LOCAL SPECTRAL EQUIDISTRIBUTION FOR SIEGEL MODULAR FORMS AND APPLICATIONS

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Abstract. We study the distribution, in the space of Satake parameters, of local components of Siegel cusp forms of genus 2 and growing weight k, subject to a specific weighting which allows us to apply results concerning Bessel models and a variant of Petersson's formula. We obtain for this family a quantitative local equidistribution result, and derive a number of consequences. In particular, we show that the computation of the density of low-lying zeros of the spinor L-functions (for restricted test functions) gives global evidence for a well-known conjecture of Böcherer concerning the arithmetic nature of Fourier coefficients of Siegel cusp forms.

1. Introduction

1.1. **Motivation.** The motivation behind this paper lies in attempts to understand what is a correct definition of a family of cusp forms, either on $\mathrm{GL}(n)$ or on some other reductive algebraic group. The basic philosophy (or strategy) underlying our work is the following form of local-global principle: given a "family" Π of cusp forms, for any finite place v, the local components π_v of the elements $\pi \in \Pi$ (which are represented as restricted tensor products $\pi = \otimes\pi_v$ over all places) should be well-behaved, and more specifically, under averaging over finite subsets of the family, (π_v) should become equidistributed with respect to a suitable measure μ_v. Readers already familiar with work on families of Dirichlet characters, classical modular forms on $\mathrm{GL}(2)$, or families of L-functions of abelian varieties, will recognize that this principle is implicit in much of these works, through the orthogonality relations for Dirichlet characters, the trace formula (or the Petersson formula) and the "vertical" Sato-Tate laws over finite fields. The expected outcome is that, for instance, averages over the family of values of L-functions $L(s_0, \pi)$ at some point s_0, at least on the right of the critical line, should be directly related to the Euler product corresponding to local averages computed using μ_v. (For a general informal survey of this point of view, see [29]).

One of our goals is to give an example where this strategy can be implemented in the case of holomorphic cusp forms on $\mathrm{GSp}(4)$ (i.e., Siegel modular forms), and to derive some applications of it. In particular, we will prove:[1]

Theorem 1.1. *For $k \geqslant 2$, let $\mathcal{S}_k^*$ be a Hecke basis of the space of Siegel cusp forms on* $\mathrm{Sp}(4,\mathbb{Z})$*, and let $\mathcal{S}_k^\flat$ be the set of those $F \in \mathcal{S}_k^*$ which are* not *Saito-Kurokawa lifts. For $F \in \mathcal{S}_k^*$, let*

$$F(Z) = \sum_{T>0} a(F,T) e(\mathrm{Tr}(TZ))$$

be its Fourier expansion, where T runs over symmetric positive-definite semi-integral matrices, and let

$$\omega_k^F = \sqrt{\pi}(4\pi)^{3-2k}\Gamma(k-\tfrac{3}{2})\Gamma(k-2)\frac{|a(F,1)|^2}{4\langle F,F\rangle}. \tag{1.1.1}$$

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[1] Unfamiliar notation will be explained later.

Let $L(F,s)$ denote the finite part of the spin L-function of F, an Euler product of degree 4 over primes with local factors of the form

$$L_p(F,s) = (1-a_p p^{-s})^{-1}(1-b_p p^{-s})^{-1}(1-a_p^{-1}p^{-s})^{-1}(1-b_p^{-1}p^{-s})^{-1}, \qquad a_p,\ b_p \in (\mathbb{C}^\times)^2.$$

Then, for any $s \in \mathbb{C}$ such that $\mathrm{Re}(s) > 1$, we have

$$\lim_{k\to+\infty} \sum_{F\in \mathcal{S}_{2k}^\flat} \omega_{2k}^F L(F,s) = \zeta(s+\tfrac{1}{2})L(\chi_4, s+\tfrac{1}{2}), \tag{1.1.2}$$

where $\zeta(s)$ denotes the Riemann zeta function and $L(\chi_4,s)$ is the L-function associated to the unique Dirichlet character of conductor 4. For $s \neq 3/2$, one can replace $\mathcal{S}_{2k}^\flat$ with $\mathcal{S}_{2k}^*$.

More generally, for all primes p there exist measures μ_p on $(\mathbb{C}^\times)^2$, which are in fact supported on $(S^1)^2$, with the following property: for any irreducible r-dimensional representation ρ of $\mathrm{GSp}(4,\mathbb{C})$, let $L(F,\rho,s)$ denote the associated Langlands L-function, an Euler product of degree $r \geqslant 1$ over primes with local factors of the form

$$L_p(F,\rho,s) = \prod_{i=1}^{r} (1-Q_i(a_p,b_p)p^{-s})^{-1}, \qquad a_p,\ b_p \in (\mathbb{C}^\times)^2.$$

where $Q_i(x,y)$ is a polynomial in x, y, x^{-1}, y^{-1}. Then for any $s \in \mathbb{C}$ such that $\mathrm{Re}(s) > s_0$, with s_0 depending on ρ, we have

$$\lim_{k\to+\infty} \sum_{F\in \mathcal{S}_{2k}^*} \omega_{2k}^F L(F,\rho,s) = \prod_p \int \prod_{i=1}^{r} (1-Q_i(a,b)p^{-s})^{-1} d\mu_p(a,b), \tag{1.1.3}$$

where the right-hand side converges absolutely.

The weight ω_k^F which is introduced in this theorem is natural because of our main tool, which is a (rather sophisticated) extension of the classical Petersson formula to the case of Siegel modular forms of genus 2; see Propositions 3.3 and 3.6. In fact, we can work with more general weights $\omega_{k,d,\Lambda}^F$ (as defined in the next section) which involve averages of $a(F,T)$ over positive definite T with a fixed discriminant.

The quantitative local equidistribution leads naturally to a result on the distribution of low-lying zeros:

Theorem 1.2 (Low-lying zeros)**.** *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be an even Schwartz function such that the Fourier transform*

$$\hat{\varphi}(t) = \int_{\mathbb{R}} \varphi(x) e^{-2i\pi x t} dx$$

has compact support contained in $[-\alpha,\alpha]$, where $\alpha < 4/15$. For $F \in \mathcal{S}_{2k}^*$, assume the Riemann Hypothesis: the zeros of $L(F,s)$ in the critical strip $0 < \mathrm{Re}(s) < 1$ are of the form

$$\rho = \frac{1}{2} + i\gamma$$

with $\gamma \in \mathbb{R}$. Define

$$D_\varphi(F) = \sum_\rho \varphi\Big(\frac{\gamma}{\pi}\log k\Big),$$

where ρ ranges over the zeros of $L(F,s)$ on the critical line, counted with multiplicity.

Then we have

$$\lim_{k\to+\infty} \sum_{F\in \mathcal{S}_{2k}^*} \omega_{2k}^F D_\varphi(F) = \int_{\mathbb{R}} \varphi(x) d\sigma_{Sp}(x), \tag{1.1.4}$$

where σ_{Sp} is the "symplectic symmetry" measure given by

$$d\sigma_{Sp} = dx - \frac{\delta_0}{2}, \qquad \delta_0 \text{ Dirac mass at 0.}$$

This result raises interesting questions concerning the notion of "family" of cusp forms, especially from the point of view of the notion of symmetry type that has arisen from the works of Katz-Sarnak [25]. Indeed, the limit measure above is the one that arises from symplectic symmetry types, i.e., from the distribution of eigenvalues close to 1 of symplectic matrices of large size, when renormalized to have averaged spacing equal to 1. In general, it is expected that some cusp forms will exhibit this symmetry when some kind of infinite symplectic group occurs as "monodromy group" for the family, in the way that generalizes the Chebotarev and Deligne equidistribution theorems.

We do not believe that this is the case here, and rather expect that the limit measure in the theorem is due in part to the presence of the weight ω_{2k}^F used in the averages involved. Precisely, we expect that the correct symmetry type, without weight, is *orthogonal*, in the sense that for φ with support in $]-1,1[$, we should have[2]

$$\frac{1}{|\mathcal{S}_{2k}^*|}\sum_{F\in\mathcal{S}_{2k}^*} D_\varphi(F) \longrightarrow \int_{\mathbb{R}} \varphi(x)d\sigma_O(x) \tag{1.1.5}$$

where $d\sigma_O(x) = dx + \frac{\delta_0}{2}$ is the corresponding measure for eigenvalues close to 1 of orthogonal matrices.

Intuitively, this should be related to the fact that the point $1/2$ is a critical special value – in the sense of Deligne – for the spin L-functions of cusp forms $F \in \mathcal{S}_k^*$ (whereas $1/2$ is not for real quadratic characters for example, which are the typical example where symplectic symmetry is expected), similar to the special role of the eigenvalue 1 for orthogonal matrices, but not for symplectic ones.

Now the natural question is why should the weight ω_{2k}^F have such an effect? (This is especially true because it may look, at first, just like an analogue of the weight involving the Petersson norm of classical modular forms which has been used very frequently without exhibiting any such behavior, e.g., in the works of Iwaniec, Luo and Sarnak [24] and Duenez-Miller [11].)

The point is that this weight ω_k^F itself *contains arithmetic information related to central L-values of the Siegel cusp forms.* Indeed, we will see in Section 5.4 that Theorem 1.2 can be interpreted convincingly – assuming an orthogonal symmetry as in (1.1.5) – as evidence for a beautiful conjecture of Böcherer (see [3] or [19, Intr.]) which suggests in particular a relation of the type

$$|a(F,1)|^2 \simeq L(F,1/2)L(F\times\chi_4,1/2) \tag{1.1.6}$$

(where the $\simeq$ sign means equality up to non-zero factors "unrelated to central critical values"; this version of the conjecture is that proposed by Furusawa and Martin [16, §1, (1.4)]). We therefore consider that Theorem 1.2 provides suggestive global evidence towards these specific variants of Böcherer's conjecture. Note that, at the current time, this conjecture is not rigorously known for any cusp form in $\mathcal{S}_{2k}^*$ which is not a Saito-Kurokawa lift.

Remark 1.3. One can easily present analogues of the phenomenon in Theorem 1.2, as we understand it, in the setting of random matrices. For instance, if μ_n denotes the probability Haar measure on $\mathrm{SO}_{2n}(\mathbb{R})$, one may consider the measures

$$d\nu_n(g) = c_n \det(1-g)d\mu_n(g),$$

[2] We do not try to predict whether odd or even orthogonal symmetry should occur; these could be distinguished most simply by computing the 2-level density for test functions with restricted support as done in [33] for classical modular forms (one can also attempt to study the low-lying zeros for test functions with support larger than $]-1,1[$, as done in [24].

where $c_n > 0$ is the constant that ensures that ν_n is a probability measure.[3] The distribution of the low-lying eigenvalues of $g \in \mathrm{SO}_{2n}(\mathbb{R})$, when computed using this measure, will clearly differ from that arising from Haar measure (intuitively, by diminishing the influence of matrices with an eigenvalue close to 1, the factor $\det(1-g)$ will produce a repulsion effect similar to what happens for symplectic matrices.)

Readers familiar with the case of GL(2)-modular forms but not with Siegel modular forms (or with their representation-theoretic interpretation) may look at the Appendix where we discuss briefly the analogies and significant differences between our results and some more elementary GL(2)-versions.

For orientation, we add the following quick remarks: (1) the spin L-function of $F \in \mathcal{S}_{2k}^*$ has analytic conductor (see [23, p. 95] for the definition) of size k^2; (2) the cardinality of $\mathcal{S}_{2k}^*$ (i.e., the dimension of $\mathcal{S}_{2k}$) is of order of magnitude k^3 (see, e.g., [27, Cor. p. 123] for the space $\mathcal{M}_{2k}$ of all Siegel modular forms of weight $2k$, and [27, p. 69] for the size of the "correction term" $\mathcal{M}_{2k}/\mathcal{S}_{2k}$); (3) as already mentioned, the spin L-function is self-dual with functional equation involving the sign $+1$ for all $F \in \mathcal{S}_{2k}^*$.

Apart from the treatment of low-lying zeros, we do not "enter the critical strip" in this paper. However, we hope to come back to the problem of extending Theorem 1.1 to averages at points inside the critical strip, and we may already remark that, if a statement like (1.1.6) is valid, the weight already involves some critical values of L-functions (in fact, of an L-function of degree 8.

1.2. **Local equidistribution statement.** In order to state our main result on local spectral equidistribution of Siegel modular forms, we begin with some preliminary notation concerning cuspidal automorphic representations of $G(\mathbb{A}) = \mathrm{GSp}(4, \mathbb{A})$.

Let π be a cuspidal automorphic representation of $G(\mathbb{A})$, which we assume to be unramified at all finite places and with trivial central character. It is isomorphic to a restricted tensor product $\pi = \otimes_v \pi_v$ where, for all places, π_v is an irreducible admissible unitary representation of the local group $G(\mathbb{Q}_v)$.

By our assumption π_p is unramified for all primes p and so the natural underlying space for local equidistribution at p (when considering families) is the set X_p of unramified unitary infinite-dimensional irreducible representations of $G(\mathbb{Q}_p)$ with trivial central character. This set has a natural topology, hence a natural σ-algebra.

We now proceed quite concretely to give natural coordinates on X_p from which the measurable structure is obvious. By [7], any $\pi_p \in X_p$ can be identified with the unique unramified constituent of a representation $\chi_1 \times \chi_2 \rtimes \sigma$ induced from a character of the Borel subgroup which is defined as follows using unramified (not necessarily unitary!) characters χ_1, χ_2, σ of $\mathbb{Q}_p^\times$:

$$\begin{pmatrix} a_1 & * & * & * \\ & a_2 & * & * \\ & & \lambda a_1^{-1} & \\ & & * & \lambda a_2^{-1} \end{pmatrix} \mapsto \chi_1(a_1)\chi_2(a_2)\sigma(\lambda).$$

Having trivial central character means that

$$\chi_1\chi_2\sigma^2 = 1,$$

and since the characters are unramified, it follows that π_p is characterized by the pair $(a, b) = (\sigma(p), \sigma(p)\chi_1(p)) \in \mathbb{C}^* \times \mathbb{C}^*$. The classification of local representations of $G(\mathbb{Q}_p)$ (see for instance [38, Proposition 3.1]) implies that the local parameters satisfy

$$0 < |a|,\ |b| \leqslant \sqrt{p}. \tag{1.2.1}$$

[3] In fact, although this seems irrelevant, a computation with the moments of characteristic polynomials of orthogonal matrices shows that $c_n = 1/2$ for all n.

There are some identifications between the representations associated to different (a, b), coming from the Weyl W group of order 8 generated by the transformations

$$(a, b) \mapsto (b, a), \quad (a, b) \mapsto (a^{-1}, b), \quad (a, b) \mapsto (a, b^{-1}). \tag{1.2.2}$$

We will denote by Y_p the quotient of the set of (a, b) satisfying the upper-bounds (1.2.1), modulo the action of W. This has the quotient topology and quotient σ-algebra, and we identify X_p with a subset of Y_p using the parameters (a, b) described above. We will also denote by $X \subset X_p$ the subset of tempered representations; under the identification of X_p with a subset of Y_p, the set X corresponds precisely to $|a| = |b| = 1$. Note that this subset is indeed independent of p.

In applications to L-functions, the local-global nature of automorphic representations is reflected not only in the existence of local components, but in their "independence" (or product structure) when p varies. To measure this below, we will also need to consider, for any finite set of primes $\boldsymbol{S}$, the maps

$$\pi \mapsto (\pi_p)_{p \in \boldsymbol{S}}$$

which have image in the space

$$X_{\boldsymbol{S}} = \prod_{p \in \boldsymbol{S}} X_p$$

and can be identified with a subset of

$$Y_{\boldsymbol{S}} = \prod_{p \in \boldsymbol{S}} Y_p.$$

Now we come back to Siegel modular forms. Let $\mathcal{S}_k = \mathcal{S}_k(\mathrm{Sp}(4, \mathbb{Z}))$ be the space of Siegel cusp forms of degree 2, level 1 and weight k. By adélization (as described in more detail in the next section), there is a cuspidal automorphic representation π_F canonically attached to F; the assumption that the level is 1 and there is no nebentypus means that π_F is unramified at finite places with trivial central character, as above. Thus we have local components $\pi_p(F) \in X_p$ and corresponding parameters $(a_p, b_p) \in Y_p$ for every prime p.

The generalized Ramanujan conjecture has been proved in this setting by Weissauer [50]: it states that, if F is not a Saito-Kurokawa lift (these forms are defined in [13] for example; at the beginning of Section 5.2, we recall the description in terms of L-functions), we have $\pi_p(F) \in X$ for all p, i.e., $|a_p| = |b_p| = 1$. On the other hand, if F is a Saito-Kurokawa lift, then $|a| = 1$ and

$$\{|b|, |b|^{-1}\} = \{p^{1/2}, p^{-1/2}\}.$$

Remark 1.4. Partly because our paper is meant to explore the general philosophy of families of cusp forms, we will not hesitate to use this very deep result of Weissauer when this helps in simplifying our arguments. But it will be seen that the proof of the local equidistribution property itself does not invoke this result, and it seems quite likely that, with some additional work, it could be avoided in most, if not all, of the applications (in similar questions of local equidistribution for classical Maass cusp forms on GL(2), one can avoid the unproved Ramanujan-Petersson conjecture).

Denote by $\mathcal{S}_k^*$ any fixed Hecke-basis of $\mathcal{S}_k$. Although this is not known to be unique, the averages we are going to consider turn out to be independent of this choice. In fact, all final results could be phrased directly in terms of automorphic representations, avoiding such a choice (at least seemingly).

We next proceed to define our way of weighting the cusp forms in $\mathcal{S}_k^*$. This generalizes the ω_k^F in the statement of the first theorem, and the reader may assume below that the parameters introduced are $d = 4$ and $\Lambda = 1$.

Let $d > 0$ be a positive integer such that $-d$ is a fundamental discriminant of an imaginary quadratic field (i.e., one of the following holds: (1) d is congruent to 3 (mod 4) and is square-free; or (2) $d = 4m$ where m is congruent to 1 or 2 (mod 4) and m is square-free). Let Cl_d denote the ideal class group of this field, $h(-d)$ denote the class number, i.e., the cardinality of Cl_d, and $w(-d)$ denote the number of roots of unity. Finally, fix a character Λ of Cl_d.

There is a well-known natural isomorphism, to be recalled more precisely in Section 2.2, between Cl_d and the $\mathrm{SL}(2,\mathbb{Z})$-equivalence classes of primitive semi-integral two by two positive definite matrix with determinant equal to $d/4$. By abuse of notation, we will also use Cl_d to denote the set of equivalence classes of such matrices.

Define normalizing factors

$$c_{k,d} = \left(\frac{d}{4}\right)^{\frac{3}{2}-k} \frac{4c_k}{w(-d)h(-d)},$$

where

$$c_k = \frac{\sqrt{\pi}}{4}(4\pi)^{3-2k}\Gamma(k-\tfrac{3}{2})\Gamma(k-2).$$

We note that, using Dirichlet's class number formula, one can also write

$$c_{k,d} = \left(\frac{d}{4}\right)^{1-k} \frac{4\pi c_k}{w(-d)^2 L(1,\chi_d)}$$

where χ_d is the real primitive Dirichlet character associated to the extension $\mathbb{Q}(\sqrt{-d})$. Now, for each $F \in \mathcal{S}_k^*$, we have a Fourier expansion

$$F(Z) = \sum_{T>0} a(F,T)e(\mathrm{Tr}(TZ)),$$

where T runs over positive definite symmetric semi-integral matrices of size 2:

$$T = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$$

with $(a,b,c) \in \mathbb{Z}$. It follows easily from the modularity property that $a(F,T)$ depends only on the equivalence class of T modulo $\mathrm{SL}(2,\mathbb{Z})$. In fact, when k is even, $a(F,T)$ depends only on the equivalence class of T modulo $\mathrm{GL}(2,\mathbb{Z})$.

We now let

$$\omega_{k,d,\Lambda}^F = c_{k,d} \cdot d_\Lambda \cdot \frac{|a(d,\Lambda;F)|^2}{\langle F,F\rangle}$$

where we put

$$d_\Lambda = \begin{cases} 1 & \text{if } \Lambda^2 = 1, \\ 2 & \text{otherwise.} \end{cases}$$

and

$$a(d,\Lambda;F) = \sum_{c\in \mathrm{Cl}_d} \overline{\Lambda(c)}a(F,c), \tag{1.2.3}$$

a quantity which is well-defined in view of the invariance of Fourier coefficients under $\mathrm{SL}(2,\mathbb{Z})$.

Remark 1.5. We will often consider (d,Λ) to be fixed, and simplify the notation by writing ω_k^F only. Note that if $d=4$ and $\Lambda=1$, the weight ω_k^F is the same as the one defined in Theorem 1.1.

Next, we define the local spectral measures associated to the family $\mathcal{S}_k^*$; we will show that they become equidistributed as $k \to +\infty$ over even integers. Let $\boldsymbol{S}$ be a finite set of primes. We have the components $\pi_{\boldsymbol{S}}(F) = (\pi_p(F))_{p\in\boldsymbol{S}} \in X_{\boldsymbol{S}}$, and we define the measure $\nu_{\boldsymbol{S},k}$ on $X_{\boldsymbol{S}}$ by

$$d\nu_{\boldsymbol{S},k} = d\nu_{\boldsymbol{S},k,d,\Lambda} = \sum_{F\in\mathcal{S}_k^*} \omega_{k,d,\Lambda}^F \delta_{\pi_{\boldsymbol{S}}(F)}$$

where $\delta_\bullet$ is the Dirac mass at the given point. As we will see, the normalization used has the effect that $\nu_{\boldsymbol{S},k}$ is asymptotically a probability measure on $X_{\boldsymbol{S}}$: we will show that

$$\lim_{k\to+\infty} \nu_{\boldsymbol{S},2k}(X_{\boldsymbol{S}}) = 1.$$

The local equidistribution problem — which can clearly be phrased for very general families of cusp forms — is to determine if these measures have limits as $k \to +\infty$, to identify their limit, to see in particular if the limit for a given $\boldsymbol{S}$ is the product of the limits for the subsets $\{p\}$, $p \in \boldsymbol{S}$ (corresponding to independence of the restrictions), and finally — if possible — to express the resulting equidistribution in quantitative terms.

