In this section of* Resonance, *we invite readers to pose questions likely to be raised in a classroom situation. We may suggest strategies for dealing with them, or invite responses, or both. "Classroom" is equally a forum for raising broader issues and sharing personal experiences and viewpoints on matters related to teaching and learning science.

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Sums of Powers of the Primitive Roots of a Prime

In [1], the construction of regular polygons by a ruler and a compass is discussed. In the last section of the article, the notion of cyclotomic polynomials is employed to evaluate the sum of the primitive roots of a prime p. This turns out to be $\mu(p-1)$ where μ is the Möbius function. The general question of evaluating the sum of the m-th powers of the primitive roots is also raised. Here, we answer this question in an elementary manner. Recall that a natural number a is a primitive root of a prime p if $p-1$ is the smallest natural number for which $a^{p-1} \equiv 1 \bmod p$. Let $1 \leq r_1, r_2, \ldots, r_k \leq p-1$ be the integers that are co-prime to $p-1$. Then if w is a primitive root of p, we know that $w^{r_1}, w^{r_2}, \ldots, w^{r_k}$ are all the primitive roots.

We wish to evaluate the sum $S = \sum_{i=1}^{k} (w^{r_i})^m$. Let us note that as primitive roots are defined only modulo p, this sum will be evaluated only modulo p.

Keywords
Primitive roots, inclusion-exclusion principle.

Here and elsewhere in this proof, we write $a = b$ to mean $a \equiv b \bmod p$. Thus S is simply the congruence class modulo p to which $\sum_{i=1}^{k}(w^{r_i})^m$ belongs.

Let us start with the useful observation (here and elsewhere (a,b) denotes the GCD of two natural numbers):

Lemma. *For an integer q, let $(p-1,q) = d$. Then, if t divides $p-1$,*

$$\sum_{\ell=1}^{(p-1)/t} w^{tq\ell} = \begin{cases} 0 & \text{if } \frac{p-1}{d} \nmid t \\ \frac{p-1}{t} & \text{if } \frac{p-1}{d} | t \end{cases}$$

Proof.

$$w^{tq} = 1 \Leftrightarrow p-1|tq \Leftrightarrow \frac{p-1}{d}|t$$

In this case $\sum_{\ell=1}^{(p-1)/t} w^{tq\ell} = 1 + 1 + \ldots + 1 = \frac{p-1}{t}$.

If $w^{tq} \neq 1$, then

$$\begin{aligned}\sum_{\ell=1}^{(p-1)/t} w^{tq\ell} &= w^{tq} + w^{2tq} + \ldots + w^{(p-1)q} \\ &= \frac{w^{tq}(w^{(tq)\cdot\frac{p-1}{t}} - 1)}{w^{tq} - 1} \\ &= 0.\end{aligned}$$

We shall prove:

Theorem. *The sum S of m-th powers of primitive roots for p is given by $S = \mu(g)\frac{\phi(p-1)}{\phi(g)}$ where $g = \frac{p-1}{(m,p-1)}$.*

Here ϕ and μ are Euler's phi function and the Möbius function respectively. We shall evaluate S by using the inclusion-exclusion principle.

Proof. Let $p_1, p_2, \ldots, p_s$ be the various distinct prime divisors of $p-1$. Thus

$$S = \sum_{i=1}^{k}(w^{r_i})^m = \sum_{i=1}^{p-1} w^{im} - \sum_{j=1}^{s}\sum_{i=1}^{\frac{p-1}{p_j}}(w^{ip_j})^m$$

$$+\sum_{j_1<j_2}\sum_{i=1}^{(p-1)/(p_{j_1}p_{j_2})}(w^{ip_{j_1}p_{j_2}})^m$$

$$-\ldots+(-1)^u\sum_{j_1<\ldots<j_u}\sum_{i=1}^{(p-1)/(p_{j_1}p_{j_2}\cdots p_{j_u})}(w^{ip_{j_1}p_{j_2}\cdots p_{j_u}})^m$$

$$\pm\ldots+(-1)^s\sum_{i=1}^{(p-1)/(p_1p_2\cdots p_s)}(w^{ip_1p_2\cdots p_s})^m \qquad \spadesuit$$

The above equality is deduced as follows. Let $T = \{1,2,\ldots,p-1\}$ and let T_f denote the subset of T consisting of those integers from T which are divisible by f. Then by the inclusion-exclusion principle, one gets:

$$S=\sum_{(x,p-1)=1}(w^x)^m=\sum_{x\in T}(w^x)^m-\{\sum_{x\in T_{p_1}}(w^x)^m+\ldots$$

$$+\sum_{x\in T_{p_s}}(w^x)^m\}+\sum_{i<j}\sum_{x\in(T_{p_i}\cap T_{p_j})}(w^x)^m-\ldots$$

$$+(-1)^s\sum_{x\in(T_{p_1}\cap T_{p_2}\cap\ldots\cap T_{p_s})}(w^x)^m.$$

Finally, as it is clear that

$$\sum_{x\in(T_{p_{j_1}}\cap T_{p_{j_2}}\cap\ldots\cap T_{p_{j_u}})}(w^x)^m=\sum_{i=1}^{(p-1)/(p_{j_1}p_{j_2}\cdots p_{j_u})}(w^{ip_{j_1}p_{j_2}\cdots p_{j_u}})^m$$

we obtain the expression ♠ for S.

