m, \mathcal{M})$ and furthermore, if σ is an automorphism of $\mathbb{C}$, then $A(m, \mathcal{M})^\sigma = A(m, \mathcal{M}^\sigma)$. In this subsection, we will prove critical value results in the spirit of the above conjecture for L-functions associated to a Siegel cusp form of full level.

For any subring $A \subset \mathbb{C}$, let $S_l\big(\mathrm{Sp}_4(\mathbb{Z}), A\big)$ be the A-module consisting of the holomorphic Siegel cusp forms $F(Z) = \sum_S a(F, S) e^{2\pi i \, \mathrm{tr}(SZ)}$ of weight l for $\mathrm{Sp}_4(\mathbb{Z})$ for which all the Fourier coefficients

$a(F, S)$ lie in A. For $F \in S_l\big(\mathrm{Sp}_4(\mathbb{Z}), \mathbb{C}\big)$ and $\sigma \in \mathrm{Aut}(\mathbb{C})$, define $^\sigma F$ by

$$^\sigma F(Z) = \sum_S \sigma(a(F, S)) e^{2\pi i \operatorname{tr}(SZ)}.$$

By work of Shimura [81], we know that $^\sigma F \in S_l\big(\mathrm{Sp}_4(\mathbb{Z}), \mathbb{C}\big)$ and

$$S_l\big(\mathrm{Sp}_4(\mathbb{Z}), \mathbb{Q}\big) \otimes_{\mathbb{Q}} \mathbb{C} = S_l\big(\mathrm{Sp}_4(\mathbb{Z}), \mathbb{C}\big).$$

Also, if F is a Hecke eigenform, so is $^\sigma F$; see Kurokawa [50].

Now, let $F \in S_l\big(\mathrm{Sp}_4(\mathbb{Z}), \mathbb{C}\big)$ be an eigenform for all the Hecke operators and let π_F be the associated cuspidal, automorphic representation of $\mathrm{GSp}_4(\mathbb{A})$. We assume that F is not of Saito-Kurokawa type, so that the hypothesis of Theorems 5.1.2 is satisfied. Let Π_F be the resulting cuspidal, automorphic representation of $\mathrm{GL}_4(\mathbb{A})$. The representation Π_F is regular and algebraic in the sense of [10]. We define the σ-twist $^\sigma \Pi_F$ as in [10] or [88]. This can be described locally. If $\Pi_F = \otimes_p \Pi_{F,p} \otimes \Pi_{F,\infty}$, then $^\sigma \Pi_F = \otimes_p {}^\sigma \Pi_{F,p} \otimes \Pi_{F,\infty}$, where for any finite place p,

$$^\sigma \mathrm{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}_4(\mathbb{Q}_p)}(\chi_1 \otimes \ldots \otimes \chi_4) = \mathrm{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}_4(\mathbb{Q}_p)}({}^\sigma \chi_1' \otimes \ldots \otimes {}^\sigma \chi_4'). \tag{272}$$

Here B is the standard Borel of GL_4, $\chi_1, \ldots, \chi_4$ are characters of $\mathbb{Q}_p^\times$ and for any such χ,

$$^\sigma \chi'(x) = \sigma(\chi(x)|x|^{\frac{1}{2}})|x|^{-\frac{1}{2}}.$$

(See Waldspurger's example on [88, p. 125].) We have the following lemma.

5.3.1 Lemma. *Let F be a holomorphic Siegel cusp form for $\mathrm{Sp}_4(\mathbb{Z})$ that is an eigenfunction for all the Hecke operators and σ an automorphism of $\mathbb{C}$. Suppose that F is not of Saito-Kurokawa type. Then $^\sigma F$ is not of Saito-Kurokawa type. Furthermore, if $\Pi_{\sigma F}$ is the cuspidal, automorphic representation of $\mathrm{GL}_4(\mathbb{A})$ obtained from $^\sigma F$ by Theorem 5.1.2, then*

$$\Pi_{\sigma F} = {}^\sigma \Pi_F.$$

Proof. First of all, note that the condition of F being of Saito-Kurokawa type is equivalent to simple relations among the Fourier coefficients of F as in [15, p. 76]. These relations are preserved under the action of σ. This proves the first part of the lemma. For the second part, we need to show that $\Pi_{\sigma F,p} = {}^\sigma \Pi_{F,p}$ for any prime p. Fix such a prime p. Suppose that

$$\Pi_{\sigma F,p} = \mathrm{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}_4(\mathbb{Q}_p)}(\chi_1'' \otimes \ldots \otimes \chi_4'').$$