To state our theorem, we now define the limiting measures. First of all, we define a generalized Sato-Tate measure μ on each X_p by first taking the probability Haar measure on the space of conjugacy classes of the compact unitary symplectic group USp(4), then pushing this to a probability measure on X by means of the map

$$\begin{pmatrix} e^{i\theta_1} & & & \\ & e^{i\theta_2} & & \\ & & e^{-i\theta_1} & \\ & & & e^{-i\theta_2} \end{pmatrix} \mapsto (e^{i\theta_1}, e^{i\theta_2}) \in X, \qquad (\theta_1,\theta_2) \in [0,\pi]^2, \quad \theta_1 \leq \theta_2,$$

and finally extending it to X_p by defining it equal to 0 outside X.

In terms of the coordinates (θ_1, θ_2) on X, the resulting measure μ is explicitly given by

$$d\mu(\theta_1,\theta_2) = \frac{4}{\pi^2}(\cos\theta_1 - \cos\theta_2)^2 \sin^2\theta_1 \sin^2\theta_2 d\theta_1 d\theta_2 \tag{1.2.4}$$

from the Weyl integration formula [14]. We can also interpret this measure as coming in the same way from conjugacy classes of the unitary spin group USpin$(5,\mathbb{C})$, because of the "exceptional isomorphism" $\mathrm{Sp}(4) \simeq \mathrm{Spin}(5)$.

For each finite set of primes $\boldsymbol{S}$, and d, Λ as above, we now define the measure $\mu_{\boldsymbol{S}} = \mu_{\boldsymbol{S},d,\Lambda}$ on $X_{\boldsymbol{S}}$ by the formula

$$d\mu_{\boldsymbol{S}} = \bigotimes_{p\in\boldsymbol{S}} d\mu_{p,d,\Lambda}$$

where, for a single prime, we have

$$d\mu_{p,d,\Lambda} = \Big(1 - \Big(\frac{-d}{p}\Big)\frac{1}{p}\Big)\Delta_p^{-1} d\mu$$

(recall that μ is defined on X_p, but has support on X only; the same is therefore true of $\mu_{\boldsymbol{S}}$) and the density functions $\Delta_p = \Delta_{p,d,\Lambda}$ are given by

$$\Delta_p(\theta_1,\theta_2) = \begin{cases} \big((1+\frac{1}{p})^2 - \frac{4\cos^2\theta_1}{p}\big)\big((1+\frac{1}{p})^2 - \frac{4\cos^2\theta_2}{p}\big) & \text{if } p \text{ inert,} \\ \big((1-\frac{1}{p})^2 + \frac{1}{p}(2\cos\theta_1\sqrt{p} - \lambda_p)(\frac{2\cos\theta_1}{\sqrt{p}} - \lambda_p)\big) & \\ \quad \times\big((1-\frac{1}{p})^2 + \frac{1}{p}(2\cos\theta_2\sqrt{p} - \lambda_p)(\frac{2\cos\theta_2}{\sqrt{p}} - \lambda_p)\big) & \text{if } p \text{ split,} \\ \big(1 - \frac{2\lambda_p\cos\theta_1}{\sqrt{p}} + \frac{1}{p}\big)\big(1 - \frac{2\lambda_p\cos\theta_2}{\sqrt{p}} + \frac{1}{p}\big) & \text{if } p \text{ ramified,} \end{cases} \tag{1.2.5}$$

where the behavior of primes refers to the field $\mathbb{Q}(\sqrt{-d})$, and in the second and third cases, we put

$$\lambda_p = \sum_{N(\mathfrak{p})=p} \Lambda(\mathfrak{p}),$$

the sum over the (one or two) prime ideals of norm p in $\mathbb{Q}(\sqrt{-d})$.

Although we have written down this concrete, but unenlightening, expression, there is a more intrinsic definition of these measures $\mu_p = \mu_{p,d,\Lambda}$ and this will in fact be the way they will naturally occur (and the way we will prove the results): they are precisely what Furusawa and Shalika [18] call the *Plancherel measure for the local Bessel model associated to the data* (d, Λ). In particular, this description shows that they are probability measures, which is not quite obvious from the definition. On the other hand, the following property, which is of great relevance to global applications, is immediate: as $p \to +\infty$, the measures μ_p converge weakly to the measure μ, which has a group-theoretic interpretation.

Our local equidistribution result can now be stated:

Theorem 1.6 (Local equidistribution and independence)**.** *Fix any d, Λ as above. For any finite set of primes $\boldsymbol{S}$, the measures $\nu_{\boldsymbol{S},k}$ on $X_{\boldsymbol{S}}$ converge weak-$*$ to $\mu_{\boldsymbol{S}}$ as $k \to +\infty$ over even integers, i.e., for any continuous function φ on $Y_{\boldsymbol{S}}$, we have*

$$\lim_{k\to+\infty} \sum_{F \in \mathcal{S}_{2k}^*} \omega_{2k}^F \varphi((a_p(F), b_p(F))_{p\in\boldsymbol{S}}) = \int_{Y_{\boldsymbol{S}}} \varphi(x) d\mu_{\boldsymbol{S}}(x).$$

In particular, if

$$\varphi((y_p)_{p\in\boldsymbol{S}}) = \prod_{p\in\boldsymbol{S}} \varphi_p(y_p)$$

is a product function, we have

$$\lim_{k\to+\infty} \sum_{F \in \mathcal{S}_{2k}^*} \omega_{2k}^F \varphi((a_p(F), b_p(F))_{p\in\boldsymbol{S}}) = \prod_{p\in\boldsymbol{S}} \int_{Y_p} \varphi_p(x) d\mu_p(x).$$

Moreover, assume φ is of product form, and that φ_p is a Laurent polynomial in (a, b, a^{-1}, b^{-1}), invariant under the action of the group W given by (1.2.2), and of total degree d_p as a polynomial in $(a + a^{-1}, b + b^{-1})$, then we have

$$\sum_{F \in \mathcal{S}_{2k}^*} \omega_{2k}^F \varphi((a_p(F), b_p(F))_{p\in\boldsymbol{S}}) = \int_{Y_{\boldsymbol{S}}} \varphi(x) d\mu_{\boldsymbol{S}}(x) + O\Big(k^{-2/3} L^{1+\varepsilon} \|\varphi\|_\infty\Big) \tag{1.2.6}$$

for any $\varepsilon > 0$, where

$$L = \prod_{p\in\boldsymbol{S}} p^{d_p},$$

and $\|\varphi\|_\infty$ is the maximum of $|\varphi|$ on $X^{|\boldsymbol{S}|} \subset Y_{\boldsymbol{S}}$. The implied constant depends only on d and ε.

Remark 1.7. It is possible to extend our results to odd k. However, this requires a slightly different definition of the weights ω_k^F. For simplicity, we only consider k even in this paper.

Remark 1.8. In a recent preprint, Sug Woo Shin [47] has proved a related result. In Shin's work, the weights ω_k^F are not present; instead the cusp forms are counted with the natural weight 1. Using the trace formula, he proves a qualitative result that for suitable families of cusp forms on connected reductive groups over totally real fields, there is local equidistribution at a given place; when the level grows the sum of the point measures associated to the forms of fixed level converges towards the *Plancherel* measure on the unitary dual of the local group.

In fact, from the viewpoint of automorphic representations on reductive groups, our result is essentially a (quantitative) *relative trace formula* analogue of what Shin (and others before him, such as DeGeorge-Wallach [9], Clozel [8], Savin [45], Serre [46], Sauvageot [44]) did using the trace formula.

We expect that the methods of this paper would suffice to prove the equidistribution result for Siegel modular forms of level N coprime to the set of places $\boldsymbol{S}$ as $N + k \to \infty$. It would also be interesting to see if our results can be generalized to the case of automorphic forms on the split special orthogonal groups, using the formulas for the Bessel model there from [6] (at least qualitatively). We hope to treat these questions elsewhere.

We briefly explain the structure of this paper. In Chapter 2, we introduce the Bessel model, explain its relation to the Fourier coefficients and derive a result relating the Fourier coefficients to Satake parameters. In Chapter 3 we recall the definition of Poincaré series in this context and derive a Petersson-type quantitative orthogonality formula for the Siegel cusp forms (this involves non-trivial adaptations of the method of Kitaoka [26]). In Chapter 4, we put the above results together to deduce our main theorem (Theorem 1.6) on local equidistribution. Finally, in Chapter 5, we prove Theorems 1.1 and 1.2 as well as provide several other applications of the results of the previous chapters.

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1.3. **Notation.** We introduce here some notation used in the paper.

- The symbols $\mathbb{Z}$, $\mathbb{Z}_{\geq 0}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}_p$ and $\mathbb{Q}_p$ have the usual meanings. $\mathbb{A}$ denotes the ring of adèles of $\mathbb{Q}$. For a complex number z, $e(z)$ denotes $e^{2\pi i z}$.
- For any commutative ring R and positive integer n, $M(n,R)$ denotes the ring of n by n matrices with entries in R and $\mathrm{GL}(n,R)$ denotes the group of invertible matrices in $M(n,R)$. If $A \in M(n,R)$, we let ${}^tA$ denote its transpose. We use $R^\times$ to denote $\mathrm{GL}(1,R)$.
- For matrices A and B, we use $A[B]$ to denote ${}^tBAB$, whenever the matrices are of compatible sizes.
- We say that a symmetric matrix in $M(n,\mathbb{Z})$ is semi-integral if it has integral diagonal entries and half-integral off-diagonal ones.
- Denote by J_n the $2n$ by $2n$ matrix given by

$$J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

  We use J to denote J_2.
- For a positive integer n, define the algebraic group $\mathrm{GSp}(2n)$ over $\mathbb{Z}$ by

$$\mathrm{GSp}(2n,R) = \{g \in \mathrm{GL}(2n,R) | {}^tgJ_ng = \mu_n(g)J_n, \mu_n(g) \in R^\times\}$$

  for any commutative ring R.

  Define $\mathrm{Sp}(2n)$ to be the subgroup of $\mathrm{GSp}(2n)$ consisting of elements $g_1 \in \mathrm{GSp}(2n)$ with $\mu_n(g_1) = 1$.

  The letter G will always stand for $\mathrm{GSp}(4)$. The letter Γ will always stand for the group $\mathrm{Sp}(4,\mathbb{Z})$.
- The Siegel upper-half space is defined by

$$\mathbb{H}_n = \{Z \in M_n(\mathbb{C}) | Z = {}^tZ,\ \mathrm{Im}(Z) \text{ is positive definite}\}.$$

  For

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G(\mathbb{R}),$$

  and $Z \in \mathbb{H}_2$, we denote

$$J(g,Z) = CZ + D.$$

- For a prime p, the maximal compact subgroup K_p of $G(\mathbb{Q}_p)$ is defined by

$$K_p = G(\mathbb{Q}_p) \cap \mathrm{GL}(4,\mathbb{Z}_p).$$

- For a quadratic extension $\mathcal{K}$ of $\mathbb{Q}$ and p a prime, define $\mathcal{K}_p = \mathcal{K} \otimes_{\mathbb{Q}} \mathbb{Q}_p$; we let $\mathbb{Z}_{\mathcal{K}}$ denote the ring of integers of $\mathcal{K}$ and $\mathbb{Z}_{\mathcal{K},p}$ its p-closure in $\mathcal{K}_p$.

2. Bessel models

2.1. Global Bessel models. We recall the definition of the Bessel model of Novodvorsky and Piatetski-Shapiro [35] following the exposition of Furusawa [15].

Let $S \in M_2(\mathbb{Q})$ be a symmetric matrix.[4] Define $\mathrm{disc}(S) = -4\det(S)$ and put $d = -\mathrm{disc}(S)$. For

$$S = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix},$$

we define the element

$$\xi = \xi_S = \begin{pmatrix} b/2 & c \\ -a & -b/2 \end{pmatrix}.$$

Let $\mathcal{K}$ denote the subfield $\mathbb{Q}(\sqrt{-d})$ of $\mathbb{C}$. We always identify $\mathbb{Q}(\xi)$ with $\mathcal{K}$ via

$$\mathbb{Q}(\xi) \ni x + y\xi \mapsto x + y\frac{\sqrt{-d}}{2} \in \mathcal{K},\ \ x, y \in \mathbb{Q}. \tag{2.1.1}$$

We define a $\mathbb{Q}$-algebraic subgroup $T = T_S$ of $\mathrm{GL}(2)$ by

$$T = \{g \in \mathrm{GL}(2) | {}^t gSg = \det(g)S\}, \tag{2.1.2}$$

so that it is not hard to verify that $T(\mathbb{Q}) = \mathbb{Q}(\xi)^\times$. We identify $T(\mathbb{Q})$ with $\mathcal{K}^\times$ via (2.1.1).

We can also consider T as a subgroup of G via

$$T \ni g \mapsto \begin{pmatrix} g & 0 \\ 0 & \det(g)\ {}^t g^{-1} \end{pmatrix} \in G. \tag{2.1.3}$$

Let us further denote by U the subgroup of G defined by

$$U = \{u(X) = \begin{pmatrix} 1_2 & X \\ 0 & 1_2 \end{pmatrix} | {}^t X = X\},$$

and by R be subgroup $R = TU$ of G.

Let $\psi = \prod_v \psi_v$ be a character of $\mathbb{A}$ such that

- The conductor of ψ_p is $\mathbb{Z}_p$ for all (finite) primes p,
- $\psi_\infty(x) = e(x)$, for $x \in \mathbb{R}$,
- $\psi|_{\mathbb{Q}} = 1$.

We define the character $\theta = \theta_S$ on $U(\mathbb{A})$ by

$$\theta(u(X)) = \psi(\mathrm{Tr}(S(X))).$$

Let Λ be a character of $T(\mathbb{A})/T(\mathbb{Q})$ such that $\Lambda|\mathbb{A}^\times = 1$. Via (2.1.1) we can think of Λ as a character of $\mathcal{K}^\times(\mathbb{A})/\mathcal{K}^\times$ such that $\Lambda|\mathbb{A}^\times = 1$. Denote by $\Lambda \otimes \theta$ the character of $R(\mathbb{A})$ defined by $(\Lambda \otimes \theta)(tu) = \Lambda(t)\theta(u)$ for $t \in T(\mathbb{A})$ and $u \in U(\mathbb{A})$.

Let π be an automorphic cuspidal representation of $G(\mathbb{A})$ with trivial central character and V_π be its space of automorphic forms. For $\Phi \in V_\pi$, we define a function B_Φ on $G(\mathbb{A})$ by

$$B_\Phi(h) = \int_{R(\mathbb{Q})Z_G(\mathbb{A})\backslash R(\mathbb{A})} \overline{(\Lambda \otimes \theta)(r)} \Phi(rh) dr. \tag{2.1.4}$$

The $\mathbb{C}$-vector space of functions on $G(\mathbb{A})$ spanned by $\{B_\Phi | \Phi \in V_\pi\}$ is called the *global Bessel space* of type (S, Λ, ψ) for π; it is invariant under the regular action of $G(\mathbb{A})$ and when the space is non-zero, the corresponding representation is a model of π. Thus one says that π has a global Bessel model of type (S, Λ, ψ) if this global Bessel space is non-zero, i.e., if there exists $\Phi \in V_\pi$ such that $B_\Phi \neq 0$.

[4] The notation conflicts a bit with the sets of primes $\boldsymbol{S}$, but we hope that the boldface font of the latter and the context will prevent any confusion.

2.2. **The classical interpretation.** Let us now suppose that Φ, π come from a classical Siegel cusp form F. More precisely, for a positive integer N, define

$$\Gamma^*(N) := \{g \in \Gamma : g \equiv \begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix} \pmod{N}\}.$$

We say that $F \in \mathcal{S}_k(\Gamma^*(N))$ if it is a holomorphic function on $\mathbb{H}_2$ which satisfies

$$F(\gamma Z) = \det(J(\gamma, Z))^k F(Z)$$

for $\gamma \in \Gamma^*(N)$, $Z \in \mathbb{H}_2$, and vanishes at the cusps. It is well-known that F has a Fourier expansion

$$F(Z) = \sum_{T>0, T \in \mathcal{L}} a(F,T) e(\mathrm{Tr}(TZ)),$$

where $e(z) = \exp(2\pi i z)$ and T runs through all symmetric positive-definite matrices of size two in a suitable lattice $\mathcal{L}$ (depending on N). If $N = 1$, then $\mathcal{L}$ is just the set of symmetric, semi-integral matrices. Also, recall that $\mathcal{S}_k(\Gamma^*(1))$ is denoted simply by $\mathcal{S}_k$.

We define the adélization Φ_F of F to be the function on $G(\mathbb{A})$ defined by

$$\Phi_F(\gamma h_\infty k_0) = \mu_2(h_\infty)^k \det(J(h_\infty, iI_2))^{-k} F(h_\infty(i)) \tag{2.2.1}$$

where $\gamma \in G(\mathbb{Q}), h_\infty \in G(\mathbb{R})^+$ and

$$k_0 \in \prod_{p \nmid N} K_p \prod_{p \mid N} K_p^N$$

where

$$K_p^N = \{g \in G(\mathbb{Z}_p) : g \equiv \begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix} \pmod{N}\}$$

is the local analogue of $\Gamma^*(N)$. Then Φ_F is a well-defined function on the whole of $G(\mathbb{A})$ by strong approximation, and is an automorphic form.

We now assume $N = 1$. Let d be a positive integer such that $-d$ is the discriminant of the imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$. Define

$$S = S(-d) = \begin{cases} \begin{pmatrix} \frac{d}{4} & 0 \\ 0 & 1 \end{pmatrix} & \text{if } d \equiv 0 \pmod{4}, \\ \begin{pmatrix} \frac{1+d}{4} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} & \text{if } d \equiv 3 \pmod{4}. \end{cases} \tag{2.2.2}$$

Define the groups R, T, U as in the previous section.

Put $\mathcal{K} = \mathbb{Q}(\sqrt{-d})$. Recall that Cl_d denotes the ideal class group of $\mathcal{K}$. Let (t_c), $c \in \mathrm{Cl}_d$, be coset representatives such that

$$T(\mathbb{A}) = \coprod_c t_c T(\mathbb{Q}) T(\mathbb{R}) \Pi_{p<\infty}(T(\mathbb{Q}_p) \cap \mathrm{GL}_2(\mathbb{Z}_p)), \tag{2.2.3}$$

with $t_c \in \prod_{p<\infty} T(\mathbb{Q}_p)$. We can write

$$t_c = \gamma_c m_c \kappa_c$$

with $\gamma_c \in \mathrm{GL}(2,\mathbb{Q})$, $m_c \in \mathrm{GL}^+(2,\mathbb{R})$, and $\kappa_c \in \Pi_{p<\infty}\mathrm{GL}(2,\mathbb{Z}_p)$.

The matrices

$$S_c = \det(\gamma_c)^{-1} \ ({}^t\gamma_c) S \gamma_c$$

have discriminant $-d$, and form a set of representatives of the $\mathrm{SL}(2,\mathbb{Z})$-equivalence classes of primitive semi-integral positive definite matrices of discriminant $-d$.

Choose Λ a character of $T(\mathbb{A})/T(\mathbb{Q})T(\mathbb{R})((\Pi_{p<\infty}T(\mathbb{Z}_p))$, which we identify with an ideal class character of $\mathbb{Q}(\sqrt{-d})$.

Next, for any positive integer N, define (this is a certain ray class group)

$$\mathrm{Cl}_d(N) = T(\mathbb{A})/T(\mathbb{Q})T(\mathbb{R})\Pi_{p<\infty}(T(\mathbb{Q}_p)\cap K_p^{(0)}(N)),$$

where $K_p^{(0)}(N)$ is the subgroup of $\mathrm{GL}_2(\mathbb{Z}_p)$ consisting of elements that are congruent to a matrix of the form

$$\begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \pmod{N}.$$

As before, we can take coset representatives $(t_{c'})$, $c' \in \mathrm{Cl}_d(N)$, such that

$$T(\mathbb{A}) = \coprod_{c'} t_{c'}T(\mathbb{Q})T(\mathbb{R})\Pi_{p<\infty}(T(\mathbb{Q}_p)\cap K_p^{(0)}(N)), \tag{2.2.4}$$

with $t_{c'} \in \prod_{p<\infty} T(\mathbb{Q}_p)$, and write

$$t_{c'} = \gamma_{c'} m_{c'} \kappa_{c'}$$

with $\gamma_{c'} \in \mathrm{GL}(2,\mathbb{Q})$, $m_{c'} \in \mathrm{GL}^+(2,\mathbb{R})$, and $\kappa_{c'} \in \Pi_{p<\infty}K_p^{(0)}(N)$.

Define the matrices

$$S_{c'} = \det(\gamma_{c'})^{-1}\ ({}^t\gamma_{c'})S\gamma_{c'}.$$

For each pair of positive integers L, M, define the element $H(L,M) \in G(\mathbb{A})$ by

$$H(L,M)_\infty = 1, \qquad H(L,M)_p = \begin{pmatrix} LM^2 & & & \\ & LM & & \\ & & 1 & \\ & & & M \end{pmatrix}$$

for each prime $p < \infty$. Note that $H(1,1) = 1$.