Now $\{p_1,p_2,\ldots,p_s\}$ is the set of all prime divisors of $p-1$. Consider its subset $\{p_1,p_2,\ldots,p_t\}$, the set of prime divisors of $g=\frac{p-1}{(m,p-1)}$. Then, by the lemma, a sum of the form $\sum_{i=1}^{(p-1)/(p_{j_1}p_{j_2}\cdots p_{j_u})}(w^{ip_{j_1}p_{j_2}\cdots p_{j_u}})^m$ is not equal to 0 if and only if $g|p_{j_1}p_{j_2}\ldots p_{j_k}$. Clearly this happens only if g is squarefree. Assume g is squarefree; then $g=p_1p_2\ldots p_t$. So, in evaluating S, we only have to find the sum of all terms of the form

$$(-1)^u\sum_{i=1}^{(p-1)/(p_{j_1}p_{j_2}\cdots p_{j_u})}(w^{ip_{j_1}p_{j_2}\cdots p_{j_u}})^m$$

where $\{1,2,\ldots,t\} \subseteq \{j_1,j_2,\ldots,j_u\}$. But, the lemma gives us

$$\sum_{i=1}^{(p-1)/(p_{j_1}p_{j_2}\cdots p_{j_u})} (w^{ip_{j_1}p_{j_2}\cdots p_{j_u}})^m = \frac{p-1}{p_{j_1}p_{j_2}\cdots p_{j_u}}$$

whenever $\{1,2,\ldots,t\} \subseteq \{j_1,j_2,\ldots j_u\}$. Hence, we have

$$S = (-1)^t \frac{p-1}{p_1p_2\cdots p_t} +$$

$$(-1)^{t+1}\left[\frac{p-1}{p_1p_2\cdots p_tp_{t+1}} + \ldots + \frac{p-1}{p_1p_2\cdots p_tp_s}\right]$$

$$+(-1)^{t+2}\left[\frac{p-1}{p_1p_2\cdots p_t}\left(\frac{1}{p_{t+1}p_{t+2}} + \frac{1}{p_{t+1}p_{t+3}} + \ldots + \frac{1}{p_{s-1}p_s}\right)\right]$$

$$\pm \ldots + (-1)^s \frac{p-1}{p_1p_2\cdots p_s}$$

$$= (-1)^t \frac{p-1}{p_1p_2\cdots p_t}\left[1 - \left(\frac{1}{p_{t+1}} + \ldots + \frac{1}{p_s}\right)\right.$$

$$\left. + \left(\frac{1}{p_{t+1}p_{t+2}} + \ldots + \frac{1}{p_{s-1}p_s}\right) + \ldots + \frac{(-1)^{s-t}}{p_{t+1}p_{t+2}\cdots p_s}\right]$$

$$= (-1)^t \frac{p-1}{p_1p_2\cdots p_t}\left(1 - \frac{1}{p_{t+1}}\right)\left(1 - \frac{1}{p_{t+2}}\right)\ldots\left(1 - \frac{1}{p_s}\right)$$

$$= (-1)^t\left(\frac{p-1}{p_1p_2\cdots p_t}\right)\frac{(1-\frac{1}{p_1})(1-\frac{1}{p_2})\ldots(1-\frac{1}{p_s})}{(1-\frac{1}{p_1})(1-\frac{1}{p_2})\ldots(1-\frac{1}{p_t})}$$

$$= (-1)^t\left(\frac{p-1}{p_1p_2\cdots p_t}\right)\frac{\phi(p-1)/(p-1)}{\phi(g)/g} = \mu(g)\frac{\phi(p-1)}{\phi(g)}$$

since $g = p_1 \cdots p_t$ and $(-1)^t = \mu(g)$).

Thus whenever g is squarefree, $S = \frac{\mu(g)\phi(p-1)}{\phi(g)}$. But, if g is not squarefree, g cannot divide $p_{j_1}p_{j_2}\cdots p_{j_u}$; so each term of ♠ is 0 and $S = 0$. Also $\mu(g)\frac{\phi(p-1)}{\phi(g)} = 0$ if g is not squarefree. Therefore, in all cases $S = \mu(g)\frac{\phi(p-1)}{\phi(g)}$.

Suggested Reading

[1] B Sury, Cyclotomy and cyclotomic polynomials, *Resonance*, Vol.4, No.12, 1999.