Let $\lambda_{F,m}$ be the eigenvalue for the Hecke operator $T(m)$ acting on F. For the exact definition of these Hecke operators, we refer the reader to Andrianov [1]. By Kurokawa [50], we know that $\sigma(\lambda_{F,m}) = \lambda_{\sigma F,m}$. By writing the local degree-4 Euler factors in terms of the Hecke eigenvalues, we conclude that the multisets $\{\chi_1''(p), \ldots, \chi_4''(p)\}$ and $\{{}^\sigma \chi_1'(p), \ldots, {}^\sigma \chi_4'(p)\}$ are identical. Hence

$$\mathrm{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}_4(\mathbb{Q}_p)}(\chi_1'' \otimes \ldots \otimes \chi_4'') = \mathrm{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}_4(\mathbb{Q}_p)}({}^\sigma \chi_1' \otimes \ldots \otimes {}^\sigma \chi_4'),$$

and therefore, $\Pi_{\sigma F} = {}^\sigma \Pi_F$. ∎

We now supply certain results on critical L-values for $\mathrm{GSp}_4 \times \mathrm{GL}_n$ where $n \in \{1, 2\}$.

Critical value result for $\mathrm{GSp}_4 \times \mathrm{GL}_1$

In [30], Grobner and Raghuram define certain periods of automorphic forms on GL_{2n} by comparing cohomologies in top degree. We refrain from giving the definition of these periods here in the interest of brevity and instead refer the reader to [30, Sec. 4] for details. When the results of [30] are combined with our Theorem 5.1.2, we obtain a special value result for twists of Siegel eigenforms by Dirichlet characters. We now briefly describe this result.

Let F be a holomorphic Siegel cusp form of weight l for $\mathrm{Sp}_4(\mathbb{Z})$ that is an eigenfunction for all the Hecke operators and is not of Saito-Kurokawa type. Let $\Pi_F = \Pi_{F,f} \otimes \Pi_{F,\infty}$ be the lift to $\mathrm{GL}_4(\mathbb{A})$; here $\Pi_{F,f}$ denotes the finite part of the automorphic representation Π_F. Let $\mathbb{Q}(\Pi_F)$ denote the rationality field of Π_F as defined in [10]. This is a totally real number field, and by the argument of Lemma 5.3.1, we know that $\mathbb{Q}(\Pi_F)$ equals the field generated by all the Hecke eigenvalues of F. For convenience we will denote $\mathbb{Q}(\Pi_F)$ by $\mathbb{Q}(F)$. For χ a Hecke character of $\mathbb{A}$ of finite order, let $\mathbb{Q}(\chi)$ denote the number field generated by the image of χ and let $\mathbb{Q}(F,\chi)$ denote the compositum of $\mathbb{Q}(F)$ and $\mathbb{Q}(\chi)$. Define $\mathbb{Q}(^\sigma F)$, $\mathbb{Q}(^\sigma \chi)$ and $\mathbb{Q}(^\sigma F, {}^\sigma \chi)$ similarly.

5.3.2 Remark. By Mizumoto [59], it is known that for any integer l, there exists an orthogonal basis $\{F_1, F_2, \ldots, F_d\}$ comprising of Hecke eigenfunctions for $S_l(\mathrm{Sp}_4(\mathbb{Z}), \mathbb{C})$ such that each $F_i \in S_l(\mathrm{Sp}_4(\mathbb{Z}), \mathbb{Q}(F_i))$.

Let $\omega^+(\Pi_{F,f})$ and $\omega^-(\Pi_{F,f})$ be the periods as defined in [30, Sect. 4]. For convenience, let us denote them by $\omega^+(F)$ and $\omega^-(F)$ respectively. These are non-zero complex numbers obtained from comparing cohomologies in top degree. Also let $c(\Pi_{F,\infty,0})$ be as in [30, Sect. 6.6] and denote $c(\Pi_{F,\infty},0)^{-1}$ by $\omega_\infty(l)$; this notation is justified because $c(\Pi_{F,\infty},0)$ depends only on the weight l. Then, applying the main theorem of [30] to the representation Π_F leads to the following special value result.

5.3.3 Theorem. ([30], Corollary 8.3.1) *Let F be a cuspidal Siegel eigenform of weight l for $\mathrm{Sp}_4(\mathbb{Z})$ that is not of Saito-Kurokawa type and let χ be a Hecke character of $\mathbb{A}$ of finite order. Let $\epsilon_\chi \in \{+,-\}$ denote the sign of $\chi(-1)$, $\mathcal{G}(\chi_f)$ denote the Gauss sum for χ and $L_f(s, \pi_F \times \chi) = \prod_{p<\infty} L(s, \pi_{F,p} \times \chi_p)$ denote the finite part of the L-function. Define*

$$A(F,\chi) = \frac{L_f(\frac{1}{2}, \pi_F \times \chi)}{\omega^{\epsilon_\chi}(F)\omega_\infty(l)\mathcal{G}(\chi_f)^2}.$$

Then we have

   i) $A(F,\chi) \in \mathbb{Q}(F,\chi)$,

   ii) For any automorphism σ of $\mathbb{C}$, we have $\sigma(A(F,\chi)) = A(^\sigma F, {}^\sigma \chi)$.