For any symmetric matrix T, we let $T^{L,M}$ denote the matrix

$$T^{L,M} = \begin{pmatrix} L & \\ & L \end{pmatrix}\begin{pmatrix} M & \\ & 1 \end{pmatrix} T \begin{pmatrix} M & \\ & 1 \end{pmatrix}. \tag{2.2.5}$$

Define the quantity $B(L,M;\Phi_F)$ by

$$B(L,M;\Phi_F) = B_\Phi(H(L,M)) = \int_{R(\mathbb{Q})Z_G(\mathbb{A})\backslash R(\mathbb{A})} \overline{(\Lambda\otimes\theta)(r)}\Phi_F(rH(L,M))dr. \tag{2.2.6}$$

The next proposition proves an important relation which expresses the quantity $B(L,M;\Phi_F)$ in terms of Fourier coefficients. The proof is fairly routine, but to our best knowledge it has not appeared in print before.

Proposition 2.1. *Let $F \in \mathcal{S}_k$ have the Fourier expansion*

$$F(Z) = \sum_{T>0} a(F,T)e(\mathrm{Tr}(TZ)).$$

Then we have

$$B(L,M;\Phi_F) = r \cdot e^{-2\pi\mathrm{Tr}(S)}(LM)^{-k}\frac{1}{|\,\mathrm{Cl}_d(M)|}\sum_{c\in\mathrm{Cl}_d(M)} \overline{\Lambda(c)}a(F,S_c^{L,M}).$$

where r is a non-zero constant depending only on the normalization used for the Haar measure on R.

Proof. For the purpose of this proof, we shorten $H(L,M)$ to H whenever convenient. Note that

$$B(L,M;\Phi_F) = B(1,1;\Phi_F^{L,M})$$

where the automorphic form $\Phi_F^{L,M}$ is given by

$$\Phi_F^{L,M}(g) = \Phi_F(gH).$$

Define

$$H_\infty = \begin{pmatrix} LM^2 & & & \\ & LM & & \\ & & 1 & \\ & & & M \end{pmatrix} \in G(\mathbb{R})^+,$$

and define the Siegel modular form

$$F'(Z) = (LM)^{-k} F(H_\infty^{-1} Z), \tag{2.2.7}$$

which is in $\mathcal{S}_k(\Gamma^*(LM^2))$, as one can easily show.

Let $\Phi_{F'}$ be the adélization of F' as defined by (2.2.1). We claim that $\Phi_F^{L,M} = \Phi_{F'}$. To see this, since both functions are right invariant under the group

$$\prod_{p \nmid LM^2} K_p \prod_{p \mid LM^2} K_p^{LM^2},$$

it is enough to show that $\Phi_F^{L,M}(g_\infty) = \Phi_{F'}(g_\infty)$ for $g_\infty \in G(\mathbb{R})^+$. This is shown by the following computation:

$$\begin{aligned}
\Phi_F^{L,M}(g_\infty) &= \Phi_F(g_\infty H) \\
&= \Phi_F(H_\infty^{-1} g_\infty) \\
&= \mu_2(H_\infty^{-1} g_\infty)^k \det(J(H_\infty^{-1} g_\infty, iI_2))^{-k} F(H_\infty^{-1} g_\infty(i)) \\
&= (LM)^{-k} \mu_2(g_\infty)^k \det(J(g_\infty, iI_2))^{-k} F(H_\infty^{-1} g_\infty(i)) \\
&= \Phi_{F'}(g_\infty).
\end{aligned}$$

Hence we are left with the problem of evaluating $B(1,1;\Phi_{F'})$. Note that $\Phi_{F'}$ is right invariant under $K_p^{(0)}(M)$ (where we think of GL_2 as a subgroup of GSp_4 via (2.1.3)). Using (2.2.4), the same arguments as in [42, Prop. 2.8.5], give us

$$B(1,1;\Phi_{F'}) = e^{-2\pi \mathrm{Tr}(S)} \frac{1}{|\mathrm{Cl}_d(M)|} \sum_{c \in \mathrm{Cl}_d(M)} \overline{\Lambda(c)} a(F', S_c)$$

for a suitably normalized Haar measure, where $a(F',T)$ denotes the Fourier coefficients of F'. Using (2.2.7), one can easily check that

$$a(F', S_c) = (LM)^{-k} a(F, S_c^{L,M}),$$

and this completes the proof. □

Remark 2.2. In an earlier preprint version of this paper, we had claimed such a result with a sum over c in Cl_d instead of $\mathrm{Cl}_d(M)$. This was incorrect (when $M \neq 1$); the mistake in the proof was to assume that $\Phi_{F'}$ is invariant under the bigger subgroup $\mathrm{GL}_2(\mathbb{Z}_p)$ when arguing as in [42, Prop. 2.8.5].

Remark 2.3. The above result is one of the three crucial ingredients that are required for the proof of the asymptotic Petersson-type formula (Proposition 3.6) which forms the technical heart of this paper. The other two ingredients are Sugano's formula (Theorem 2.5) and the asymptotic orthogonality for Poincare series (Proposition 3.3).

2.3. **Local Bessel models and Sugano's formula.** Let $\pi = \otimes_v \pi_v$ be an irreducible automorphic cuspidal representation of $G(\mathbb{A})$ with trivial central character and V_π be its space of automorphic forms. We assume that π is unramified at all finite places. Let S be a positive definite, symmetric, semi-integral matrix such that $-d = -4\det(S)$ is the discriminant of the imaginary quadratic field $\mathcal{K} = \mathbb{Q}(\sqrt{-d})$. Let ψ, Λ be defined as in the previous Section. Define the groups R, T, U as before and the Bessel function B_Φ on $G(\mathbb{A})$ as in (2.1.4), for a function $\Phi = \prod_v \Phi_v$ which is a pure tensor in π.

For a finite prime p, we use $\left(\frac{\mathcal{K}}{p}\right)$ to denote the Legendre symbol; thus $\left(\frac{\mathcal{K}}{p}\right)$ equals -1, 0 or 1 depending on whether the prime is inert, ramified or split in $\mathcal{K}$. In the latter two cases, we use $p_{\mathcal{K}}$ to denote an element of $\mathcal{K}_p = \mathcal{K} \otimes_{\mathbb{Q}} \mathbb{Q}_p$ such that $N_{\mathcal{K}/\mathbb{Q}}(p_{\mathcal{K}}) \in p\mathbb{Z}_p^\times$.

Outside $v = \infty$, the local representations are unramified spherical principal series. Therefore, by the uniqueness of the Bessel model for G, due to Novodvorsky and Piatetski-Shapiro [35], we have

$$B_\Phi(g) = B_\Phi(g_\infty) \prod_p B_p(g_p) \tag{2.3.1}$$

where B_p is a local Bessel function on $G(\mathbb{Q}_p)$, the definition of which we will now recall.

Remark 2.4. If the global Bessel space is zero, then both sides of (2.3.1) are zero. In particular, (2.3.1) remains valid regardless of whether our choice of S and Λ ensures a non-zero Bessel model.

To describe the local Bessel function B_p for a prime p, let $\mathcal{B}$ be the space of locally constant functions φ on $G(\mathbb{Q}_p)$ satisfying

$$\varphi(tuh) = \Lambda_p(t)\theta_p(u)\varphi(h), \text{ for } t \in T(\mathbb{Q}_p), u \in U(\mathbb{Q}_p), h \in G(\mathbb{Q}_p).$$

Then by Novodvorsky and Piatetski-Shapiro [35], there exists a unique subspace $\mathcal{B}(\pi_p)$ of $\mathcal{B}$ such that the right regular representation of $G(\mathbb{Q}_p)$ on $\mathcal{B}(\pi_p)$ is isomorphic to π_p. Let B_p be the unique K_p-fixed vector in $\mathcal{B}(\pi_p)$ such that $B_p(1) = 1$. Therefore we have

$$B_p(tuhk) = \Lambda_p(t)\theta_p(u)B_p(h), \tag{2.3.2}$$

for $t \in T(\mathbb{Q}_p), u \in U(\mathbb{Q}_p), h \in G(\mathbb{Q}_p), k \in K_p$.

Let $h_p(l, m) \in G(\mathbb{Q}_p)$ be the matrix defined as follows:

$$h_p(l,m) := \begin{pmatrix} p^{l+2m} & & & \\ & p^{l+m} & & \\ & & 1 & \\ & & & p^m \end{pmatrix}.$$

As explained in [15], the local Bessel function B_p is completely determined by its values on $h_p(l, m)$. An explicit formula for $B_p(h_p(l, m))$ in terms of the Satake parameters is stated in [6]. This formula can be neatly encapsulated in a generating function, due to Sugano [48], which we now explain.

Because π_p is spherical, as recalled earlier, it is the unramified constituent of a representation $\chi_1 \times \chi_2 \rtimes \sigma$ induced from a character of the Borel subgroup associated to unramified characters χ_1, χ_2, σ of $\mathbb{Q}_p^\times$, and because it has trivial central character (since π does) we have $\chi_1\chi_2\sigma^2 = 1$. Let us put $(a_p, b_p) = (\sigma(p), \sigma(p)\chi_1(p)) \in Y_p$, and as in the definition of the measure μ_p, let

$$\lambda_p = \sum_{\substack{x \in \mathcal{K}_p^\times/\mathbb{Z}_{\mathcal{K},p}^\times \\ N(x)=p}} \Lambda_p(x),$$

where the number of terms in the sum is $1 + \left(\frac{\mathcal{K}}{p}\right)$.

The next Theorem is due to Sugano [48, p. 544] (the reader may also consult [15, (3.6)]).

Theorem 2.5 (Sugano)**.** *Let π be an unramified spherical principal series representation of $G(\mathbb{Q}_p)$ with associated local parameters $(a,b)\in Y_p$ and spherical Bessel function B_p as above. Then we have*

$$B_p(h_p(l,m)) = p^{-2m-\frac{3l}{2}}U_p^{l,m}(a,b) \tag{2.3.3}$$

where for each $l,m\geqslant 0$, the function

$$U_p^{l,m}(a,b) = U_p^{l,m}(a,b;\mathcal{K}_p,\Lambda_p)$$

is a Laurent polynomial in $\mathbb{C}[a,b,a^{-1},b^{-1}]$, invariant under the action of the Weyl group (1.2.2), which depends only on p, $\left(\frac{\mathcal{K}}{p}\right)$ and λ_p.

More precisely, the generating function

$$C_p(X,Y) = C_p(X,Y;a,b) = \sum_{l\geq 0}\sum_{m\geq 0} U_p^{l,m}(a,b)X^mY^l \tag{2.3.4}$$

is a rational function given by

$$C_p(X,Y) = \frac{H_p(X,Y)}{P_p(X)Q_p(Y)} \tag{2.3.5}$$

where

$$\begin{aligned}
P_p(X) &= (1-abX)(1-ab^{-1}X)(1-a^{-1}bX)(1-a^{-1}b^{-1}X),\\
Q_p(Y) &= (1-aY)(1-bY)(1-a^{-1}Y)(1-b^{-1}Y),\\
H_p(X,Y) &= (1+XY^2)\big(M_1(X)(1+X)+p^{-1/2}\lambda_p\sigma(a,b)X^2\big)\\
&\quad - XY\big(\sigma(a,b)M_1(X)-p^{-1/2}\lambda_pM_2(X)\big) - p^{-1/2}\lambda_pP_p(X)Y + p^{-1}\left(\frac{\mathcal{K}}{p}\right)P_p(X)Y^2,
\end{aligned}$$

in terms of auxiliary polynomials given by

$$\begin{aligned}
&\sigma(a,b) = a+b+a^{-1}+b^{-1}, \qquad \tau(a,b) = 1+ab+ab^{-1}+a^{-1}b+a^{-1}b^{-1},\\
&M_1(X) = 1-\left(p-\left(\frac{\mathcal{K}}{p}\right)\right)^{-1}\left(p^{1/2}\lambda_p\sigma(a,b)-\left(\frac{\mathcal{K}}{p}\right)(\tau(a,b)-1)-\lambda_p^2\right)X - p^{-1}\left(\frac{\mathcal{K}}{p}\right)X^2,\\
&M_2(X) = 1-\tau(a,b)X-\tau(a,b)X^2+X^3.
\end{aligned}$$

Remark 2.6. For instance, we note that

$$U_p^{0,0}(a,b) = 1$$

and that

$$U_p^{1,0}(a,b) = \sigma(a,b)-p^{-1/2}\lambda_p = a+b+a^{-1}+b^{-1}-p^{-1/2}\lambda_p. \tag{2.3.6}$$

We also note that taking $X=0$ leads to the simple formula

$$\sum_{l\geq 0}U_p^{l,0}(a,b)Y^l = \frac{1-p^{-\frac{1}{2}}\lambda_pY+p^{-1}\left(\frac{\mathcal{K}}{p}\right)Y^2}{Q_p(Y)}, \tag{2.3.7}$$

which we will use later on. The formula for $Y=0$ is more complicated, but we note (also for further reference) that it implies the formula

$$U_p^{0,1}(a,b) = \tau(a,b)-\left(p-\left(\frac{\mathcal{K}}{p}\right)\right)^{-1}\left(p^{1/2}\lambda_p\sigma(a,b)-\left(\frac{\mathcal{K}}{p}\right)(\tau(a,b)-1)-\lambda_p^2\right). \tag{2.3.8}$$

As was the case for the definition of the measures μ_p, we have written down a concrete formula. These are not very enlightening by themselves (though we will use the special cases above), and the intrinsic point of view is that of *Macdonald polynomials* [32] associated to a root system. In particular, this leads to the following important fact:

Proposition 2.7. *Let (d, Λ) be as before. For any fixed prime p, the functions*

$$(a, b) \mapsto U_p^{l,m}(a, b)$$

where l, m run over non-negative integers, form a basis of the space of Laurent polynomials in $\mathbb{C}[a, b, a^{-1}, b^{-1}]$ which are invariant under the group W generated by the three transformations above.

Moreover, any such Laurent polynomial φ which has total degree d as polynomial in the variables $(a + a^{-1}, b + b^{-1})$ can be represented as a combination of polynomials $U_p^{l,m}(a, b)$ with $l + 2m \leqslant d$.

Proof. Because of Theorem 2.5, we can work with the Bessel functions $B_p(h_p(l, m))$ instead. But as shown in detail in [17, §3], these unramified Bessel functions are (specializations of) Macdonald polynomials associated to the root system of G, in the sense of [32]. By the theory of Macdonald polynomials, these unramified Bessel functions form a basis for the Laurent polynomials in two variables that are symmetric under the action of the Weyl group W.

The last statement, concerning the $U_p^{l,m}$ occurring in the decomposition of φ of bidegree (d, d), can be easily proved by induction from the corresponding fact for the coefficients (say $\tilde{U}_{l,m}(a, b)$) of the simpler generating series

$$\frac{1}{P_p(X)Q_p(Y)} = \sum_{l,m \geqslant 0} \tilde{U}_{l,m}(a, b) X^m Y^l,$$

for which the stated property is quite clear. (It is also a standard fact about the characters of representations of $\mathrm{USp}(4, \mathbb{C})$, since $\sigma(a, b)$ and $\tau(a, b)$ are the characters of the two fundamental representations acting on a maximal torus.) □

Lemma 2.8. *Let (d, Λ) be as before. Let $\boldsymbol{S}$ be a finite set of primes and (l_p), (m_p) be tuples of non-negative integers, indexed by $\boldsymbol{S}$. There exists an absolute constant $C \geqslant 0$ such that for every $(x_p)_{p \in \boldsymbol{S}} = (a_p, b_p) \in X^{\boldsymbol{S}}$, i.e., parameters of tempered representations, we have*

$$\Big| \prod_{p \in \boldsymbol{S}} U_p^{l_p, m_p}(a_p, b_p) \Big| \leqslant C^{|\boldsymbol{S}|} \prod_{p \in \boldsymbol{S}} (l_p + 3)^3 (m_p + 3)^3.$$

Proof. It is enough to prove this when $\boldsymbol{S} = \{p\}$ is a single prime, and $l_p = l$, $m_p = m \geqslant 0$. Then by Sugano's formula, the polynomial $U_p^{l,m}(a, b)$ is a linear combination of at most 14 polynomials of the type arising in the expansion of the denominator only, i.e., of

$$\frac{1}{P_p(X)Q_p(Y)},$$

and moreover the coefficients in this combination are absolutely bounded as p varies (they are either constants or involve quantities like $p^{-1/2}$).

Expanding in geometric series and using $|a| = 1$, $|b| = 1$, the coefficient of $X^m Y^l$ in the expansion of the denominator is a product of the coefficient of X^m and that of Y^l; each of them is a sum, with coefficient $+1$, of $\leqslant (m + 3)^3$ (resp. $(l + 3)^3$) terms of size $\leqslant 1$. The result follows from this. □

Remark 2.9. Sugano's formula explicitly computes the Bessel function in terms of Satake parameters in the case of an unramified representation. The other case where an explicit formula for the Bessel function at a finite place is known is when π_p is Steinberg, see [41], [37].

2.4. **The key relation.** We consider now Siegel modular forms again. Let

$$F(Z) = \sum_{T>0} a(F,T)e(\mathrm{Tr}(TZ)) \in \mathcal{S}_k$$

be an eigenfunction for all the Hecke operators. Define its adélization $\Phi_F(g)$ by (2.2.1). This is a function on $G(\mathbb{Q})\backslash G(\mathbb{A})$ and we may consider the representation of $G(\mathbb{A})$ generated by it under the right-regular action. Because we do not have strong multiplicity one for G, we can only say that this representation is a *multiple* of an irreducible representation π_F. However, the unicity of π_F, as an isomorphism class of representations of $G(\mathbb{A})$, is enough for our purposes.[5]

We can factor $\pi_F = \otimes \pi_v(F)$ where the local representations π_v are given by:

$$\pi_v(F) = \begin{cases} \text{holomorphic discrete series} & \text{if } v = \infty, \\ \text{unramified spherical principal series} & \text{if } v \text{ is finite,} \end{cases}$$

and we denote by $(a_p(F), b_p(F)) \in Y_p$, the local parameters corresponding to the local representation $\pi_p(F)$ at a finite place.

Let once more d be a positive integer such that $-d$ is a fundamental discriminant and define S as in (2.2.2). Choose an ideal class character Λ of $\mathcal{K}$. Let the additive character ψ, the groups R, T, U and the matrices S_c, $S_{c'}^{L,M}$ be defined as in Section 2.2. For positive integers L, M, define $B(L, M; \Phi_F)$ by (2.2.6). Then, by the uniqueness of the Bessel model (i.e., (2.3.1), we have

$$B(L,M;\Phi_F) = B(1,1;\Phi_F)\prod_p B_p(h_p(l_p,m_p)) = B(1,1;\Phi_F)\prod_{p|LM} B_p(h_p(l_p,m_p)),$$

where l_p and m_p are the p-adic valuations of L and M respectively. Now, using Sugano's formula (2.3.3) and twice Proposition 2.1 – which has the effect of canceling the constant $r \neq 0$ that appears in the latter –, we deduce:

Theorem 2.10. *Let (d, Λ) be as before, let p be prime and let $U_p^{l,m}(a,b)$ be the functions defined in Theorem 2.5. For any $F \in \mathcal{S}_{2k}^*$ and integers L, $M \geqslant 1$, we have*

$$\frac{|\,\mathrm{Cl}_d\,|}{|\,\mathrm{Cl}_d(M)|}\sum_{c'\in \mathrm{Cl}_d(M)} \overline{\Lambda(c')}a(F,S_{c'}^{L,M}) = L^{k-\frac{3}{2}}M^{k-2}\sum_{c\in\mathrm{Cl}_d}\overline{\Lambda(c)}a(F,S_c)\prod_{p|LM}U_p^{l_p,m_p}(a_p(F),b_p(F)),$$

where l_p and m_p are the p-adic valuations of L and M respectively.

The point of this key result is that it allows us to study functions of the Satake parameters of π_F using Fourier coefficients of F, although there is no direct identification of Hecke eigenvalues with Fourier coefficients.

Remark 2.11. This relation holds for every Λ, but we can not remove the sum over c by Fourier inversion because the functions $U_p^{l_p,m_p}$ *depend on* Λ.

3. Poincaré series, Petersson formula and orthogonality

The relation given by Theorem 2.10 between Fourier coefficients of F on the one hand, and functions of the spectral Satake parameters of π_F on the other, will enable us to deduce equidistribution results for Satake parameters from asymptotics for Fourier coefficients. For this, we need a way to understand averages of Fourier coefficients of Siegel forms F in a suitable family; this will be provided by a variant of the classical Petersson formula. In order to prove the latter, we follow the standard approach: we consider Poincaré series and study their Fourier coefficients.

[5] Added in proof: in a recent preprint, Narita, Pitale and Schmidt show that Φ_F does indeed generate an irreducible representation.

3.1. **Poincaré series and the Petersson formula.** Given a symmetric semi-integral positive-definite matrix Q of size two, the Q-th Poincaré series of weight k, denoted $P_{k,Q}$, is defined as follows:

$$P_{k,Q}(Z) = \sum_{\gamma\in\Delta\backslash\Gamma} \det(J(\gamma, Z))^{-k} e(\mathrm{Tr}(Q\gamma(Z)))$$

where Δ is the subgroup of Γ consisting of matrices of the form $\begin{pmatrix} 1 & U \\ 0 & 1 \end{pmatrix}$, with U symmetric.

