A NOTE ON FOURIER COEFFICIENTS OF POINCARÉ SERIES

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ABSTRACT. We give a short and "soft" proof of the asymptotic orthogonality of Fourier coefficients of Poincaré series for classical modular forms as well as for Siegel cusp forms, in a qualitative form.

1. INTRODUCTION

The Petersson formula (see, e.g., [2, Ch. 14]) is one of the most basic tools in the analytic theory of modular forms on congruence subgroups of $\mathrm{SL}(2, \mathbf{Z})$. One of its simplest consequences, which explains its usefulness, is that it provides the asymptotic orthogonality of distinct Fourier coefficients for an orthonormal basis in a space of cusp forms, when the analytic conductor is large (e.g., when the weight or the level is large). From the proof of the Petersson formula, we see that this orthonormality is equivalent (on a qualitative level) to the assertion that the n-th Fourier coefficient of the m-th Poincaré series is essentially the Kronecker symbol $\delta(m, n)$.

In this note, we provide a direct "soft" proof of this fact in the more general context of Siegel modular forms when the main parameter is the weight k. Although this is not sufficient to derive the strongest applications (e.g., to averages of L-functions in the critical strip), it provides at least a good motivation for the more quantitative orthogonality relations required for those. And, as we show in our paper [4] concerning the local spectral equidistribution of Satake parameters for certain families of Siegel modular forms of genus $g = 2$, the "soft" proof suffices to derive some basic consequences, such as the analogue of "strong approximation" for cuspidal automorphic representations, and the determination of the conjectural "symmetry type" of the family. See Corollary 3 for a simple example of this when $g = 1$.

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2. CLASSICAL MODULAR FORMS

In this section, we explain the idea of our proof for classical modular forms; we hope this will be useful as a comparison point in the next section,

especially for readers unfamiliar with Siegel modular forms. Let $k \geqslant 2$ be an even integer, $m \geqslant 1$ an integer. The m-th Poincaré series of weight k is defined by

$$P_{m,k}(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} (cz+d)^{-k} e(m\gamma \cdot z),$$

where $\Gamma = SL(2, \mathbf{Z})$, acting on the Poincaré upper half-plane $\mathbf{H}$,

$$\Gamma_\infty = \left\{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbf{Z} \right\}$$

is the stabilizer of the cusp at infinity, and we write

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad (a, b, c, d) \in \mathbf{Z}^4.$$

It is well known that for $k \geqslant 4$, $m \geqslant 1$, this series converges absolutely and uniformly on compact sets, and that it defines a cusp form of weight k for $\Gamma = \mathrm{SL}(2, \mathbf{Z})$. We denote by $p_{m,k}(n)$, $n \geqslant 1$, the Fourier coefficients of this Poincaré series, so that

$$P_{m,k}(z) = \sum_{n \geqslant 1} p_{m,k}(n) e(nz)$$

for all $z \in \mathbf{H}$.

Proposition 1 (Asymptotic orthogonality of Fourier coefficients of Poincaré series)**.** *With notation as above, for fixed $m \geqslant 1$, $n \geqslant 1$, we have*

$$\lim_{k \to +\infty} p_{m,k}(n) = \delta(m, n).$$

Proof. The idea is to use the definition of Fourier coefficients as

$$p_{m,k}(n) = \int_U P_{m,k}(z) e(-nz) dz$$

where U is a suitable horizontal interval of length 1 in $\mathbf{H}$, and dz is the Lebesgue measure on such an interval; we then let $k \to +\infty$ under the integral sign, using the definition of the Poincaré series to understand that limit.

We select

$$U = \{x + iy_0 \mid |x| \leqslant 1/2\}$$

for some fixed $y_0 > 1$. The Lebesgue measure is then of course dx.

Consider a term

$$(cz+d)^{-k} e(m\gamma \cdot z)$$

in the Poincaré series as $k \to +\infty$. We have

$$\left| (cz+d)^{-k} e(m\gamma \cdot z) \right| \leqslant |cz+d|^{-k}$$

for all $z \in \mathbf{H}$ and $\gamma \in \mathrm{SL}(2, \mathbf{Z})$, since $m \geqslant 0$ and $\gamma \cdot z \in \mathbf{H}$. But for $z \in U$, we find

(1) $$|cz+d|^2 = (cx+d)^2 + c^2 y_0^2 \geqslant c^2 y_0^2.$$

If $c \neq 0$, since c is an integer, the choice of $y_0 > 1$ leads to $c^2 y_0^2 > 1$, and hence

$$\left| (cz+d)^{-k} e(m\gamma \cdot z) \right| \to 0$$

as $k \to +\infty$, uniformly for $z \in U$ and $\gamma \in \Gamma$ with $c \neq 0$. On the other hand, if $c = 0$, we have $\gamma \in \Gamma_\infty$; this means this corresponds to a single term which we take to be $\gamma = \mathrm{Id}$, and we then have

$$(cz+d)^{-k} e(m\gamma \cdot z) = e(mz)$$

for all k and $z \in U$.