5.3.4 Remark. In [32], Harris defined certain "occult" periods for GSp_4 by comparing rational structures on Bessel models and rational structures on coherent cohomology and used these to study the critical values of the degree 4 L-function for GSp_4.

As a corollary to Theorem 5.3.3, we immediately obtain the following result.

5.3.5 Corollary. *Let d_1 and d_2 be two fundamental discriminants of the same sign, and let χ_{d_1}, χ_{d_2} be the associated quadratic Dirichlet characters. Let F be a cuspidal Siegel eigenform of weight l for $\mathrm{Sp}_4(\mathbb{Z})$ that is not of Saito-Kurokawa type. Then we have*

$$L_f(\frac{1}{2}, \pi_F \times \chi_{d_1}) \sim_{\mathbb{Q}(F)} L_f(\frac{1}{2}, \pi_F \times \chi_{d_2}),$$

where $"\sim_{\mathbb{Q}(F)}"$ means up to multiplication by an element in the number field $\mathbb{Q}(F)$.

5.3.6 Remark. In [5], Böcherer made a remarkable conjecture that expresses the central values $L_f(\frac{1}{2}, \pi_F \times \chi_d)$, as d varies over negative fundamental discriminants, in terms of the Fourier coefficients of F of discriminant d. In particular, Böcherer's conjecture implies Corollary 5.3.5 above for the case that d_1, d_2 are both negative. Thus Corollary 5.3.5 can be read as providing evidence towards Böcherer's conjecture.

Critical value result for $\mathrm{GSp}_4 \times \mathrm{GL}_2$

Next, we provide a critical value result for $\mathrm{GSp}_4 \times \mathrm{GL}_2$. This result will not use our lifting theorem, but instead will follow from the integral representation (Theorem 3.6.1) using the methods of [73].

5.3.7 Theorem. *Let F be a cuspidal Siegel eigenform of weight l for $\mathrm{Sp}_4(\mathbb{Z})$ such that $F \in S_l(\mathrm{Sp}_4(\mathbb{Z}), \mathbb{Q}(F))$. Let $g \in S_l(N, \chi)$ be a primitive Hecke eigenform of level N and nebentypus χ; here N is any positive integer, and χ a Dirichlet character mod N. Let π_F and τ_g be the irreducible, cuspidal, automorphic representations of $\mathrm{GSp}_4(\mathbb{A})$ and $\mathrm{GL}_2(\mathbb{A})$ corresponding to F and g. Let $\mathbb{Q}(F, g, \chi)$ be the field generated by the Hecke eigenvalues of F, the Hecke eigenvalues of g and the values taken by χ. For a positive integer k, $1 \le k \le \frac{\ell}{2} - 2$, define*

$$A(F, g; k) = \frac{L_f(\frac{\ell}{2} - k, \pi_F \times \pi_g)}{\pi^{5\ell - 4k - 4} \langle F, F \rangle \langle g, g \rangle}.$$

Then we have,

i) *$A(F, g; k) \in \mathbb{Q}(F, g, \chi)$,*

ii) *For an automorphism σ of $\mathbb{C}$, $\sigma(A(F, g; k)) = A(^\sigma F, ^\sigma g; k)$.*

5.3.8 Remark. Note that the first claim of the above theorem actually follows from the second.

5.3.9 Remark. Partial results towards the above theorem have been previously obtained by Böcherer–Heim [6], Furusawa [19], and various combinations of the authors [64, 66, 72, 73].

Proof. The proof is essentially identical to that of Theorem 8.1 of [73] which proved the above result under certain restrictions on N, χ and F. More precisely, in [73], N was assumed to be squarefree and all its prime divisors inert in a certain quadratic field, χ was assumed to be trivial, and F was assumed to satisfy a certain non-vanishing condition on the Fourier coefficients. These restrictions were necessary because the relevant integral representation [73, Thm. 6.4] in that paper was proved only under these assumptions. The special value result in that paper followed from the integral representation by first rewriting the integral representation in classical language and then using results of Garrett and Harris and the theory of nearly holomorphic functions due to Shimura.

However, in the current paper, the second integral representation (Theorem 3.6.1) works for general N and χ and the non-vanishing assumption on F is always satisfied, as shown in [75]. Now, Theorem 5.3.7 follows in an identical manner as in [73], because the remaining ingredients (the theory of nearly holomorphic functions and the results of Garrett and Harris) are true for general N and χ. It is worth noting, however, that we still need to assume that the weights of F and g are equal (even though the integral representation, Theorem 3.6.1, works for arbitrary g) because otherwise the Eisenstein series $E_\Upsilon(g, s)$ at the right-most critical point (corresponding to $s = \frac{l}{6} - \frac{1}{2}$) is no longer holomorphic. ∎

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