It is known that $P_{k,Q}$ is absolutely and locally uniformly convergent for $k \geqslant 6$, and defines an element of $\mathcal{S}_k$ (as first proved by Maass). In fact, any Siegel cusp form $F \in \mathcal{S}_k$ is a linear combination of various $P_{k,Q}$ (with Q varying). This follows from the basic property of Poincaré series: they represent, in terms of the Petersson inner product, the linear forms on $\mathcal{S}_k$ given by Fourier coefficients. Precisely, for $F \in \mathcal{S}_k$ with Fourier expansion

$$F(Z) = \sum_{T>0} a(F, T) e(\mathrm{Tr}(TZ)),$$

we have the crucial identity

$$\langle F, P_{k,T}\rangle = 8c_k(\det T)^{-k+3/2} a(F, T), \tag{3.1.1}$$

where

$$c_k = \frac{\sqrt{\pi}}{4}(4\pi)^{3-2k}\Gamma(k - \tfrac{3}{2})\Gamma(k - 2). \tag{3.1.2}$$

(see [28] or [27, p. 90] for instance).

We are interested in the limiting behavior of $a(k; c, c', L, M)$ as $k \to +\infty$. The following qualitative result was proved in [31]:

Proposition 3.1 (Asymptotic orthogonality, qualitative version)**.** *For L, $M \geqslant 1$, $c \in \mathrm{Cl}_d$ and $c' \in \mathrm{Cl}_d(M)$, let*

$$a(k; c, c', L, M) = a(P_{k,S_c}, S_{c'}^{L,M})$$

denote the $S_{c'}^{L,M}$-th Fourier coefficient of the Poincaré series P_{k,S_c}. Then, we have

$$a(k; c, c', L, M) \to |\operatorname{Aut}(c)| \cdot \delta(c, c'; L; M)$$

as $k \to +\infty$ over the even integers. Here

$$\delta(c, c'; L; M) = \begin{cases} 1 & \text{if } L = 1, M = 1 \text{ and } c \text{ is } \mathrm{GL}(2,\mathbb{Z})\text{-equivalent to } c', \\ 0 & \text{otherwise,} \end{cases}$$

and $|\operatorname{Aut}(c)|$ is the finite group of integral points in the orthogonal group $O(T)$ of the quadratic form defined by c.

Remark 3.2. Since this will be a subtle point later on, we emphasize that $\delta(c, c'; L, M) = 1$ when c and c' are invariant under $\mathrm{GL}(2,\mathbb{Z})$, not under $\mathrm{SL}(2,\mathbb{Z})$.

This is sufficient for some basic applications, but (for example) to handle the low-lying zeros, we require a quantitative version. We prove the following:

Proposition 3.3 (Asymptotic orthogonality, quantitative version)**.** *With notation as above, we have*

$$a(k; c, c', L, M) = |\operatorname{Aut}(c)| \cdot \delta(c, c'; L; M) + L^{k-3/2}M^{k-2}A(k; c, c', L, M)$$

where

$$A(k; c, c', L, M) \ll L^{1+\varepsilon}M^{3/2+\varepsilon}k^{-2/3}$$

for any $\varepsilon > 0$, the implied constant depending only on ε and d.

In the proof, for conciseness, we will write $|A|$ for the determinant of a matrix. The basic framework of the argument is contained in the work of Kitaoka [26], who proved an estimate for the Fourier coefficients $a(P_{k,Q}, T)$ of Poincaré series for fixed Q and $k \geqslant 6$, in terms of the determinant $\det(T)$ (and deduced from this an estimate for Fourier coefficients of arbitrary Siegel cusp forms in $\mathcal{S}_k$, since the space is spanned by Poincaré series).

However, Kitaoka considered k to be fixed; our goal is to have a uniform estimate in terms of L, M and k, and this requires more detailed arguments.

In particular, we will require the following quite standard asymptotics for Bessel functions:

$$J_k(x) \ll \frac{x^k}{\Gamma(k+1)}, \quad \text{if } k \geqslant 1,\ 0 \leqslant x \ll \sqrt{k+1}, \tag{3.1.3}$$

$$J_k(x) \ll \min(1, xk^{-1})k^{-1/3}, \quad \text{if } k \geqslant 1,\ x \geqslant 1, \tag{3.1.4}$$

$$J_k(x) \ll \frac{2^k}{\sqrt{x}}, \quad \text{if } k \geqslant 1,\ x > 0, \tag{3.1.5}$$

where the implied constants are absolute (the first inequality follows from the Taylor expansion of $J_k(z)$ at $z = 0$, the second is [24, (2.11)], and the third, which is very rough, by combining $|J_k(x)| \leqslant 1$ when $x \leqslant 2k$, and, e.g., [24, (2.11')] for $x \geqslant 2k$).

Proof of Proposition 3.3. Let

$$T = S_{c'}^{L,M}, \qquad Q = S_c,$$

so that we must consider the T-th Fourier coefficients of $P_{k,Q}$ (this notation, which clashes a bit with the earlier one for the torus T, is chosen to be the same as that in [26], in order to facilitate references). Before starting, we recall that since we consider d to be fixed, so is the number of ideal classes, and hence Q varies in a fixed finite set, and may therefore be considered to be fixed. Also note that

$$\det(T) = dL^2M^2/4, \tag{3.1.6}$$

and we seek estimates involving $\det(T)$. Thus, compared with Kitaoka, the main difference is the dependency on k, which we must keep track of. In particular, we modify and sharpen Kitaoka's method, so that any implicit constants that appear depend only on d.

Because we think of d as fixed, throughout the proof we drop the subscript d from the symbols $\ll, \gg, \asymp$. The reader should not be misled into thinking that the implied constants are independent of d.

Since the proof is rather technical, the reader is encouraged to assume first that $d = 4$, $M = 1$ (so that there is a single class $c = c'$, and moreover $Q = S_c = S_{c'} = 1$) and also[6] by (2.2.5), T is a simple diagonal matrix

$$T = \begin{pmatrix} L & 0 \\ 0 & L \end{pmatrix}.$$

In principle, we now follow the formula for $a(P_{k,Q}, T)$ which is implicit in [26]. Given a system of representatives $\mathfrak{h}$ of $\Gamma_1(\infty)\backslash\Gamma/\Gamma_1(\infty)$, Kitaoka defines certain incomplete Poincaré series $H_k(M, Z)$ such that

$$P_{k,Q}(Z) = \sum_{M\in\mathfrak{h}} H_k(M, Z).$$

Denoting the T-th Fourier coefficient of $H_k(M, Z)$ by $h_k(M, T)$, we have

$$a(k; c, c', L, M) = a(P_{k,Q}, T) = \sum_{M\in\mathfrak{h}} h_k(M, T).$$

[6] This is the only case needed in Theorem 1.1 for averaging the spin L-function; however, this is *not* sufficient for Theorem 1.2, although the latter is also concerned only with the spinor L-function.

We write

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A, B, C and D are matrices in $M(2,\mathbb{Z})$, and we now divide the sum above depending on the rank of C. We denote the component corresponding to rank i by R_i, so that

$$a(P_{k,Q},T) = R_0 + R_1 + R_2.$$

Step 1 (rank 0). First of all, we consider R_0. By [26, p. 160] (or direct check), we have

$$R_0 = \sum_{\substack{U\in \mathrm{GL}(2,\mathbb{Z}) \\ UT^tU=Q}} 1,$$

which is 0 unless T is $\mathrm{GL}(2,\mathbb{Z})$-equivalent to Q, in which case it is equal to $|\operatorname{Aut}(T)| = |O(T,\mathbb{Z})|$ (where T is viewed as defining a quadratic form and $O(T)$ is the corresponding orthogonal group). In our case, looking at the determinant we find that $R_0 = 0$ unless $L = M = 1$, and then it is also 0 except if c is $\mathrm{GL}(2,\mathbb{Z})$-equivalent to c', and is then $|\operatorname{Aut}(c)|$. In other words, we have

$$R_0 = |\operatorname{Aut}(c)| \cdot \delta(c,c';L;M),$$

and hence, by definition, the remainder is therefore

$$R_1 + R_2 = L^{k-3/2}M^{k-2}A(k;c,c',L,M), \tag{3.1.7}$$

and — having isolated our main term — we must now estimate the two remaining ones.

Step 2 (rank 1). Following the computations in Kitaoka (specifically, Lemma 4, p. 159, Lemma 1, p. 160, and up to line 2 on p. 163 in [26]), but keeping track of the dependency on k by keeping the factor $Q^{3/4-k/2}$ (which Kitaoka considers as part of his implied constant), we find that

$$|R_1| \ll_\varepsilon \sum_{c,m\geqslant 1} |T|^{k/2-3/4}|Q|^{3/4-k/2}A(m,T)m^{-1/2+\varepsilon}(m,c)^{1/2}\Big|J_{k-3/2}\Big(4\pi\frac{\sqrt{|T||Q|}}{mc}\Big)\Big| \tag{3.1.8}$$

where $A(m,T)$ is the number of times T, seen as a quadratic form, represents m.

Now recall that $|Q| = d/4$ and $|T| = L^2M^2(d/4)$, and observe that $A(m,T) = 0$ unless L divides m and $A(m,T) = A(m/L,S_c)$ whenever L divides m. It follows that

$$A(m,T) \ll_\varepsilon (m/L)^\varepsilon$$

for any $\varepsilon > 0$. Using (3.1.8), we get by a very rough estimate that

$$\begin{aligned} |R_1| &\ll_\varepsilon (LM)^{k-\frac{3}{2}} \sum_{\substack{c,m\geqslant 1 \\ L|m}} m^{-1/2+\varepsilon}(m/L)^\varepsilon(m,c)^{1/2}\Big|J_{k-3/2}\Big(\pi\frac{LMd}{mc}\Big)\Big| \\ &\ll (LM)^{k-\frac{3}{2}}L^\varepsilon \sum_{c,m_1\geqslant 1} m_1^{-1/2+\varepsilon}(m_1,c)^{1/2}\Big|J_{k-3/2}\Big(\pi\frac{Md}{m_1c}\Big)\Big|. \end{aligned}$$

Now we define

$$\begin{aligned} \mathcal{R}_1 &= \sum_{c,m\geqslant 1} m^{-1/2+\varepsilon}(m,c)^{1/2}\Big|J_{k-3/2}\Big(\pi\frac{Md}{mc}\Big)\Big| \\ &= \mathcal{R}_{11} + \mathcal{R}_{12} + \mathcal{R}_{13}, \end{aligned}$$

where $\mathcal{R}_{1i}$ corresponds to the sums restricted to

$$\begin{cases} mc > \pi Md, & \text{if } i = 1 \\ \pi Mdk^{-1/2} \leqslant mc \leqslant \pi MD, & \text{if } i = 2 \\ mc \leqslant \pi Mdk^{-1/2} & \text{if } i = 3. \end{cases}$$

For $i = 1$, the argument of the Bessel function is $\leqslant 1$ and by (3.1.3), we find

$$\mathcal{R}_{11} \ll \frac{1}{\Gamma(k - 3/2)} \sum_{mc > \pi Md} m^{-1/2+\varepsilon}(m, c)^{1/2} \Big(\frac{\pi Md}{mc}\Big)^{k-3/2}.$$

We can replace the exponent $k - 3/2$ in the sum with any exponent $1 + \delta$, for small $0 < \delta \leqslant 1$ (since $k \geqslant 6$ anyway), and then we can remove the summation condition, observing that the double series is then convergent, and obtain

$$\mathcal{R}_{11} \ll M^{1+\varepsilon} k^{-E},$$

(taking δ small enough in terms of ε) for any $E \geqslant 1$ and $\varepsilon > 0$, where the implied constant depends on E, ε and d.

For $i = 2$, we use (3.1.3) and find that

$$\mathcal{R}_{12} \ll \frac{k^{k/2-3/4}}{\Gamma(k - 3/2)} \sum_{cm \leqslant \pi Md} m^{-1/2+\varepsilon}(m, c)^{1/2} \ll M^{1+\varepsilon} k^{-E}$$

for $E \geqslant 1$ and $\varepsilon > 0$ again (by summing over c first and then over m, and by Stirling's formula).

Finally, using now (3.1.4), we have

$$\mathcal{R}_{13} \ll_\varepsilon k^{-1/3} \sum_{cm < \pi Mdk^{-1/2}} m^{-1/2+\varepsilon}(m, c)^{1/2} \ll k^{-1/3-1/2} M^{1+\varepsilon} = k^{-5/6} M^{1+\varepsilon},$$

(summing as for $\mathcal{R}_{12}$).

It follows that

$$\mathcal{R}_1 = \mathcal{R}_{11} + \mathcal{R}_{12} + \mathcal{R}_{13} \ll M^{1+\varepsilon} k^{-\frac{5}{6}}$$

for any $\varepsilon > 0$, and so the contribution of rank 1 is bounded by

$$|R_1| \ll (LM)^{k-\frac{3}{2}} L^\varepsilon M^{1+\varepsilon} k^{-\frac{5}{6}} \tag{3.1.9}$$

for any $\varepsilon > 0$, where the implied constant depends only on d and ε.

Step 3 (rank 2). Finally, we deal with the R_2 term, which is much more involved. The relevant set of matrices M is given by

$$M \in \left\{ \begin{pmatrix} \star & \star \\ C & D \end{pmatrix} \right\} \subset \mathrm{Sp}(4, \mathbb{Z})$$

where $|C| \neq 0$ and D is arbitrary modulo C. Denoting

$$M_2^*(\mathbb{Z}) = \{C \in M_2(\mathbb{Z}) \mid |C| \neq 0\},$$

we have then

$$R_2 = \sum_{C \in M_2^*(\mathbb{Z})} \sum_{D \,(\mathrm{mod}\, C)} h_k(M, T).$$

The inner sum was computed by Kitaoka [26, p. 165, 166]. To state the formula, let

$$P = P(C) := TQ[{}^tC^{-1}] = T({}^tC^{-1})QC^{-1},$$

and let

$$0 < s_1 \leqslant s_2 \tag{3.1.10}$$

be such that s_1^2, s_2^2 are the eigenvalues of the positive definite matrix P. Then Kitaoka proved that

$$\sum_{D\,(\mathrm{mod}\,C)} h_k(M,T) = \frac{1}{2\pi^4}\Big(\frac{|T|}{|Q|}\Big)^{k/2-3/4}|C|^{-3/2}K(Q,T;C)\mathcal{J}_k(P(C)), \tag{3.1.11}$$

where $K(Q,T;C)$ is a type of matrix-argument Kloosterman sum (see [26, §1, p. 150] for the precise definition, which we do not need here), and[7]

$$\mathcal{J}_k(P) = \int_0^{\pi/2} J_{k-3/2}(4\pi s_1 \sin\theta)J_{k-3/2}(4\pi s_2\sin\theta)\sin\theta d\theta.$$

We note that

$$|P| = |T||Q||C|^{-2} = (d/4)^2L^2M^2|C|^{-2}.$$

In order to exploit this formula (3.1.11), we must handle the sum over C. For this purpose, we use a parametrization of $M_2^*(\mathbb{Z})$ in terms of principal divisors: any $C \in M_2^*(\mathbb{Z})$ can be written uniquely

$$C = U^{-1}\begin{pmatrix} c_1 & 0 \\ 0 & c_2\end{pmatrix}V^{-1} \tag{3.1.12}$$

where

$$1 \leqslant c_1, \quad c_1 \mid c_2, \quad U \in \mathrm{GL}(2,\mathbb{Z}) \text{ and } V \in \mathrm{SL}(2,\mathbb{Z})/\Gamma^0(c_2/c_1),$$

where $\Gamma^0(n)$ denotes the congruence subgroup of $\mathrm{SL}(2,\mathbb{Z})$ (conjugate to $\Gamma_0(n)$) consisting of matrices

$$\begin{pmatrix} a & b \\ c & d\end{pmatrix}$$

with $n \mid b$. Note that there is a bijection

$$\mathrm{SL}(2,\mathbb{Z})/\Gamma^0(n) \simeq \mathbb{P}^1(\mathbb{Z}/nZ),$$

(this is denoted $S(n)$ in [26]) and in particular

$$|\mathrm{SL}(2,\mathbb{Z})/\Gamma^0(n)| = n\prod_{p\mid n}(1+p^{-1}) \ll n^{1+\varepsilon} \tag{3.1.13}$$

for any $\varepsilon > 0$.

We will first consider matrices where the last three parameters $\boldsymbol{c} = (c_1, c_2, V)$ are fixed, subject to the conditions above. The set of such triples is denoted $\mathcal{V}$, and for each $\boldsymbol{c} \in \mathcal{V}$, we fix (as we can) a matrix $U_1 \in \mathrm{GL}(2,\mathbb{Z})$ such that the matrix

$$A(\boldsymbol{c}) = A := T\Big[V\begin{pmatrix} c_1 & \\ & c_2\end{pmatrix}^{-1}U_1\Big]$$

is Minkowski-reduced. This matrix is conjugate to a diagonal matrix $H = H(\boldsymbol{c})$ of the form

$$H = \begin{pmatrix} a & \\ & c\end{pmatrix}$$

with $a \leq c$. Computing determinants and using the fact that A is Minkowski-reduced, we note also that we have

$$(d/4)\frac{L^2M^2}{c_1^2c_2^2} = ac \asymp s_1^2s_2^2 = (d/4)^2\frac{L^2M^2}{c_1^2c_2^2} \tag{3.1.14}$$

(we recall again that d is assumed to be fixed).

[7] We have made the change of variable $t = \sin(\theta)$ for convenience.

For fixed $\boldsymbol{c} \in \mathcal{V}$, the set of matrices $C \in M_2^*(\mathbb{Z})$ corresponding to $\boldsymbol{c}$ can be parameterized in the form

$$C = U^{-1}U_1^{-1}\begin{pmatrix} c_1 & \\ & c_2 \end{pmatrix} V^{-1},$$

where U varies freely over $\mathrm{GL}(2,\mathbb{Z})$ (this is a simple change of variable of the last parameter $U \in \mathrm{GL}(2,\mathbb{Z})$ in (3.1.12)). As shown in [26, p. 167], for any such C associated to $\boldsymbol{c}$, we have also

$$|P| \asymp |A|, \quad \mathrm{Tr}(P) \asymp \mathrm{Tr}(A[U]) = \mathrm{Tr}(H[U]).$$

We can now start estimating. First, for a given C parametrized by $(U, \boldsymbol{c})$, Kitaoka proved (see [26, Prop. 1]) that the Kloosterman sum satisfies

$$|K(Q,T;C)| \ll c_1^2 c_2^{1/2+\varepsilon}(c_2, T[v])^{1/2},$$

for any $\varepsilon > 0$, where v is the second column of V and the implied constant depends only on ε. Hence by (3.1.11), we obtain

$$\sum_{D \,(\mathrm{mod}\, C)} h_k(M,T) \ll (LM)^{k-3/2} c_1^{1/2} c_2^{-1+\varepsilon} (c_2, T[v])^{1/2} |\mathcal{J}_k(P(C))|.$$

In order to handle the Bessel integral $\mathcal{J}_k(P(C))$, we will partition $M_2^*(\mathbb{Z})$ in three sets $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, according to the relative sizes of the values s_1 and s_2 for the corresponding invariants $\boldsymbol{c}$. These can be determined from the size of $\mathrm{Tr}(P)$ and $|P|$; precisely, we let

$$\mathcal{C}_1 = \{C \mid \mathrm{Tr}(P) < 1\},$$
$$\mathcal{C}_2 = \{C \mid \mathrm{Tr}(P) \geqslant \max(2|P|, 1)\},$$
$$\mathcal{C}_3 = \{C \mid 1 \leqslant \mathrm{Tr}(P) < 2|P|\},$$

and we further denote by $\mathcal{C}_i(\boldsymbol{c})$ the subsets of $\mathcal{C}_i$ where C is associated with the invariants $\boldsymbol{c} = (c_1, c_2, V)$. The following lemma gives the rough size of these sets, or a weighted version that is needed below:

Lemma 3.4. *With notation as above, for any* $\boldsymbol{c} = (c_1, c_2, V)$*, we have*

$$|\mathcal{C}_1(\boldsymbol{c})| \ll (ac)^{-1/2-\varepsilon}, \tag{3.1.15}$$

$$\sum_{C \in \mathcal{C}_2(\boldsymbol{c})} |A|^{1+\delta}(\mathrm{Tr}(A[U]))^{-5/4-\delta} \ll \begin{cases} (ac)^{1/2+\delta-\varepsilon} & \text{if } ac < 1 \\ (ac)^{1/4+\varepsilon} & \text{if } ac \geqslant 1, \end{cases} \tag{3.1.16}$$

$$|\mathcal{C}_3(\boldsymbol{c})| \ll (ac)^{1/2+\varepsilon}, \tag{3.1.17}$$

for any $\varepsilon > 0$ *and* $\delta > 0$ *in the second, where the implied constants depend on* δ *and* ε.

Proof. All these are proved by Kitaoka. Precisely:

- The bound (3.1.15) comes from [26, p. 167], using the fact that in that case we have $a \ll 1$;
- The bound (3.1.16) comes from the arguments of [26, p. 168, 169] (note that in that case the summation set is infinite); to be more precise, Kitaoka argues with what amounts to taking

  $$\delta = k/2 - 7/4, \text{ so that } k/2 - 3/4 = 1 + \delta, \quad (1-k)/2 = -5/4 - \delta,$$

  but the only information required (up to [26, p. 169, line 10]) is the sign and the value of the sum of the two exponents

  $$k/2 - 3/4 + (1-k)/2 = -1/4 = 1 + \delta + (-5/4 - \delta)$$

  (this is used in [26, p. 168, line -12]). Hence Kitaoka's argument applies for $\delta > 0$.
- The bound (3.1.17) comes from [26, p. 168], using the fact that in that case we have $c \gg 1$.