Moreover, all this shows also that

$$\left| (cz+d)^{-k} e(m\gamma \cdot z) \right| \leqslant |cz+d|^{-4}$$

for $k \geqslant 4$ and $\gamma \in \Gamma_\infty \backslash \Gamma$. Since the right-hand converges absolutely and uniformly on compact sets, we derive by dominated convergence that

$$P_{m,k}(z) \to e(mz)$$

for all $z \in U$. The above inequality gives further

$$|P_{m,k}(z)| \leqslant \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} |cz+d|^{-4}$$

for $k \geqslant 4$ and $z \in U$. Since U is compact, we can integrate by dominated convergence again to obtain

$$\int_U P_{m,k}(z) e(-nz) dz \longrightarrow \int_U e((m-n)z) dz = \delta(m,n)$$

as $k \to +\infty$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

It turns out that the same basic technique works for the other most important parameter of cusp forms, the level. For $q \geqslant 1$ and $m \geqslant 1$ integers, let now

$$P_{m,q}(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(q)} (cz+d)^{-k} e(m\gamma \cdot z)$$

be the m-th Poincaré series of weight k for the Hecke group $\Gamma_0(q)$, and let $p_{m,q}(n)$ denote its Fourier coefficients.

Proposition 2 (Orthogonality with respect to the level). *With notation as above, for $k \geqslant 4$ fixed, for any fixed m and n, we have*

$$\lim_{q \to +\infty} p_{m,q}(n) = \delta(m,n).$$

Proof. We start with the integral formula

$$p_{m,q}(n) = \int_U P_{q,m}(z) e(-nz) dz$$

as before. To proceed, we observe that $\Gamma_\infty \backslash \Gamma_0(q)$ is a subset of $\Gamma_\infty \backslash \Gamma$, and hence we can write

$$P_{m,q}(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \Delta_q(\gamma)(cz+d)^{-k}e(m\gamma \cdot z),$$

where

$$\Delta_q\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{cases} 1 & \text{if } c \equiv 0 \, (\mathrm{mod} \, q), \\ 0 & \text{otherwise.} \end{cases}$$

We let $q \to +\infty$ in each term of this series. Clearly, we have $\Delta_q(\gamma) = 0$ for all $q > c$, unless if $c = 0$, in which case $\Delta_q(\gamma) = 1$. Thus

$$\Delta_q(\gamma)(cz+d)^{-k}e(m\gamma \cdot z) \to 0$$

if $c \neq 0$, and otherwise

$$\Delta_q(\gamma)(cz+d)^{-k}e(m\gamma \cdot z) = e(mz).$$

Moreover, we have obviously

$$\left| \Delta_q(\gamma)(cz+d)^{-k}e(m\gamma \cdot z) \right| \leqslant |cz+d|^{-k}$$

and since $k \geqslant 4$, this defines an absolutely convergent series for all z. We therefore obtain

$$P_{m,q}(z) \to e(mz)$$

for any $z \in U$. Finally, the function

$$z \mapsto \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} |cz+d|^{-k}$$

being integrable on U, we obtain the result after integrating. $\square$

Here is a simple application to show that such qualitative statements are not entirely content-free:

Corollary 3 ("Strong approximation" for $GL(2)$-cusp forms)**.** *Let* $\mathbf{A}$ *be the adèle ring of* $\mathbf{Q}$*. For each irreducible, cuspidal, automorphic representation* π *of* $GL(2, \mathbf{A})$ *and each prime* p*, let* π_p *be the unitary, admissible representation of* $GL(2, \mathbf{Q}_p)$ *that is the local component of* π *at* p*. Then, for any finite set of primes* S*, as* π *runs over the cuspidal spectrum of* $GL(2, \mathbf{A})$ *unramified at primes in* S*, the set of tuples* $(\pi_p)_{p \in S}$ *is dense in the product over* $p \in S$ *of the unitary tempered unramified spectrum* X_p *of* $GL(2, \mathbf{Q}_p)$*.*

Proof. This is already known, due to Serre [6] (if one uses holomorphic forms) or Sarnak [8] (using Maass forms), but we want to point out that this is a straightforward consequence of Proposition 1; for more details, see the Appendix to [4]. We first recall that the part of unitary unramified spectrum of $GL(2, \mathbf{Q}_p)$ with trivial central character can be identified with $[-2\sqrt{p}, 2\sqrt{p}]$ via the map sending Satake parameters (α, β) to $\alpha + \beta$. The subset X_p can then be identified with $[-2, 2]$, and for $\pi = \pi(f)$ attached to a cuspidal primitive form unramified at p, the local component $\pi_p(f)$ corresponds to the normalized Hecke eigenvalue $\lambda_f(p)$.