□

As shown also by Kitaoka, we have the following crucial localization properties (see [26, p. 166]):

(a) If $C \in \mathcal{C}_1$, then $s_1 \leq 1$ and $s_2 \leq 1$;
(b) If $C \in \mathcal{C}_2$, then $s_1 \leq 1$ and $s_2 \gg 1$, with absolute implied constant;
(c) If $C \in \mathcal{C}_3$, then $s_1 \gg 1$ and $s_2 \gg 1$, with absolute implied constants.

Now, by breaking up the sum over C in R_2 according to the three subsets $\mathcal{C}_i$, we can write

$$|R_2| \ll R_{21} + R_{22} + R_{23}$$

where, for $i = 1, 2, 3$, and any fixed $\varepsilon > 0$, we have

$$R_{2i} \ll (LM)^{k-\frac{3}{2}} \sum_{\boldsymbol{c}\in\mathcal{V}} c_1^{\frac{1}{2}} c_2^{-1+\varepsilon} (c_2, T[v])^{1/2} \mathcal{R}_{2i}(\boldsymbol{c}),$$

for any $\varepsilon > 0$ with

$$\mathcal{R}_{2i}(\boldsymbol{c}) = \sum_{C\in\mathcal{C}_i(\boldsymbol{c})} |\mathcal{J}_k(P(C))|,$$

the implied constant depending only on d and ε.

Accordingly, we study each of R_{21}, R_{22}, R_{23} separately.

– **Estimation of R_{21}.** Since we have $s_1 \leq 1, s_2 \leq 1$, we use (3.1.3); using the superexponential growth of the Gamma function, we obtain easily

$$|\mathcal{J}_k(P(C))| \ll_\varepsilon \frac{(s_1 s_2)^{2+\delta}}{2^k}$$

for $C \in \mathcal{C}_1$ and any fixed $\delta > 0$. On the other hand, by (3.1.15), we have

$$|\mathcal{C}_1(\boldsymbol{c})| \ll (ac)^{-\frac{1}{2}-\epsilon} \ll (s_1 s_2)^{-1-2\varepsilon},$$

for any $\epsilon > 0$, and taking it small enough we obtain

$$\mathcal{R}_{21}(\boldsymbol{c}) \ll \frac{(s_1 s_2)^{1+\delta}}{2^k} \ll (LM)^{1+\delta} \frac{(c_1 c_2)^{-1-\delta}}{2^k}, \tag{3.1.18}$$

for any fixed $\delta > 0$. For fixed c_1, c_2 first, we have

$$\sum_{V\in \mathrm{SL}(2,\mathbb{Z})/\Gamma^0(c_2/c_1)} c_1^{\frac{1}{2}} c_2^{-1+\varepsilon} (c_2, T[v])^{1/2} \mathcal{R}_{21}(\boldsymbol{c}) \ll \frac{(LM)^{1+\delta}}{2^k} \sum_V c_1^{-\frac{1}{2}-\delta} c_2^{-2-\delta+\varepsilon} (c_2, T[v])^{1/2}$$

from which one deduces easily

$$\sum_{V\in \mathrm{SL}(2,\mathbb{Z})/\Gamma^0(c_2/c_1)} c_1^{\frac{1}{2}} c_2^{-1+\varepsilon} (c_2, T[v])^{1/2} \mathcal{R}_{21}(\boldsymbol{c}) \ll \frac{(LM)^{1+\delta}}{2^k} (c_1 c_2)^{-1-\delta+\varepsilon} (c_2/c_1, LM^2)^{1/2}$$

for any $\delta > 0$, possibly different than before (using (3.1.13) and [26, Prop. 2] to handle the gcd; the exponent of c_1 was worsened by 1/2 to facilitate the use of this lemma).

Writing $c_2 = nc_1$, with $n \geqslant 1$, we can finally sum over c_1 and n; the resulting series converge for $\delta > 0$ and we obtain

$$R_{21} \ll 2^{-k} (LM)^{k-\frac{3}{2}+1+\delta} \sum_{c_1,n\geqslant 1} c_1^{-2} n^{-1-\delta+\varepsilon} (n, LM^2)^{1/2},$$

and therefore by taking, e.g., $\delta = 2\varepsilon$ (and changing notation), we derive

$$R_{21} \ll (LM)^{k-\frac{3}{2}} (LM)^{1+\varepsilon} k^{-E} \tag{3.1.19}$$

for any $\varepsilon > 0$ and $E \geqslant 1$, where the implied constant depends on d, E and ε.

– **Estimation of R_{22}.** We treat the R_{22} term next. Using (3.1.3) for the Bessel function involving s_1 and (3.1.5) for the one involving s_2, and using the fact that

$$\mathrm{Tr}(A[U]) \asymp \mathrm{Tr}(P) = s_1^2 + s_2^2 \asymp s_2^2$$

for this term, it is easy to check that

$$|\mathcal{J}_k(P(C))| \ll \frac{1}{\Gamma(k-3/2)} |A|^{\frac{k}{2}-\frac{3}{4}} (\mathrm{Tr} A[U])^{\frac{1-k}{2}}.$$

If we write

$$|A|^{\frac{k}{2}-\frac{3}{4}} (\mathrm{Tr} A[U])^{\frac{1-k}{2}} = |A|^{1+\delta} (\mathrm{Tr} A[U])^{-5/4-\delta} \Big(\frac{|A|}{\mathrm{Tr}(A[U])} \Big)^{k/2-7/4-\delta}$$

for any fixed $\delta > 0$, and observe that

$$\frac{|A|}{\mathrm{Tr}(A[U])} \asymp \frac{ac}{s_2^2} \asymp s_1^2 \ll 1,$$

it follows using the super-exponential growth of the Gamma function that

$$\mathcal{R}_{22}(\boldsymbol{c}) \ll 2^{-k} \sum_{C \in \mathcal{C}_2(\boldsymbol{c})} |A|^{1+\delta} (\mathrm{Tr} A[U])^{-\frac{5}{4}-\delta}$$

for any fixed $\delta > 0$. By (3.1.16), we have

$$\mathcal{R}_{22}(\boldsymbol{c}) \ll 2^{-k} \times \begin{cases} (ac)^{\frac{1}{2}+\delta-\epsilon} & \text{if } ac < 1, \\ (ac)^{\frac{1}{4}+\epsilon} & \text{if } ac \geq 1, \end{cases}$$

for any $\epsilon > 0$ and $\delta > 0$.

We take $\epsilon = \delta/2$ and using (3.1.14), we deduce that

$$R_{22} \ll \frac{(LM)^{k-\frac{3}{2}}}{2^k} \Big(\sum_{c_1 c_2 > d^{1/2} LM/4} \Big(\frac{LM}{c_1 c_2} \Big)^{1+\delta} \sum_{V \in \mathrm{SL}(2,\mathbb{Z})/\Gamma^0(c_2/c_1)} c_1^{\frac{1}{2}} c_2^{-1+\varepsilon} (c_2, T[v])^{1/2} + \sum_{c_1 c_2 \leqslant d^{1/2} LM/4} \Big(\frac{d^{1/2} LM}{2 c_1 c_2} \Big)^{\frac{1}{2}+\delta} \sum_{V \in \mathrm{SL}(2,\mathbb{Z})/\Gamma^0(c_2/c_1)} c_1^{\frac{1}{2}} c_2^{-1+\varepsilon} (c_2, T[v])^{1/2} \Big)$$

for any $\delta > 0$. In the second sum, we can write trivially

$$\Big(\frac{d^{1/2} LM}{2 c_1 c_2} \Big)^{\frac{1}{2}+\delta} \leq \Big(\frac{d^{1/2} LM}{2 c_1 c_2} \Big)^{1+\delta} \ll \Big(\frac{LM}{c_1 c_2} \Big)^{1+\delta},$$

so we end up with

$$R_{22} \ll \frac{(LM)^{k-\frac{3}{2}}}{2^k} \sum_{c_1 | c_2} \Big(\frac{LM}{c_1 c_2} \Big)^{1+\delta} \sum_{V \in \mathrm{SL}(2,\mathbb{Z})/\Gamma^0(c_2/c_1)} c_1^{\frac{1}{2}} c_2^{-1+\varepsilon} (c_2, T[v])^{1/2}$$

and now, using the same type of arguments leading from (3.1.18) to (3.1.19), we see that

$$R_{22} \ll (LM)^{k-3/2} (LM)^{1+\varepsilon} k^{-E} \tag{3.1.20}$$

for any $\varepsilon > 0$ and $E \geqslant 1$, the implied constant dependind on d, E and ε.

– **Estimation of R_{23}.** Recall that we have $1 \ll s_1 \leq s_2$ for $C \in \mathcal{C}_3$. We estimate the Bessel integral using

$$|\mathcal{J}_k(P)| \leqslant \Big(\int_{M_1} + \int_{M_2} \Big) \Big| J_{k-3/2}(4\pi s_1 \sin\theta) J_{k-3/2}(4\pi s_2 \sin\theta) \sin\theta \Big| d\theta$$

where

$$M_1 = \{\theta \in [0, \pi/2] \mid 4\pi s_1 \sin\theta \le 1\},$$
$$M_2 = \{\theta \in [0, \pi/2] \mid 1 \le 4\pi s_1 \sin\theta,\ 1 \le 4\pi s_2 \sin\theta\}.$$

In the first we use (3.1.3) and the super-exponential growth of the Gamma function to write

$$J_{k-3/2}(4\pi s_1 \sin\theta) \ll 2^{-k} s_1^{\delta}, \quad J_{k-3/2}(4\pi s_2 \sin\theta) \ll 1 \ll s_2^{\delta},$$

for any $\delta > 0$, and in the second we use the estimate (3.1.4) to get

$$J_{k-3/2}(4\pi s_1 \sin\theta) J_{k-3/2}(4\pi s_2 \sin\theta) \ll k^{-2/3},$$

so that

$$|\mathcal{J}_k(P(C))| \ll k^{-2/3} + 2^{-k}(s_1 s_2)^{\delta} \ll k^{-2/3}(s_1 s_2)^{\delta}$$

for any $\delta > 0$. It follows that

$$\mathcal{R}_{23}(\boldsymbol{c}) \ll k^{-2/3} \sum_{C \in \mathcal{C}_3(\boldsymbol{c})} (s_1 s_2)^{\delta},$$

which, by (3.1.17) with, e.g., $\varepsilon = \delta$, gives

$$\mathcal{R}_{23}(\boldsymbol{c}) \ll k^{-2/3}(LM)^{1+\delta}(c_1 c_2)^{-1-\delta}$$

for any $\delta > 0$. Then the same argument as that following (3.1.18) is used to sum over the parameters $\boldsymbol{c}$, and to deduce

$$R_{23} \ll (LM)^{k-3/2}(LM)^{1+\varepsilon} k^{-E} \tag{3.1.21}$$

for any $\varepsilon > 0$ and $E \geqslant 1$, the implied constant depending on d, E and ε-

Summarizing, we have

$$(LM)^{k-3/2} A(k; c, c', L, M) = R_1 + R_2 \ll R_1 + R_{21} + R_{22} + R_{23},$$

and putting together the estimates (3.1.9), (3.1.19), (3.1.20), and (3.1.21), we find that we have proved the estimate

$$A(k; c, c', L, M) \ll L^{1+\varepsilon} M^{3/2+\varepsilon} k^{-2/3}$$

for $\varepsilon > 0$, which was our goal. □

Remark 3.5. For later investigations, it may be worth pointing out that the limitation on the error term, as a function on k, arises only from the contributions R_1 and (the second part of) R_{23}. All other terms decay faster than any polynomial in k as $k \to +\infty$.

3.2. **A quasi-orthogonality relation for Siegel modular forms.** We now put together the results of the previous sections. For every $k \geqslant 1$, we fix a Hecke basis $\mathcal{S}_k^*$ of $\mathcal{S}_k$. Fix the data (d, Λ) as in Section 1.2 and let $\omega_{k,d,\Lambda}^F$ be as defined there; accordingly we have measures $\nu_{sop,k}$ defined for every finite set of primes $\boldsymbol{S}$ and weight $k \geqslant 1$ using suitable average over $F \in \mathcal{S}_k^*$.

Our main result in this section is:

Proposition 3.6. *Let $\boldsymbol{S}$ be a finite set of primes, and $l = (l_p)$, $m = (m_p)$ be $\boldsymbol{S}$-tuples of non-negative integers. Put*

$$L = \prod_{p \in \boldsymbol{S}} p^{l_p}, \qquad M = \prod_{p \in \boldsymbol{S}} p^{m_p}.$$

Then we have

$$\int_{X_{\boldsymbol{S}}} \prod_{p \in \boldsymbol{S}} U_p^{l_p, m_p}(x_p) d\nu_{\boldsymbol{S},k} = \sum_{F \in \mathcal{S}_k^*} \omega_{k,d,\Lambda}^F \prod_{p \in \boldsymbol{S}} U_p^{l_p, m_p}(a_p(F), b_p(F)) \longrightarrow \delta(l; m)$$

as $k \to \infty$ over the even *integers, where*

$$\delta(l;m) = \begin{cases} 1 & \text{if } L = M = 1, \text{ i.e. all } l_p \text{ and } m_p \text{ are } 0, \\ 0 & \text{otherwise.} \end{cases}$$

More precisely, for any even k we have

$$\sum_{F \in \mathcal{S}_k^*} \omega_{k,d,\Lambda}^F \prod_{p \in \boldsymbol{S}} U_p^{l_p,m_p}(a_p(F), b_p(F)) = \delta(l;m) + O\left(\frac{L^{1+\varepsilon} M^{3/2+\varepsilon}}{k^{\frac{2}{3}}}\right), \tag{3.2.1}$$

for any $\varepsilon > 0$, where the implied constant depends only on d and ε.

We will first prove a lemma which is easy, but where the distinction between $\mathrm{SL}(2, \mathbb{Z})$ and $\mathrm{GL}(2, \mathbb{Z})$-equivalence of quadratic forms is important.

Lemma 3.7. *For c, $c' \in \mathrm{Cl}_d$, put*

$$\delta(c, c') = \begin{cases} 1 & \text{if } c \text{ is } \mathrm{GL}(2, \mathbb{Z})\text{-equivalent to } c', \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

$$\sum_{c, c' \in \mathrm{Cl}_d} \Lambda(c) \overline{\Lambda(c')} \delta(c, c') |\operatorname{Aut}(c)| = \frac{2h(-d)w(-d)}{d_\Lambda}$$

where

$$d_\Lambda = \begin{cases} 1 & \text{if } \Lambda^2 = 1, \\ 2 & \text{otherwise.} \end{cases}$$

Proof. Let $H \subset \mathrm{Cl}_d$ be the group of 2-torsion elements. The classes c' which are $\mathrm{GL}(2, \mathbb{Z})$-equivalent to a given class c are c and c^{-1}, hence there are either one or two, depending on whether $c \in H$ or not. Similarly, $|\operatorname{Aut}(c)|$ (which is the order of $\mathrm{GL}(2, \mathbb{Z})$-automorphisms of a representative of c) equals either $2w(-d)$ or $w(-d)$, depending on whether c lies in H or not.

Therefore, we have

$$\begin{aligned} \sum_{c, c' \in \mathrm{Cl}_d} \Lambda(c) \overline{\Lambda(c')} \delta(c, c') |\operatorname{Aut}(c)| &= \sum_{c \in \mathrm{Cl}_d} \Lambda(c) |\operatorname{Aut}(c)| \sum_{c'} \overline{\Lambda(c')} \delta(c, c') \\ &= \sum_{c \in H} |\operatorname{Aut}(c)| \Lambda(c)^2 + \sum_{c \notin H} |\operatorname{Aut}(c)| \Lambda(c)(\Lambda(c) + \Lambda(c^{-1})) \\ &= w(-d) \sum_{c} (1 + \Lambda^2(c)) \end{aligned}$$

by writing $2 = 1 + \Lambda(c^2) = 1 + \Lambda^2(c)$ when $c \in H$. The result follows immediately. $\square$

Now we come to the proof of Proposition 3.6.

Proof. For brevity, we drop the subscripts d and Λ here. For $F \in \mathcal{S}_k^*$, by definition of ω_k^F, we have

$$\omega_k^F \prod_{p \in \boldsymbol{S}} U_p^{l_p,m_p}(a_p(F), b_p(F)) = \frac{4c_k d_\Lambda (d/4)^{\frac{3}{2}-k}}{w(-d)h(-d)} \frac{|a(d, \Lambda; F)|^2}{\langle F, F \rangle} \prod_{p \in \boldsymbol{S}} U_p^{l_p,m_p}(a_p(F), b_p(F)).$$

Now, we write

$$|a(d, \Lambda; F)|^2 = a(d, \Lambda; F) \overline{a(d, \Lambda; F)}$$

and using (1.2.3) to express the first term, we get

$$\frac{|a(d,\Lambda;F)|^2}{\langle F,F\rangle}\prod_{p\in\boldsymbol{S}}U_p^{l_p,m_p}(a_p(F),b_p(F))=\frac{\overline{a(d,\Lambda;F)}}{\langle F,F\rangle}\sum_{c\in\mathrm{Cl}_d}\overline{\Lambda(c)}a(F,S_c)\prod_{p\in\boldsymbol{S}}U_p^{l_p,m_p}(a_p(F),b_p(F)).$$

Now Theorem 2.10 applies to transform this into

$$\begin{aligned}\frac{|a(d,\Lambda;F)|^2}{\langle F,F\rangle}\prod_{p\in\boldsymbol{S}}U_p^{l_p,m_p}(a_p(F),b_p(F))&=\frac{|\mathrm{Cl}_d|}{|\mathrm{Cl}_d(M)|}\frac{L^{\frac{3}{2}-k}M^{2-k}\overline{a(d,\Lambda;F)}}{\langle F,F\rangle}\sum_{c'\in\mathrm{Cl}_d(M)}\overline{\Lambda(c')}a(F,S_{c'}^{L,M})\\&=\frac{|\mathrm{Cl}_d|L^{3/2-k}M^{2-k}}{|\mathrm{Cl}_d(M)|}\sum_{\substack{c\in\mathrm{Cl}_d\\c'\in\mathrm{Cl}_d(M)}}\Lambda(c)\overline{\Lambda(c')}\cdot\frac{\overline{a(F,S_c)}a(F,S_{c'}^{L,M})}{\langle F,F\rangle},\end{aligned}$$

after expanding $\overline{a(d,\Lambda;F)}$ using its definition. We are now reduced to a quantity involving only Fourier coefficients.

We then apply the basic property of the Poincaré series (3.1.1) to express these Fourier coefficients in terms of inner product with Poincaré series: we have

$$\overline{\langle F,P_{k,S_c}\rangle}=8c_k\left(\frac{d}{4}\right)^{-k+3/2}\overline{a(F,S_c)},$$

$$\langle F,P_{k,S_{c'}^{L,M}}\rangle=8c_k(LM)^{-2k+3}\left(\frac{d}{4}\right)^{-k+3/2}a(F,S_{c'}^{L,M})$$

for $c\in\mathrm{Cl}_d$, $c'\in\mathrm{Cl}_d(M)$, and multiplying out with the normalizing constants, we get

$$\omega_k^F\prod_{p\in\boldsymbol{S}}U_p^{l_p,m_p}(a_p(F),b_p(F))=\frac{M^{k-1}L^{k-3/2}d_\Lambda(d/4)^{k-\frac{3}{2}}}{16c_kw(-d)|\mathrm{Cl}_d(M)|}\sum_{\substack{c\in\mathrm{Cl}_d\\c'\in\mathrm{Cl}_d(M)}}\Lambda(c)\overline{\Lambda(c')}\frac{\overline{\langle F,P_{k,S_c}\rangle}\langle F,P_{k,S_{c'}^{L,M}}\rangle}{\langle F,F\rangle}.$$

for every $F\in\mathcal{S}_k^*$.

We now sum over F and exchange the summation to average over F first. Since $\{F/\|F\|\}$ is an orthonormal basis of the vector space $\mathcal{S}_k$, we have

$$\sum_{F\in\mathcal{S}_k^*}\frac{1}{\|F\|^2}\overline{\langle F,P_{k,S_c}\rangle}\langle F,P_{k,S_{c'}^{L,M}}\rangle=\langle P_{k,S_c},P_{k,S_{c'}^{L,M}}\rangle.$$

Now, according to (3.1.1) again, we have

$$\langle P_{k,S_c},P_{k,S_{c'}^{L,M}}\rangle=8c_k\Big(\frac{dL^2M^2}{4}\Big)^{-k+\frac{3}{2}}a(k;c,c',L,M)$$

where $a(k;c,c',L,M)$ denotes, as before, the $S_{c'}^{L,M}$-th Fourier coefficient of the Poincaré series P_{k,S_c}. Applying this and the formal definition

$$a(k;c,c',L,M)=|\mathrm{Aut}(c)|\delta(c,c';L;M)+L^{k-3/2}M^{k-2}A(k;c,c',L,M),$$

as in Proposition 3.1, we obtain first, using Lemma 3.7 that

$$\sum_{F\in\mathcal{S}_k^*}\omega_k^F\prod_{p\in\boldsymbol{S}}U_p^{l_p,m_p}(a_p(F),b_p(F))=\delta(l;m)+$$

$$L^{-k+3/2}M^{-k+2}\frac{d_\Lambda}{2h(-d)w(-d)|\mathrm{Cl}_d(M|}\sum_{\substack{c\in\mathrm{Cl}_d\\c'\in\mathrm{Cl}_d(M)}}\Lambda(c)\overline{\Lambda(c')}A(k;c,c',L,M),$$

and then Proposition 3.1 and Proposition 3.3 lead immediately to the desired result. □

Remark 3.8. In the case of cusp forms on $\mathrm{SL}(2,\mathbb{Z})$ and its congruence subgroups, one can write the Petersson formula in a way which is suitable for further transformations (with "off-diagonal terms" involving Kloosterman sums), as first investigated by Duke, Friedlander and Iwaniec. These are of crucial importance in, e.g., the extension of the range of test functions for low-lying zeros in [24]). In our case, the complexity of the analogue expansion (which is only implicit in Kitaoka's work) for Siegel cusp forms makes this a rather doubtful prospect, at least at the moment.

4. Local equidistribution

To pass from Proposition 3.6 to a local equidistribution result, we must understand how the test functions considered there relate to the space of all continuous functions on $Y_{\boldsymbol{S}}$. This is the purpose of this section.

4.1. **Symmetric functions and polynomials.**

We first observe explicitly that the Laurent polynomials

$$U_p^{l,m}(a,b) \in \mathbb{C}[a,b,a^{-1},b^{-1}]$$

of Theorem 2.5 are invariant under the transformations

$$(a,b) \mapsto (b,a), \qquad (a,b) \mapsto (a^{-1},b), \qquad (a,b) \mapsto (a,b^{-1}),$$

which means that they can be interpreted as functions (also denoted $U_p^{l,m}$) on the space Y_p or on the set X_p of unramified principal series of $G(\mathbb{Q}_p)$. We first state a simple consequence of Proposition 2.7.

Corollary 4.1. *Let $\boldsymbol{S}$ be a fixed finite set of primes, and let $Y_{\boldsymbol{S}}$ be as before. The linear span of the functions*

$$(a_p,b_p)_{p\in\boldsymbol{S}} \mapsto \prod_{p\in\boldsymbol{S}} U_p^{l_p,m_p}(a_p,b_p),$$

where (l_p), (m_p) run over non-negative integers indexed by $\boldsymbol{S}$, is dense in the space $C(Y_{\boldsymbol{S}})$ of continuous functions on $Y_{\boldsymbol{S}}$.

Proof. By the Stone-Weierstrass Theorem, this follows immediately from Proposition 2.7, using the product structure to go from a single prime to a finite set of primes. □

The point of this, in comparison with Proposition 3.6, is of course the following fact:

Proposition 4.2. *Let $\boldsymbol{S}$ be any fixed finite set of primes, and let $\mu_{\boldsymbol{S}}$ be the associated Plancherel measure on $Y_{\boldsymbol{S}}$, defined in the introduction. We have*

$$\int_{Y_{\boldsymbol{S}}} \prod_{p\in\boldsymbol{S}} U_p^{l_p,m_p}(a_p,b_p)d\mu_{\boldsymbol{S}} = \begin{cases} 1 & \text{if } l_p = m_p = 0 \text{ for all } p \in \boldsymbol{S}, \\ 0 & \text{otherwise,} \end{cases}$$

for all non-negative integers (l_p), (m_p) indexed by primes in $\boldsymbol{S}$.

Proof. Since we work with product measures and product functions, it is enough to prove this for the case $n = 1$. But that follows directly from [17, equation (8)]. □

Remark 4.3. This fact can also be proved by direct contour integration via Cauchy's formula using the generating function description for $U_p^{l,m}(a,b)$ (given by Theorem 2.5).

It is now a simple matter to conclude the proof of Theorem 1.6.

Proof of Theorem 1.6. Fix a finite set of primes $\boldsymbol{S}$. Using the Weyl equidistribution criterion, in order to prove that $\nu_{\boldsymbol{S},k}$ converges weakly to $\mu_{\boldsymbol{S}}$ as $k \to +\infty$ over even integers, it suffices to show that

$$\lim_{k\to+\infty} \int_{Y_{\boldsymbol{S}}} \varphi((x_p)) d\nu_{\boldsymbol{S},k} = \int_{Y_{\boldsymbol{S}}} \varphi(x) d\mu_{\boldsymbol{S}}(x) \tag{4.1.1}$$

for all functions φ taken from a set of continuous functions whose linear combinations span $C(Y_{\boldsymbol{S}})$. By Corollary 4.1, the functions

$$\varphi((a_p, b_p)) = \prod_{p\in\boldsymbol{S}} U_p^{l_p,m_p}(a_p, b_p),$$

where (l_p), (m_p) are non-negative integers indexed by $\boldsymbol{S}$, form such a set. But for φ of this type, the desired limit (4.1.1) is obtained by combining Proposition 4.2 and Proposition 3.6.

Now, for the proof of the quantitative version (1.2.6). First of all, we can assume that all polynomials φ_p are non-constant, i.e., that $d_p \geqslant 1$ for each $p \in \boldsymbol{S}$, by working with a smaller $\boldsymbol{S}$ if necessary (and incorporating the constant functions at a single prime). The polynomials φ_p are finite linear combinations, say

$$\varphi_p(a_p, b_p) = \sum_{0\leqslant l_p\leqslant e_p} \sum_{0\leqslant m_p\leqslant f_p} \hat{\varphi}_p(l_p, m_p) U_p^{l_p,m_p}(a, b)$$

of the basis polynomials $U_p^{l_p,m_p}$, for some e_p, $f_p \geqslant 0$ with $\max(e_p, f_p) \geqslant 1$.

Taking the product of these expressions over $\boldsymbol{S}$, summing over F, and using (3.2.1) we get

$$\sum_{F\in\mathcal{S}_{2k}^*} \omega_{2k}^F \varphi((a_p(F), b_p(F))_{p\in S}) = \int_{Y_{\boldsymbol{S}}} \varphi(x) d\mu_{\boldsymbol{S}}(x) + k^{-2/3} R \tag{4.1.2}$$

where we see that the remainder R can be bounded by

$$|R| \ll \sum_{L|L_\varphi} \sum_{M|M_\varphi} L^{1+\varepsilon} M^{3/2+\varepsilon} \prod_{p\in\boldsymbol{S}} |\hat{\varphi}_p(v_p(L), v_p(M))|$$

for any $\varepsilon > 0$, where the implied constant depends only on ε and

$$L_\varphi = \prod_{p\in\boldsymbol{S}} p^{e_p}, \qquad M_\varphi = \prod_{p\in\boldsymbol{S}} p^{f_p}.$$

The coefficients in the expansion are obtained as inner products

$$\hat{\varphi}_p(l, m) = \langle \varphi_p, U_p^{l,m} \rangle$$

in $L^2(Y_p, d\mu_p)$, by orthogonality of the polynomials $U_p^{l,m}$. Since the underlying measure μ_p is a probability measure supported on the tempered subset $X \subset Y_p$, those coefficients may be bounded by

$$|\hat{\varphi}_p(v_p(L), v_p(M))| \leqslant \|\varphi_p U_p^{l_p,m_p}\|_\infty \leqslant C(v_p(L) + 3)^3 (v_p(M) + 3)^3 \|\varphi_p\|_\infty,$$

by Lemma 2.8. Therefore, we get the estimate

$$|R| \ll L_\varphi^{1+\varepsilon} M_\varphi^{3/2+\varepsilon} \eta(LM) \prod_{p\in S} \|\varphi_p\|_\infty = L_\varphi^{1+\varepsilon} M_\varphi^{3/2+\varepsilon} \eta(LM) \|\varphi\|_\infty$$

where $\eta(n)$ is the multiplicative function such that

$$\eta(p^\nu) = C(\nu + 1)^6$$

for p prime and $\nu \geqslant 0$ (here we use the fact that $\max(e_p, f_p) \geqslant 1$ for each p). This is a divisor-like function, i.e., it satisfies

$$\eta(n) \ll n^\varepsilon$$

for any $\varepsilon > 0$, the implied constant depending only on ε. Therefore we get

$$|R| \ll L_{\varphi}^{1+\varepsilon} M_{\varphi}^{3/2+\varepsilon} \|\varphi\|_{\infty},$$

for any $\varepsilon > 0$, where the implied constant depends only on $\varepsilon > 0$.

To derive (1.2.6), we observe that, by the second part of Proposition 2.7, the linear decomposition of φ_p holds with

$$e_p + 2f_p \leqslant d_p,$$

where d_p is the total degree of φ_p as polynomial in $(a+a^{-1}, b+b^{-1})$. Thus, at the cost of worsening the factor involving M, we obtain (1.2.6). □

Remark 4.4. The proof shows that if we know that the factors φ_p are combinations of polynomials $U_p^{l,m}$ with $l \leqslant l_p$, $m \leqslant m_p$, we have the stronger estimate

$$\sum_{F \in \mathcal{S}_{2k}^*} \omega_{2k}^F \varphi((a_p(F), b_p(F))_{p \in S}) = \int_{Y_{\boldsymbol{S}}} \varphi(x) d\mu_{\boldsymbol{S}}(x) + O\Big(k^{-2/3} L^{1+\varepsilon} M^{3/2+\varepsilon} \|\varphi\|_{\infty}\Big)$$

for any $\varepsilon > 0$, where

$$L = \prod_{p \in \boldsymbol{S}} p^{l_p}, \qquad M = \prod_{p \in \boldsymbol{S}} p^{m_p}.$$

5. Applications

We now gather some applications of the local equidistribution theorem. To emphasize the general principles involved, and their expected applicability to the most general "families" of L-functions, we denote

$$\mathbf{E}_k(\alpha(F)) = \frac{1}{\sum_{F \in \mathcal{S}_k^*} \omega_k^F} \sum_{F \in \mathcal{S}_k^*} \omega_k^F \alpha(F)$$

for $k \geqslant 2$ even and for any complex numbers $(\alpha(F))$. This is the averaging operator for a probability measure depending on k, and we know from the previous results that

$$\mathbf{E}_k(\alpha(F)) \sim \sum_{F \in \mathcal{S}_k^*} \omega_k^F \alpha(F)$$

as $k \to +\infty$ over even integers. We denote by $\mathbf{P}_k(\bullet)$ the associated probability. We also recall that $\mathcal{S}_{2k}^{\flat}$ is the set of cusp forms which are not Saito-Kurokawa forms, and we denote by $\mathcal{S}_{2k}^{\sharp}$ the complementary set of Saito-Kurokawa lifts.

5.1. **Direct applications.**

We start with direct consequences of the local equidistribution. The first is partly superseded by the proof of the generalized Ramanujan conjecture in our case [50], but it may still be taken as an indication that the special Saito-Kurokawa modular forms which fail to satisfy it are "few", even when counted with our special weights.

Proposition 5.1. (1) *Fix a prime p. Then "most" $F \in \mathcal{S}_k^*$ satisfy the generalized Ramanujan conjecture at p, in the sense that we have*

$$\mathbf{P}_{2k}(\pi_p(F) \text{ is not tempered}) = \mathbf{P}_{2k}(\pi_p(F) \notin X \subset X_p) \longrightarrow 0$$

as $k \to +\infty$.

(2) *Let $\alpha(F)$ be any bounded function defined for F which are Saito-Kurokawa lifts, i.e., $F \in \mathcal{S}_k^{\sharp}$. Then we have*

$$\lim_{k \to +\infty} \mathbf{E}_{2k}(\alpha(F) \mathbf{1}_{\{F \in \mathcal{S}_{2k}^{\sharp}\}}) = 0.$$

Proof. Since the limiting measure μ_p is supported on $X \subset X_p$, this is immediate. □

In particular, it follows that the measure $\nu^{\flat}_{\boldsymbol{S},k}$ defined as $\nu_{\boldsymbol{S},k}$, but with F restricted to $\mathcal{S}^{\flat}_k$, also converge weakly, as $k \to +\infty$, to $\mu_{\boldsymbol{S}}$. We will denote by $\mathbf{E}^{\flat}_k(\alpha(F))$ the average

$$\mathbf{E}^{\flat}_k(\alpha(F)) = \mathbf{E}_k(\alpha(F)\mathbf{1}_{\{F\in\mathcal{S}^{\flat}_k\}}),$$

so that the previous result means that this is still, asymptotically, a probability average.

The next result has the feel of a "Strong Approximation" theorem:

Proposition 5.2 ("Strong approximation")**.** (1) *Let* Aut *denote the set of all cuspidal automorphic representations on* $\mathrm{GSp}(4,\mathbb{A})$*. Then for any finite set of primes* $\boldsymbol{S}$*, the local components* $\pi_{\boldsymbol{S}}$ *for those* $\pi \in \mathrm{Aut}$ *unramified at the primes in* $\boldsymbol{S}$ *form a dense subset of* $X^{\boldsymbol{S}}$*.*

(2) *Fix a finite set of prime* $\boldsymbol{S}$*, and let* $(\varepsilon_p)_{p\in\boldsymbol{S}}$ *be signs* ± 1*. There exist infinitely many Siegel cusp forms* F *of level* 1 *which are Hecke-eigenforms such that the Hecke eigenvalues at all* $p \in \boldsymbol{S}$ *have sign* ε_p*.*

Proof. (1) The support of the limiting measure $\mu_{\boldsymbol{S}}$ is $X^{\boldsymbol{S}}$, hence the result is again immediate (with the much more precise information that denseness holds already for π associated to Siegel cusp forms of full level).

(2) This sample application follows from the fact that the Hecke eigenvalue at a prime p is $a_p + a_p^{-1} + b_p + b_p^{-1}$ for $F \in \mathcal{S}^*_k$, and it is clear from the formulas for the Haar measure μ (1.2.4) and for the density function Δ_p (1.2.5) that

$$\mu_p(\{(a_p, b_p)_{p\in\boldsymbol{S}} \mid \text{the sign of } a_p + a_p^{-1} + b_p + b_p^{-1} \text{ is } \varepsilon_p\}) > 0$$

for any prime p and hence

$$\mu_{\boldsymbol{S}}(\{(a_p, b_p)_{p\in\boldsymbol{S}} \mid \text{the sign of } a_p + a_p^{-1} + b_p + b_p^{-1} \text{ is } \varepsilon_p \text{ for } p \in \boldsymbol{S}\}) > 0$$

(and hence, for k large enough, at least one $F \in \mathcal{S}^*_k$ satisfies those local conditions). □

Remark 5.3. In fact, our local equidistribution shows much more. For instance, for any non-negligible subset of T of $X^{\boldsymbol{S}}$ and any fundamental discriminant $-d$, our result shows that one can find infinitely many Siegel modular forms whose local components at $\boldsymbol{S}$ lie in T and the sum of whose Fourier coefficients of discriminant $-d$ is non-zero. Since such sums of Fourier coefficients of Siegel modular forms are conjecturally related to central critical L values of the twisted forms — see Section 5.4 — this can be interpreted as a (conditional) result on the plentitude of Siegel modular forms with prescribed local behavior and non-vanishing central critical values.

5.2. **Averaging L-functions.** The spin L-function associated to $F \in \mathcal{S}^*_k$ is defined, in terms of the Satake parameters (a_p, b_p), by the Euler product

$$L(F,s) = \prod_p (1-a_p p^{-s})^{-1}(1-b_p p^{-s})^{-1}(1-a_p^{-1}p^{-s})^{-1}(1-b_p^{-1}p^{-s})^{-1} ;$$

it is explained in [2] that this is a Langlands L-function associated with the Spin representation $\mathrm{Spin}(5,\mathbb{C}) \to \mathrm{GL}(4,\mathbb{C})$. This Spin group is the Langlands dual group of $\mathrm{Sp}(4)$, and since $\mathrm{Spin}(5) \simeq \mathrm{Sp}(4)$, this is natural in view of the parametrization of the local representations in terms of semisimple conjugacy classes of $\mathrm{USp}(4,\mathbb{C})$ which is described in the introduction. From the point of view of $\mathrm{Sp}(4)$, this is the Langlands L-function corresponding to the natural representation $\mathrm{GSp}(4) \subset \mathrm{GL}(4)$.

The idea is that in the region of absolute convergence (which is $\mathrm{Re}(s) > 1$, for F not a Saito-Kurokawa lift), the average of such a product is the average of asymptotically independent random variables, and hence will be the product of the averages for the limiting distributions at each p. Saito-Kurokawa lifts, being asymptotically negligible, do not cause much trouble in that case.

To go to the details, we first recall that, from work of Andrianov [1], it is known that $L(F,s)$ has the basic standard analytic properties expected from an L-function; it is self-dual with root number $(-1)^k$ and unramified at finite places; it extends to a meromorphic function of s, and the completed L-function

$$\Lambda(F,s) = (2\pi)^{-2s}\Gamma(s+1/2)\Gamma(s+k-3/2)L(F,s)$$

satisfies

$$\Lambda(F,s) = (-1)^k\Lambda(F,1-s).$$

Furthermore, if k is odd, the L-function is entire; otherwise, it may have poles at $s = 3/2$ and $s = -1/2$, and this happens precisely when F is a Saito-Kurokawa lift (i.e., if $F \in \mathcal{S}_k^\sharp$, using the notation of Theorem 1.1). In that case, the L-function is given by

$$L(F,s) = \zeta(s-1/2)\zeta(s+1/2)L(f,s) \tag{5.2.1}$$

for some classical cusp form f of weight $2k-2$ on $\mathrm{SL}(2,\mathbb{Z})$ (the L-function of which is also normalized so that the critical line is $\mathrm{Re}(s) = 1/2$). Note that although, in general, there are other automorphic forms on $\mathrm{GSp}(4)$ where the L-function have poles, the Saito-Kurokawa lifts are the only ones which are holomorphic with level 1 (we refer to Piatetski-Shapiro's paper [36] for more details).

Remark 5.4. For completeness, even though we do not need it here, let us recall the corresponding results for the other "standard" L-function, which is the degree-five L-function coming from the projection

$$\mathrm{pr} : \mathrm{Spin}(5,\mathbb{C}) \to \mathrm{SO}(5,\mathbb{C}) \subset \mathrm{GL}(5,\mathbb{C}).$$

This L-function has the form

(5.2.2)

$$L(F,\mathrm{pr},s) = \prod_p \Big((1-p^{-s})(1-a_pb_pp^{-s})(1-a_pb_p^{-1}p^{-s})(1-a_p^{-1}b_pp^{-s})(1-(a_pb_p)^{-1}p^{-s}\Big)^{-1}.$$

From work of Mizumoto [34], it is known that $L(F,\mathrm{pr},s)$ has the basic standard analytic properties expected from an L-function; it is self-dual with root number 1 and unramified at finite places; it extends to an *entire* function of s, and the completed L-function

$$\Lambda(F,\mathrm{pr},s) = 2^{-2s}\pi^{-5s/2}\Gamma\left(\frac{s+1}{2}\right)\Gamma(s+k-1)\Gamma(s+k-2)L(F,s)$$

satisfies

$$\Lambda(F,\mathrm{pr},s) = \Lambda(F,\mathrm{pr},1-s).$$

Let us now return to the spin L-function. To average the L-function, we express it in additive terms. For this purpose, we denote by

$$\pi_p : \mathcal{S}_k^* \to X_p$$

the map $F \mapsto \pi_p(F)$, the local component of the automorphic representation π_F associated with F as described earlier, which we identify with $(a_p(F), b_p(F)) \in Y_p$.

Expanding the Euler factors in powers of p^{-s} and then expanding the product into Dirichlet series, we find the expression

$$L(F,s) = \sum_{n\geqslant 1} \lambda(F,n)n^{-s}$$

in the region of absolute convergence, where

$$\lambda(F,n) = \prod_{p\mid n} H_{v_p(n)}(\pi_p(F)), \quad \text{for} \quad n = \prod_{p\mid n} p^{v_p(n)},$$

in terms of functions H_m, $m \geqslant 0$, on Y_p given by the symmetric functions

$$H_m(a,b) = \sum_{k_1+k_2+k_3+k_4=m} a^{k_1-k_3}b^{k_2-k_4}$$

(note that H_m is independent of p, though that is not crucial in what follows, and that it is well-defined on X_p since it is invariant under the Weyl group).

If $\mathrm{Re}(s) > 1$, the series $L(F,s)$ converge absolutely at s for $F \in \mathcal{S}^{\flat}_{2k}$, and we have

$$\mathbf{E}^{\flat}_{2k}(L(F,s)) = \sum_{n\geqslant 1} \mathbf{E}^{\flat}_{2k}(\lambda(F,n))n^{-s}.$$

Fix n first, and factor it as before

$$n = \prod_{p\mid n} p^{v_p}.$$

By our local equidistribution theorem applied to $\nu^{\flat}_{\boldsymbol{S},2k}$, we have

$$\mathbf{E}^{\flat}_{2k}(\lambda(F,n)) = \mathbf{E}^{\flat}_{2k}\Bigl(\prod_{p\mid n} H_{v_p}(\pi_p(F))\Bigr) \to \prod_{p\mid n} \int_{X_p} H_{v_p}(x)d\mu_p(x)$$

as $k \to +\infty$. Therefore, by the dominated convergence theorem, we have

$$\sum_{n\geqslant 1} \mathbf{E}^{\flat}_{2k}(\lambda(F,n))n^{-s} \to \sum_{n\geqslant 1} \Bigl(\prod_{p\mid n} \int_{X_p} H_{v_p}(x)d\mu_p(x)\Bigr)n^{-s}$$

since, using the formula

$$\frac{1}{6}(m+1)(m+2)(m+3)$$

for the number of monomials of degree m in 4 variables (the number of terms in H_m) we have

$$\Bigl|\mathbf{E}^{\flat}_{2k}(\lambda(F,n))n^{-s}\Bigr| \leqslant n^{-\sigma} \prod_{p\mid n} (v_p+3)^3$$

for all $n \geqslant 1$ and k, which defines an absolutely convergent series for $\sigma = \mathrm{Re}(s) > 1$. (We use here the generalized Ramanujan conjecture, proved in this case by Weissauer [50].)

Now we refold back the limiting expression as an Euler product:

$$\begin{aligned}\sum_{n\geqslant 1} \Bigl(\prod_{p\mid n} \int_{X_p} H_{v_p}(x)d\mu_p(x)\Bigr)n^{-s} &= \prod_p \sum_{l\geqslant 0} p^{-ls} \int_{X_p} H_l(x)d\mu_p(x)\\ &= \prod_p \int_{X_p} L_p(x,s)d\mu_p(x),\end{aligned}$$

where $L_p(x,s)$ is the local L-factor of a local representation $x = (a,b) \in X_p$ defined in the lemma above (the Euler expansion is justified again by the fact that the series on the left is absolutely convergent, as we checked in the lemma). Thus we have proved

$$\lim_{k\to+\infty} \mathbf{E}^{\flat}_{2k}(L(F,s)) = \prod_p \int_{X_p} L_p(x,s)d\mu_p(x). \tag{5.2.3}$$

Now assume $\mathrm{Re}(s) > 1$ and $s \neq 3/2$. Then all spin L-functions of Saito-Kurokawa lifts are well-defined at s, and we therefore also want to have average formulas involving them. If $\mathrm{Re}(s) > 3/2$, this is immediate by the previous argument. Otherwise, we have

$$\sum_{F\in\mathcal{S}^{\sharp}_{2k}} \omega^F_{2k} L(F,s) = \zeta(s-1/2)\zeta(s+1/2) \sum_{F\in\mathcal{S}^{\sharp}_{2k}} \omega^F_{2k} L(f_F,s),$$

where f_F is a classical modular form from which F arises. The L-function $L(f_F, s)$ is now absolutely convergent, and its values are bounded for all Saito-Kurokawa lifts (by the generalized Ramanujan conjecture, for instance). Thus we have

$$\lim_{k \to +\infty} \sum_{F \in \mathcal{S}_{2k}^{\sharp}} \omega_{2k}^F L(F, s) = 0$$

by Proposition 5.1, (2), and this combined with (5.2.3) gives the result

$$\lim_{k \to +\infty} \mathbf{E}_{2k}(L(F, s)) = \prod_p \int_{X_p} L_p(x, s) d\mu_p(x).$$

At this point, it is clear how to extend this to other Langlands L-functions. Indeed, let

$$\rho : \mathrm{GSp}(4, \mathbb{C}) \to \mathrm{GL}(r, \mathbb{C})$$

be an algebraic representation. The Langlands L-function is defined by

$$L(F, \rho, s) = L(\pi_F, \rho, s) = \prod_p \det(1 - \rho(x_p(F))p^{-s})^{-1}$$

where

$$x_p(F) = x_p(a_p, b_p) = \begin{pmatrix} a_p & & & \\ & b_p & & \\ & & a_p^{-1} & \\ & & & b_p^{-1} \end{pmatrix}$$

is the semisimple conjugacy class of $\mathrm{GSp}(4, \mathbb{C})$ associated with $\pi_p(F)$. We can expand

$$\det(1 - \rho(x_p(F))p^{-s})^{-1} = \prod_{1 \leqslant j \leqslant r} (1 - \alpha_j(a_p, b_p)p^{-s})^{-1}$$

for some polynomial functions α_j on X_p, and then we can repeat the same argument to derive (1.1.3) with

$$\lim_{k \to +\infty} \mathbf{E}_{2k}(L(F, \rho, s)) = \prod_p \int_X \det(1 - \rho(x_p(a, b))p^{-s})^{-1} d\mu_p(a, b),$$

when s is in the region of common absolute convergence of all F.

Finally, to get the precise expression in Theorem 1.1 for the spin L-function, we note that the special case (2.3.7) of Sugano's formula (Theorem 2.5), with $Y = p^{-s}$, gives the explicit decomposition

$$L_p(x, s) = \Big(1 - \lambda_p p^{-1/2-s} + \Big(\frac{\mathcal{K}}{p}\Big) p^{-1-2s}\Big)^{-1} \Big(\sum_{l \geq 0} U_p^{l,0}(a, b) p^{-ls}\Big)$$

for any prime p. Applying Proposition 4.2, we get therefore the simple expression

$$\int_{X_p} L_p(x, s) d\mu_p(x) = \frac{1}{1 - \lambda_p p^{-1/2-s} + \left(\frac{\mathcal{K}}{p}\right) p^{-1-2s}},$$

and (using the definition of λ_p) we recognize that this is

$$L(\Lambda, s + 1/2),$$

where Λ is the class group character of $\mathcal{K} = \mathbb{Q}(\sqrt{-d})$ defining our fixed Bessel models. When $d = 4$ and Λ is trivial, this is $\zeta(s + 1/2)L(\chi_4, s + 1/2)$, which is the formula (1.1.2).

Remark 5.5. In fact, this second argument for the spin L-function can be used to bypass the first one (which therefore requires only that we work with the family of functions $U_p^{l,0}(a, b)$).

Remark 5.6. Note that although Theorem 1.1 was stated in the introduction only for averages with respect to the weight $\omega_k^F := \omega_{k,d,\Lambda}^F$ in the special case $d = 4$, $\Lambda = 1$, our proof actually works for general d and Λ.

The proof also gives the following fact concerning the limit averages:

Lemma 5.7. *For p prime, let μ_p be the limiting measure in the local equidistribution result and let*

$$L_p(x, s) = \prod_p (1 - ap^{-s})^{-1}(1 - bp^{-s})^{-1}(1 - a^{-1}p^{-s})^{-1}(1 - b^{-1}p^{-s})^{-1}$$

be the local L-function for $x \in X_p$. Then the Euler product

$$\prod_p \int_{X_p} L_p(x, s) d\mu_p(x)$$

is absolutely convergent for $\mathrm{Re}(s) > 1/2$.

Proof. According to what we have said, we have

$$\int_{Y_p} L_p(x, s) d\mu_p(x) = \frac{1}{1 - \lambda_p p^{-1/2-s} + \left(\frac{\mathcal{K}}{p}\right) p^{-1-2s}}$$

and the result is then obvious. □

5.3. **Weights and averages over Saito-Kurokawa lifts.** In this section, we will explicitly compute $\omega_k^F := \omega_{k,d,\Lambda}^F$ when F is a Saito-Kurokawa lift. This will lead to a stronger version of the second part of Proposition 5.1. This simple fact is included because it may be helpful for further investigations.

Let $k > 2$ be even and let $\mathcal{H}_{2k-2}^*$ denote the Hecke basis of the space of holomorphic cusp forms on GL(2) of weight $2k - 2$ and full level. Let $F \in \mathcal{S}_k^*$ be the (unique) Saito-Kurokawa lift of $f \in \mathcal{H}_{2k-2}^*$, so that the spinor L-function is given by (5.2.1). As usual, we let

$$F(Z) = \sum_{T>0} a(F, T) e(\mathrm{Tr}(TZ))$$

be the Fourier expansion of F. It is well-known (see [13] for instance) that $a(F, T)$ then depends only on the determinant of T. In particular, it follows that $\omega_{k,d,\Lambda}^F = 0$ whenever $\Lambda \neq 1$. So, we assume that $\Lambda = 1$ and shorten $\omega_{k,d,1}^F$ to ω_k^F.

Let

$$\tilde{f}(z) = \sum_{n>0} c(n) e(nz)$$

be the cusp form of half-integer weight $k - \frac{1}{2}$ on $\Gamma_0(4)$ that is associated to f via the Shimura correspondence. Then, by [13, Th. 6.2, equation (6)], we have

$$a(T) = c(d) \tag{5.3.1}$$

for any positive-definite semi-integral matrix T of determinant $d/4$. On the other hand, by [4], we have

$$\langle F, F\rangle = \frac{k-1}{2^4 \cdot 3^2 \cdot \pi} \cdot \frac{|c(d)|^2}{d^{k-\frac{3}{2}}} \cdot \frac{L(f, 1)}{L(f \times \chi_d, \frac{1}{2})} \langle f, f\rangle, \tag{5.3.2}$$

whenever $c(d)$ is non-zero.

Using (5.3.1), (5.3.2) and the definition of ω_k^F, it follows that

$$\omega_k^F = \frac{(48\pi)^2 h(-d)}{w(-d)(k-1)(k-2)} \frac{\Gamma(2k-3)}{(4\pi)^{2k-3}\langle f, f\rangle} \frac{L(f \times \chi_d, \frac{1}{2})}{L(f, 1)}.$$

Now we consider the average

$$\sum_{f\in\mathcal{H}^*_{2k-2}} \frac{\Gamma(2k-3)}{(4\pi)^{2k-3}\langle f,f\rangle}\frac{L(f\times\chi_d,\frac{1}{2})}{L(f,1)}.$$

It is very likely that one can prove an asymptotic formula for this quantity as $k\to+\infty$ (possibly using the methods of Ramakrishnan and Rogawski in [40]). However, to deal with it quickly, we observe first that $L(f\times\chi_d,\frac{1}{2})$ and $L(f,1)$ are both non-negative (e.g., because $L(f,s)$ is real-valued, has no zero with $\mathrm{Re}(s)>1$ and tends to 1 as $s\to+\infty$, and the ratio is non-negative by the above). Then, using the fact that $L(f,s)$ has no Siegel zeros (a result of Hoffstein and Ramakrishnan [20]), one gets in the usual way a lower bound

$$L(f,1)\gg\frac{1}{\log k},$$

and therefore

$$\sum_{f\in\mathcal{H}^*_{2k-2}} \frac{\Gamma(2k-3)}{(4\pi)^{2k-3}\langle f,f\rangle}\frac{L(f\times\chi_d,\frac{1}{2})}{L(f,1)}\ll(\log k)\sum_{f\in\mathcal{H}^*_{2k-2}} \frac{\Gamma(2k-3)}{(4\pi)^{2k-3}\langle f,f\rangle}L(f\times\chi_d,\frac{1}{2}).$$

Next, from the results of Duke [12], one gets

$$\sum_{f\in\mathcal{H}^*_{2k-2}} \frac{\Gamma(2k-3)}{(4\pi)^{2k-3}\langle f,f\rangle}L(f\times\chi_d,\frac{1}{2})\ll 1$$

for $k\geqslant 2$, where the implied constant depends on d. The following Proposition, which strengthens Proposition 5.1, (2), is then an immediate consequence:

Proposition 5.8. *Suppose $\alpha(F)$ is a complex valued function defined for Saito-Kurokawa lifts and satisfying for some $\delta>0$ the inequality*

$$\alpha(F)\ll k^{2-\delta}$$

for $F\in\mathcal{S}^\sharp_{2k}$. Then

$$\lim_{k\to+\infty}\mathbf{E}_k(\alpha(F)\mathbf{1}_{\{F\,\in\,\mathcal{S}^\sharp_k\}})=0.$$

Using weak bounds, like the convexity bound, this applies for instance to $\alpha(F)=L(F,1/2+it)$ for fixed $t\neq 0$.

5.4. **Low-lying zeros, Katz-Sarnak symmetry type and Böcherer's conjecture.** The determination of the distribution of low-lying zeros of the spin L-functions (assuming the Generalized Riemann Hypothesis) for restricted test functions is not difficult once a quantitative equidistribution statement is known. Conjecturally, the answer indicates which "symmetry type" (in the sense of Katz-Sarnak) arises for the family. However, we will see that the answer in our case is surprising, and gives some global evidence towards a well known conjecture of Böcherer.

We now prove Theorem 1.2. This type of computation is quite standard by now, and is known to succeed as soon as "approximate orthogonality" has been proved with a power saving with respect to the analytic conductor, which is the case thanks to our quantitative equidistribution theorem (precisely, from (3.2.1)). We may therefore be brief, as far as technical details are concerned (we refer to, e.g., [11], where families derived from classical GL(2) cusp forms are treated with respect to the weight). However, since the main term arising from this computation has some meaning, we must justify it carefully.

As before, note that although Theorem 1.2 was stated in the introduction only for averages with respect to the weight $\omega_k^F:=\omega_{k,d,\Lambda}^F$ in the special case $d=4$, $\Lambda=1$, we actually prove it for any d (we stick to $\Lambda=1$).

Proof of Theorem 1.2. Throughout this proof, ω_k^F denotes $\omega_{k,d,1}^F$. For given F, we write

$$-\frac{L'}{L}(F,s)=\sum_{n\geqslant 1}c(F,n)\Lambda(n)n^{-s}$$

the logarithmic derivative of the spinor L-function which is supported on powers of primes and where $\Lambda(n)$ is the von Mangoldt function and, for $n=p^m$, $m\geqslant 1$, we have

$$c(F,p^m)=a_p^m+a_p^{-m}+b_p^m+b_p^{-m}=\mathrm{Tr}(\pi_p(F)^m),$$

where $\pi_p(F)$ is interpreted as a conjugacy class in $\mathrm{USp}(4,\mathbb{C})$.

We can apply the following form of the "explicit formula" (see, e.g., [23, Th. 5.12]) to relate sums over zeros to sums over primes involving those coefficients: denoting

$$\gamma(s)=(2\pi)^{-2s}\Gamma(s+1/2)\Gamma(s+k-3/2)$$

the common gamma factor for all $L(F,s)$, for any test-function ψ which is even and of Schwartz class on $\mathbb{R}$, we have

$$\sum_\rho \psi\Big(\frac{\gamma}{2\pi}\Big)=\frac{1}{2\pi}\int_{\mathbb{R}}\Big(\frac{\gamma'}{\gamma}(1/2+it)+\frac{\gamma'}{\gamma}(1/2-it)\Big)\psi(x)dx-2\sum_n\hat{\psi}(\log n)\frac{c(F,n)\Lambda(n)}{\sqrt{n}}.$$

We apply this to

$$\psi(x)=\varphi\Big(\frac{x}{2\pi}\log(k^2)\Big),\qquad \hat{\psi}(t)=\frac{\pi}{\log k}\hat{\varphi}\Big(\frac{\pi t}{\log k}\Big)$$

where φ is an even Schwartz function with Fourier transform supported in $[-\alpha,\alpha]$. After treating the gamma factor using the formula

$$\frac{\Gamma'}{\Gamma}(k-1+it)+\frac{\Gamma'}{\Gamma}(k-2-it)=2\log k+O(t^2k^{-2}),$$

which follows from Stirling's formula (see, e.g., [11, §3.1.1, §3.1.2] for precise details of these computations) and spelling out the von Mangoldt function, we obtain

$$D_\varphi(F)=\int_{\mathbb{R}}\varphi(x)dx-\frac{2}{\log(k^2)}\sum_{m\geqslant 1}\sum_p\frac{\log p}{p^{m/2}}c(F,p^m)\hat{\varphi}\Big(m\frac{\log p}{\log(k^2)}\Big)+O((\log k)^{-1}). \tag{5.4.1}$$

Averaging over F leads to

$$\mathbf{E}_k(D_\varphi(F))=\hat{\varphi}(0)-\frac{2}{\log(k^2)}\sum_{m\geqslant 1}\sum_p\frac{\log p}{p^{m/2}}\mathbf{E}_k(c(F,p^m))\hat{\varphi}\Big(m\frac{\log p}{\log(k^2)}\Big)+O((\log k)^{-1}).$$

As usual, easy estimates give

$$\lim_{k\to+\infty}\frac{1}{\log(k^2)}\sum_{m\geqslant 3}\sum_p\frac{\log p}{p^{m/2}}\mathbf{E}_k(c(F,p^m))\hat{\varphi}\Big(m\frac{\log p}{\log(k^2)}\Big)=0$$

(the series over primes being convergent even without the compactly-supported test function).

In the term $m=1$, we have

$$\begin{aligned}\mathbf{E}_k(c(F,p))&=\mathbf{E}_k(a_p+b_p+a_p^{-1}+b_p^{-1})\\ &=\mathbf{E}_k(U_p^{1,0}(\pi_p(F)))+\frac{\lambda_p}{\sqrt{p}}=\frac{\lambda_p}{\sqrt{p}}+O(p^{1+\varepsilon}k^{-2/3})\end{aligned}\tag{5.4.2}$$

by (2.3.6) and (3.2.1), and hence the contribution of $m=1$, which is given by

$$\frac{2}{\log(k^2)}\sum_p\frac{\log p}{p^{1/2}}\mathbf{E}_k(c(F,p))\hat{\varphi}\Big(\frac{\log p}{\log(k^2)}\Big)$$

is equal to

$$\frac{2}{\log(k^2)}\sum_p \frac{\lambda_p \log p}{p}\hat{\varphi}\Big(\frac{\log p}{\log(k^2)}\Big) + O\Big(\frac{1}{k^{2/3}\log k}\sum_{p\leqslant k^{2\alpha}} p^{1/2+\varepsilon}\Big) = M_k(\varphi) + O(k^{5\alpha/2-2/3+\varepsilon}).$$

for any $\varepsilon > 0$, where

$$\begin{aligned} M_k(\varphi) &= \frac{2}{\log(k^2)}\sum_p \frac{\lambda_p \log p}{p}\hat{\varphi}\Big(\frac{\log p}{\log(k^2)}\Big) \\ &= 2\int_{[1,+\infty[} \hat{\varphi}\Big(\frac{\log y}{\log(k^2)}\Big)\frac{1}{\log(k^2)}\frac{dy}{y} + o(1) \\ &= 2\int_{[0,+\infty[} \hat{\varphi}(x)dx = \varphi(0) + o(1), \end{aligned}$$

(since φ is even), by summation by parts using the Prime Number Theorem and the fact that $\lambda_p = 2$ or 0 for primes with asymptotic density $1/2$ each, so the average value is 1 (see, e.g., [11, Lemma 2.7]).

Now we consider the term $m = 2$. Although we could appeal to the general estimate (1.2.6), we will use an explicit decomposition and (3.2.1). First, using (2.3.7), we have

$$U_p^{2,0}(\pi_p(F)) = 1 - \frac{\lambda_p}{\sqrt{p}}U_p^{1,0}(\pi_p(F)) + c(F,p^2) + \tau(\pi_p(F)) + O(p^{-1}) \tag{5.4.3}$$

where the implied constant is absolute and

$$\tau(a,b) = 1 + ab + ab^{-1} + a^{-1}b + (ab)^{-1}$$

as in Theorem 2.5. By (2.3.8), we have

$$\tau(\pi_p(F))(1+\alpha_p) = U_p^{0,1}(\pi_p(F)) + \beta_p\sigma(\pi_p(F)) + O(p^{-1})$$

where the quantities $\alpha_p \ll p^{-1}$, $\beta_p \ll p^{-1/2}$ do not depend on F, and the implied constants are absolute. Averaging (with $\mathbf{E}_k(\cdot)$) from this last formula and using (5.4.2), we find

$$\mathbf{E}_k(\tau(\pi_p(F))) \ll p^{-1} + p^{3/2+\varepsilon}k^{-2/3},$$

and from (5.4.3), we therefore derive

$$\mathbf{E}_k(c(F,p^2)) = -1 + O(p^{-1} + p^{2+\varepsilon}k^{-2/3})$$

for any $\varepsilon > 0$. Consequently, we see that the term $m = 2$, after averaging, is given by

$$\begin{aligned} \frac{2}{\log(k^2)}\sum_p \frac{\log p}{p}\mathbf{E}_k(c(F,p^2))\hat{\varphi}\Big(2\frac{\log p}{\log(k^2)}\Big) &= -\frac{2}{\log(k^2)}\sum_p \frac{\log p}{p}\hat{\varphi}\Big(2\frac{\log p}{\log(k^2)}\Big) + \\ &\quad + O\Big(\frac{1}{k^{2/3}\log k}\sum_{p\leqslant k^{\alpha}} p^{1+\varepsilon} + (\log k)^{-1}\Big) \\ &= -N_k(\varphi) + O((\log k)^{-1} + k^{2\alpha-2/3+\varepsilon}), \end{aligned}$$

where

$$N_k(\varphi) = \frac{2}{\log(k^2)}\sum_p \frac{\log p}{p}\hat{\varphi}\Big(2\frac{\log p}{\log(k^2)}\Big) = \frac{\varphi(0)}{2} + o(1),$$

by computations similar to that of $M(\varphi)$ before.

We notice that the contribution of the main terms for $m = 1$ and $m = 2$ together are

$$-\varphi(0) + \frac{\varphi(0)}{2} = -\frac{\varphi(0)}{2},$$

which gives a main term

$$\hat{\varphi}(0) - \frac{\varphi(0)}{2} = \int_{\mathbb{R}} \varphi(x) d\sigma_{Sp}(x).$$

Moreover, the error terms in both are negligible as long as $5\alpha/2 - 2/3 < 0$, i.e., $\alpha < 4/15$. Under this condition, we obtain therefore

$$\mathbf{E}_k(D_\varphi(F)) = \int_{\mathbb{R}} \varphi(x) dx + o(1),$$

as $k \to +\infty$, which is the desired conclusion. □

Remark 5.9. It is interesting to note that, to understand the logarithmic derivative of the spin L-function $L(F, s)$, one needs to involve the average of the quantity

$$\tau(\pi_p(F)) = 1 + a_p b_p + a_p b_p^{-1} + a_p^{-1} b_p + (a_p b_p)^{-1},$$

which is the coefficient of p^{-s} in the *projection* L-function $L(F, \mathrm{pr}, s)$ (see (5.2.2)). This illustrates the fact that, in the study of automorphic forms on groups of rank $r \geqslant 2$, all Langlands L-functions (or, at least, those associated with the r fundamental representations of the dual group) are intimately linked, and must be considered together.

We now comment on the relation of Theorem 1.2 with Böcherer's conjecture. In the literature, a density of low-lying zeros given by the measure $d\sigma_{Sp}$ (as we proved is the case) is taken as an indication of *symplectic symmetry type* (the basic example being the family of real Dirichlet characters). Intuitively, these are families of central L-values of self-dual L-functions with sign of functional equation $+1$ for which the central point of evaluation has no special meaning. However, although our families are indeed self-dual, a symplectic symmetry seems very unlikely for our family, for at least two reasons: first, $1/2$ is a critical point in the sense of Deligne, and second, the forms of odd weight have functional equations with sign -1.

There is a natural explanation for the discrepancy: the Fourier coefficient $|a(d, 1; F)|^2$ appearing in the weight

$$\omega_F^k = c_{k,d} \frac{|a(d, 1; F)|^2}{\langle F, F \rangle}.$$

involves probably deeper arithmetic content than one might naively think. Indeed, in [3], Böcherer made the following remarkable conjecture:

Conjecture (Böcherer's Conjecture)**.** *For any $F \in \mathcal{S}_{2k}^*$, there exists a constant C_F depending only on F such that for all fundamental discriminants $-d < 0$ one has*

$$L(F \times \chi_d, \frac{1}{2}) = C_F \cdot d^{1-2k} w(-d)^{-2} \cdot |a(d, 1; F)|^2,$$

where χ_d denotes the quadratic character associated to the extension $\mathbb{Q}(\sqrt{-d})$.

Böcherer proved this conjecture for Eisenstein series and Saito-Kurokawa lifts in [3]. Later, he and Schulze-Pillot proved an analogue of this conjecture (for Siegel modular forms with level) in the case of Yoshida lifts. More recently, works of Furusawa-Shalika [19], Furusawa-Martin [16] and Furusawa-Martin-Shalika [17] have tried to tackle this problem using the relative trace formula.

Böcherer did not make any speculation about the value of the quantity C_F. However recent works such as [16] give some inkling of what to expect.

We now show that a certain assumption on C_F explains our result on low-lying zeroes. To be more definite, we make the following hypothesis.

Hypothesis. *For non-Saito-Kurokawa forms $F \in \mathcal{S}^*_{2k}$, we have*

$$\omega_F^{2k} = L(F, \frac{1}{2})L(F \times \chi_d, \frac{1}{2})L(\chi_d, 1)^{-1}\gamma(F) \tag{5.4.4}$$

in terms of spinor L-functions, where $\gamma(F) > 0$ is "well-behaved", in particular

$$\sum_{F \in \mathcal{S}^*_{2k}} \gamma(F)$$

has a positive limiting average value as $k \to +\infty$, and $\gamma(F)$ is asymptotically independent of the central special L-values.

In terms of Fourier coefficients, this hypothesis is equivalent to the following specific variant of Böcherer's conjecture: for all $F \in \mathcal{S}^*_{2k}$ that is not a Saito-Kurokawa lift, we should have

$$L(F, \frac{1}{2})L(F \times \chi_d, \frac{1}{2}) = 4\pi c_{2k}\gamma(F)^{-1}(d/4)^{1-2k}w(-d)^{-2}\frac{|a(d, 1; F)|^2}{\langle F, F\rangle}. \tag{5.4.5}$$

Remark 5.10. Such a formulation (involving two central values, or in other words a central value for the base-change of F to the quadratic field $\mathbb{Q}(\sqrt{-d})$) is strongly suggested by [16, (1.4)] and [39]. It is also compatible with a conjecture of Prasad and Takloo–Bighash [39], which itself is an analogue (for the case of Bessel periods) of the remarkable Ichino-Ikeda conjecture [21] dealing with $(SO(n), SO(n-1))$ periods. In this context, it is also worth mentioning that the question of vanishing of the Bessel period, i.e., the vanishing of $a(d, \Lambda; F)$, is closely tied with the Gross-Prasad conjecture for $(SO(5), SO(2))$.

Under our stated hypothesis (5.4.4), we consider the crucial average

$$\sum_{F \in \mathcal{S}^*_{2k}} \omega_F^{2k} c(F, p)$$

for a fixed prime p. Our goal is to show that this allows us to recover naturally the formula (5.4.2) from which the "mock-symplectic" symmetry-type arose in the proof of Theorem 1.2 (the contribution of p^2 was consistent with the expected orthogonal symmetry type). Thus, assuming (5.4.4), we need to compute the average

$$\sum_{F \in \mathcal{S}^\sharp_{2k}} \gamma(F)L(F, \frac{1}{2})L(F \times \chi_d, \frac{1}{2})L(\chi_d, 1)^{-1}c(F, p),$$

Since $c(F, p) = \lambda(F, p)$ is also the p-th coefficient of the Dirichlet series $L(F, s)$, and since the analytic conductor of both L-functions is about k^2, we see by applying a suitable Approximate Functional Equation (and recalling that the sign of the functional equation is 1 for both L-functions) that this is roughly

$$2L(\chi_d, 1)^{-1} \sum_{m,n \leqslant k} \frac{\chi_d(n)}{\sqrt{mn}} \sum_{F \in \mathcal{S}^\sharp_{2k}} \gamma(F)\lambda(F, m)\lambda(F, n)\lambda(F, p)$$

(the sums should involve a smooth cutoff).

Under the (reasonable) assumption that the coefficients of the Dirichlet series are asymptotically orthogonal under this average,[8] one is led to the guess that the terms which contribute are those with $m = np$ or $n = mp$, and thus one should have

$$\sum_{F \in \mathcal{S}^\sharp_{2k}} \gamma(F)L(F, \frac{1}{2})L(F \times \chi_d, \frac{1}{2})L(\chi_d, 1)^{-1}c(F, p) \approx \frac{1 + \chi_d(p)}{\sqrt{p}} = \frac{\lambda_p}{\sqrt{p}},$$

[8] This depends on the hypothesis that $\gamma(F)$ is innocuous.

as $k \to +\infty$, where the $L(1, \chi_d)$ has cancelled out with

$$\sum_m \frac{\chi_d(m)}{\sqrt{m}} \approx L(1, \chi_d)$$

(again with a smooth cutoff).

But this is exactly what we proved in (5.4.2), and what led to Theorem 1.2. We therefore interpret this as a (global, averaged) confirmation of Böcherer's conjecture in the form (5.4.5).

Appendix: comparison with GL(2)-families

This section is intended to summarize some basic facts about holomorphic Siegel modular forms and their adélic counterparts, by comparison with the case of classical modular forms for congruence subgroups of $\mathrm{SL}(2, \mathbb{Z})$. We also give references for the $\mathrm{SL}(2)$-analogues of the results in this paper.

- The closest analogue of our family of cusp forms is the set $\mathcal{H}_k^*$ of primitive holomorphic cusp forms of weight k for $\mathrm{SL}(2, \mathbb{Z})$, with trivial nebentypus, counted according to the weight given by

  $$\omega_f = \frac{\Gamma(k-1)}{(4\pi)^{k-1}} \frac{1}{\langle f, f\rangle}.$$

  In contrast with $\mathcal{S}_k^*$, this set is the unique Hecke-eigenbasis of the space $\mathcal{H}_k$ of cusp forms of weight k and level 1; in our context, the corresponding multiplicity one theorem is not known (because the Fourier coefficients are not functions of the Hecke eigenvalues), and so the Hecke basis $\mathcal{S}_k^*$ is not necessarily unique in $\mathcal{S}_k$.

  Another obvious distinction is the presence of the Fourier coefficient $a(F, 1)$ in (1.1.1). As we saw, this has crucial arithmetic content. A way to insert this aspect "by hand" into the classical case is to consider the twisted weights

  $$\omega'_f = \alpha\omega_f L(f, 1/2), \quad \text{or} \quad \omega'_f = \alpha\omega_f L(f, 1/2)L(f \times \chi_d, 1/2),$$

  where $\alpha > 0$ is a constant such that

  $$\sum_{f \in \mathcal{H}_k} \omega'_f \to 1,$$

  as $k \to +\infty$. (The existence of the limit that makes this normalization possible is essentially already in Duke's paper [12, Prop. 2], where the limit is with respect to the level.)
- The local equidistribution theorem for $\mathcal{H}_k^*$, as $k \to +\infty$, is the following: for any prime p, the map sending f to the p-component of the associated automorphic representation of $\mathrm{GL}(2, \mathbb{A})$ can be identified with the map

  $$p \mapsto \lambda_f(p) \in \mathbb{R}$$

  where $p^{\frac{k-1}{2}}\lambda_f(p)$ is the p-th Hecke eigenvalue, or equivalently the p-th Fourier coefficient. By Hecke's bound, we have $\lambda_f(p) \in [-2\sqrt{p}, 2\sqrt{p}]$, and the associated representation of $\mathrm{GL}(2, \mathbb{Q}_p)$ is the unramified principal series obtained from the unramified characters α, β of $\mathbb{Q}_p^\times$ such that

  $$\alpha(p)\beta(p) = 1, \qquad \alpha(p) + \beta(p) = \lambda_f(p).$$

  Then, for any finite set of primes $\boldsymbol{S}$, the measures $m_{\boldsymbol{S},k}$ defined as the sum of Dirac measures at $\lambda_f(p)$ for $p \in \mathcal{H}_k^*$ converge weakly to the measure

  $$n_{\boldsymbol{S}} = \prod_{p \in \boldsymbol{S}} \mu_{ST}$$

where μ_{ST} is the Sato-Tate measure, supported on $[-2,2]$, given there by

$$\frac{2}{\pi}\sqrt{1-x^2/4}dx.$$

- The above fact is quite easy to prove. First, the Hecke relations describe $\lambda_f(L)$ in terms of the factorization of $L \geqslant 1$, namely

$$\lambda_f(L) = \prod_{p|L} U_{l_p}(\lambda_f(p)),$$

where U_l is the l-th Chebychev polynomial defined by

$$U_l(2\cos\theta) = \frac{\sin((l+1)\theta)}{\sin\theta}.$$

These form a basis of polynomials in one variable, and hence span a dense subset of $C([-2\sqrt{p}, 2\sqrt{p}])$, with

$$\int U_l(x) d\mu_{ST}(x) = \delta(l,1).$$

- The second ingredient is the Petersson formula; indeed, for any $l_p \geqslant 0$, let

$$L = \prod_{p\in \boldsymbol{S}} p^{l_p},$$

and then we have

$$\sum_{f\in\mathcal{H}_k^*} \omega_f \prod_{p\in\boldsymbol{S}} U_{l_p}(\lambda_f(p)) = \sum_{f\in\mathcal{H}_k^*} \omega_f\lambda_f(L)$$
$$= \delta(L,1) - 2\pi i^{-k}\sum_{c\geqslant 1} c^{-1}S(L,1;c)J_{k-1}\Big(\frac{4\pi\sqrt{L}}{c}\Big) \longrightarrow \delta(L,1)$$

as $k\to+\infty$, where $S(L,1;c)$ denotes the standard Kloosterman sum. This gives the local equidistribution statement. Note that, in contrast with our results, the limiting measure at p is independent of p in this case.
- The Hecke relations are analogues of Sugano's formula (Theorem 2.5); the Petersson formula and the related orthogonality are the analogues of Proposition 3.6. On the other hand, the necessary work to go from Fourier coefficients (controlled by Poincaré series) to Hecke eigenvalues is completely absent from the classical case.
- Analogues of the direct applications of Section 5.1 were proved first, essentially, by Bruggeman [5] (analogue of Proposition 5.1 for Maass forms, where the Ramanujan-Petersson conjecture is not yet known); analogues of Proposition 5.2 are due to Sarnak [43] (Maass forms) and Serre (holomorphic forms), though both counted the cusp forms with the natural weight 1, and used the trace formula instead of the Petersson formula (correspondingly, their limiting distributions was different: at p they obtained the Plancherel measure for the unramified principal series of $\mathrm{GL}(2,\mathbb{Q}_p)$ with trivial central character).
- Computations tantamount to working with the twisted weight ω'_f are also classical (in particular, the computation of

$$\sum_f \omega'_f\lambda_f(m)$$

for a fixed m is a special case of the first moment computation in [30], in the case where the level goes to infinity and the Rankin-Selberg convolution is the weight 1 theta series with L-function $\zeta(s)L(s,\chi_4)$).

- The analogue of Böcherer's conjecture for $\mathcal{H}_k^*$ is the famous formula of Waldspurger [49] which relates the value $L(f \times \chi_d, \frac{1}{2})$ for $f \in \mathcal{H}_k^*$ to the squares of Fourier coefficients of modular forms of half-integral weight. However, these special values do not appear in the standard weights used for averaging L-functions. However, a weighted averaged version of Waldspurger's formula was proved by Iwaniec [22] using identities for Kloosterman sums, and this may be considered as somewhat analogue to our Theorem 1.1.
- The Saito-Kurokawa forms have no analogue in $\mathcal{H}_k^*$. Indeed, all cusp forms of GL(2) (or GL(n)) are expected to satisfy the Ramanujan-Petersson conjecture; for forms in $\mathcal{H}_k^*$, this is a theorem of Deligne [10]. On the other hand, Saito-Kurokawa forms do not satisfy the generalized Ramanujan conjecture; this is due to the fact that they are CAP representations [36].

References

[1] Anatoli N. Andrianov. Euler products that correspond to Siegel's modular forms of genus 2. *Uspehi Mat. Nauk*, 29(3 (177)):43–110, 1974.

[2] Mahdi Asgari and Ralf Schmidt. Siegel modular forms and representations. *Manuscripta Math.*, 104(2):173–200, 2001.

[3] Siegfried Böcherer. Bemerkungen über die Dirichletreihen von Koecher und Maass. *Mathematica Gottingensis*, 68:36 pp., 1986.

[4] Jim Brown. An inner product relation on Saito-Kurokawa lifts. *Ramanujan J.*, 14(1):89–105, 2007.

[5] Roelof Bruggeman. Fourier coefficients of cusp forms. *Invent. math.*, 45:1–18, 1978.

[6] Daniel Bump, Solomon Friedberg, and Masaaki Furusawa. Explicit formulas for the Waldspurger and Bessel models. *Israel J. Math.*, 102:125–177, 1997.

[7] Pierre Cartier. Representations of p-adic groups: a survey. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1*, Proc. Sympos. Pure Math., XXXIII, pages 111–155. Amer. Math. Soc., Providence, R.I., 1979.

[8] Laurent Clozel. On limit multiplicities of discrete series representations in spaces of automorphic forms. *Invent. math.*, 83(2):265–284, 1986.

[9] David L. DeGeorge and Nolan R. Wallach. Limit formulas for multiplicities in $L^2(\Gamma\backslash G)$. II. The tempered spectrum. *Ann. of Math. (2)*, 109(3):477–495, 1979.

[10] Pierre Deligne. La conjecture de Weil. I. *Inst. Hautes Études Sci. Publ. Math.*, (43):273–307, 1974.

[11] Eduardo Dueñez and Steven J. Miller. The low lying zeros of a $GL(4)$ and a $GL(6)$ family of L-functions. *Compositio Mathematica*, 142:1403—-1425, 2006.

[12] William Duke. The critical order of vanishing of automorphic L-functions with large level. *Invent. math.*, 119:165 – 174, 1995.

[13] Martin Eichler and Don Zagier. *The theory of Jacobi forms*, volume 55 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1985.

[14] William Fulton and Joe Harris. *Representation theory*, volume 129 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991. A first course, Readings in Mathematics.

[15] Masaaki Furusawa. On L-functions for $\mathrm{GSp}(4) \times \mathrm{GL}(2)$ and their special values. *J. Reine Angew. Math.*, 438:187–218, 1993.

[16] Masaaki Furusawa and Kimball Martin. On central critical values of the degree four L-functions for GSp(4): The fundamental lemma, II. *American J. Math., to appear*, 2010.

[17] Masaaki Furusawa, Kimball Martin, and Joseph Shalika. On central critical values of the degree four L-functions for GSp(4): The fundamental lemma III. Forthcoming.

[18] Masaaki Furusawa and Joseph A. Shalika. On inversion of the Bessel and Gelfand transforms. *Trans. Amer. Math. Soc.*, 354(2):837–852 (electronic), 2002.

[19] Masaaki Furusawa and Joseph A. Shalika. *On central critical values of the degree four L-functions for* GSp(4), volume 782 of *Memoirs of the American Mathematical Society*. American Mathematical Society, Providence, 2003.

[20] Jeffrey Hoffstein and Dinakar Ramakrishnan. Siegel zeros and cusp forms. *International Math. Research Notices*, (6):279–308, 1995.

[21] Atsushi Ichino and Tamotsu Ikeda. On the periods of automorphic forms on special orthogonal groups and the Gross-Prasad conjecture. *Geom. Funct. Anal.*, 19(5):1378–1425, 2010.

[22] Henryk Iwaniec. On Waldspurger's theorem. *Acta Arith.*, 49(2):205–212, 1987.

[23] Henryk Iwaniec and Emmanuel Kowalski. *Analytic number theory*, volume 53 of *Colloquium Publications*. American Mathematical Society, Providence, 2004.
[24] Henryk Iwaniec, Wenzhi Luo, and Peter Sarnak. Low lying zeros of families of L-functions. *Publications Mathématiques de l'I.H.É.S*, 91:55–131, 2000.
[25] Nicholas M. Katz and Peter Sarnak. *Random matrices, Frobenius eigenvalues and symmetry*, volume 45 of *Colloquium Publications*. American Mathematical Society, Providence, 1999.
[26] Yoshiyuki Kitaoka. Fourier coefficients of Siegel cusp forms of degree two. *Nagoya Math. J.*, 93:149–171, 1984.
[27] Helmut Kligen. *Introductory lectures on Siegel modular forms*, volume 20 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1990.
[28] Winfried Kohnen. On Poincaré series of exponential type on Sp_2. *Abh. Math. Sem. Univ. Hamburg*, 63:283–297, 1993.
[29] E. Kowalski. Families of cusp forms. *Preprint*, 2011.
[30] Emmanuel Kowalski, Philippe Michel, and Jeffrey VanderKam. Rankin-selberg L-functions in the level aspect. *Duke Mathematical Journal*, 114:123 – 191, 2002.
[31] Emmanuel Kowalski, Abhishek Saha, and Jacob Tsimerman. A note on Fourier coefficients of Poincaré series. *Mathematika*, 57(1):31–40, 2011.
[32] Ian G. Macdonald. Orthogonal polynomials associated with root systems. *Sém. Lothar. Combin.*, 45:Art. B45a, 40 pp. (electronic), 2000/01.
[33] S. J. Miller. 1- and 2-level densities for rational families of elliptic curves: evidence for the underlying group symmetries. *Composition Math.*, 140(4):952–994, 2004.
[34] Shin-ichiro Mizumoto. Fourier coefficients of generalized Eisenstein series of degree two. I. *Invent. math.*, 65(1):115–135, 1981/82.
[35] Mark E. Novodvorski and Ilya I. Piatetski-Shapiro. Generalized Bessel models for the symplectic group of rank 2. *Mat. Sb. (N.S.)*, 90(132):246–256, 326, 1973.
[36] Ilya I. Piatetski-Shapiro. On the Saito-Kurokawa lifting. *Invent. math.*, 71:309–338, 1983.
[37] Ameya Pitale. Steinberg representation of $GSp(4)$: Bessel models and integral representation of L-functions. *Pacific J. Math.*, 250(2):365–406, 2011.
[38] Ameya Pitale and Ralf Schmidt. Ramanujan-type results for Siegel cusp forms of degree 2. *J. Ramanujan Math. Soc.*, 24(1):87–111, 2009.
[39] Dipendra Prasad and Ramin Takloo-Bighash. Bessel models for GSp(4). *J. Reine Angew. Math.*, 2011:189–243.
[40] Dinakar Ramakrishnan and Jonathan Rogawski. Average values of modular L-series via the relative trace formula. *Pure Appl. Math. Q.*, 1(4, part 3):701–735, 2005.
[41] Abhishek Saha. L-functions for holomorphic forms on $\mathrm{GSp}(4) \times \mathrm{GL}(2)$ and their special values. *Int. Math. Res. Not. IMRN*, 2009(10):1773–1837.
[42] Abhishek Saha. *On critical values of L-functions for holomorphic forms on $GSp(4) \times GL(2)$*. PhD thesis, Caltech, 2009. Available at "`http://thesis.library.caltech.edu/`".
[43] Peter Sarnak. Statistical properties of eigenvalues of the Hecke operator. In *Analytic Number Theory and Diophantine Problems*, volume 60 of *Progress in Mathematics*, pages 75–102. Birkäuser, Boston, 1987.
[44] François Sauvageot. Principe de densité pour les groupes réductifs. *Compositio Math.*, 108(2):151–184, 1997.
[45] Gordan Savin. Limit multiplicities of cusp forms. *Invent. Math.*, 95(1):149–159, 1989.
[46] Jean-Pierre Serre. Répartition asymptotique des valeurs propres de l'opérateur de Hecke T_p. *Journal A.M.S*, 10(1):75–102, 1997.
[47] Sug Woo Shin. Plancherel density theorem for automorphic representations. *Preprint*, 2010.
[48] Takashi Sugano. On holomorphic cusp forms on quaternion unitary groups of degree 2. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 31(3):521–568, 1985.
[49] Jean-Loup Waldspurger. Sur les coefficients de Fourier des formes modulaires de poids demi-entier. *J. Math. Pures Appl. (9)*, 60(4):375–484, 1981.
[50] Rainer Weissauer. *Endoscopy for* GSp(4) *and the cohomology of Siegel modular threefolds*, volume 1968 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2009.

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