Now the (well-known) point is that for any integer of the form

$$m = \prod_{p \in S} p^{n(p)} \geqslant 1,$$

and any cusp form f of weight k with Fourier coefficients $n^{(k-1)/2}\lambda_f(n)$, the characteristic property

$$\langle f, P_{m,k}(\cdot) \rangle = \frac{\Gamma(k-1)}{(4\pi m)^{k-1}} m^{(k-1)/2} \lambda_f(m)$$

of Poincaré series (see, e.g., [2, Lemma 14.3]) implies that

$$p_{m,k}(1) = \sum_{f \in H_k} \omega_f \lambda_f(m) = \sum_{f \in H_k} \omega_f \prod_{p \in S} U_{n(p)}(\lambda_f(p)), \quad \omega_f = \frac{\Gamma(k-1)}{(4\pi)^{k-1}} \frac{1}{\|f\|^2},$$

where H_k is the Hecke basis of weight k and level 1, U_n denotes Chebychev polynomials, and $\|f\|$ is the Petersson norm. Because the linear combinations of Chebychev polynomials are dense in $C([-2p^{1/2}, 2p^{1/2}])$ for any prime p, the fact that

$$\lim_{k \to +\infty} p_{m,k}(1) = \delta(m, 1) = \begin{cases} 1 & \text{if all } n(p) \text{ are zero,} \\ 0 & \text{otherwise,} \end{cases}$$

(given by Proposition 1) shows, using the Weyl equidistribution criterion, that $(\pi_p(f))_{f \in H_k}$, when counted with weight ω_f, becomes equidistributed as $k \to +\infty$ with respect to the product of Sato-Tate measures over $p \in S$. Since each factor has support equal to $[-2, 2] = X_p$, this implies trivially the result. $\qquad \square$

3. Siegel modular forms

We now proceed to generalize the previous result to Siegel cusp forms; although some notation will be recycled, there should be no confusion. For $g \geqslant 1$, let $\mathbf{H}_g$ denote the Siegel upper half-space of genus g

$$\mathbf{H}_g = \{z = x + iy \in M(g, \mathbf{C}) \mid {}^t z = z, \quad y \text{ is positive definite}\},$$

on which the group $\Gamma_g = \mathrm{Sp}(2g, \mathbf{Z})$ acts in the usual way

$$\gamma \cdot z = (az + b)(cz + d)^{-1}$$

(see, e.g., [3, Ch. 1] for such basic facts; we always write

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

for symplectic matrices, where the blocks are themselves $g \times g$ matrices). Let A_g denote the set of symmetric, positive-definite matrices in $M(g, \mathbf{Z})$ with integer entries on the main diagonal and half-integer entries off it. Further, let

$$\Gamma_\infty = \left\{ \pm \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} : s \in M(g, \mathbf{Z}), \ s = {}^t s \right\}.$$

For $k \geqslant 2$, even,[1] and a matrix $s \in A_g$, the Poincaré series $\mathcal{P}_{s,k}$ is defined by

$$\mathcal{P}_{s,k}(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_g} \det(cz+d)^{-k} e(\text{Tr}(s\ (\gamma \cdot z)))$$

for z in $\mathbf{H}_g$. This series converges absolutely and uniformly on compact sets of $\mathbf{H}_g$ for $k > 2g$; indeed, as shown by Maass [5, (32), Satz 1]), the series

$$\mathcal{M}_{s,k}(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_g} |\det(cz+d)|^{-k} \exp(-2\pi \text{Tr}(s\ \text{Im}(\gamma \cdot z)))$$

which dominates it termwise converges absolutely and uniformly on compact sets (see also [3, p. 90]; note that, in contrast with the case of $SL(2, \mathbf{Z})$, one can not ignore the exponential factor here to have convergence). The Poincaré series $\mathcal{P}_{s,k}$ is then a Siegel cusp form of weight k for Γ_g. Therefore, it has a Fourier expansion

$$\mathcal{P}_{s,k}(z) = \sum_{t \in A_g} p_{s,k}(t) e(\text{Tr}(tz)),$$

which converges absolutely and uniformly on compact subsets of $\mathbf{H}_g$.

Theorem 4 (Orthogonality for Siegel-Poincaré series)**.** *With notation as above, for any fixed s, $t \in A_g$, we have*

$$\lim_{k \to +\infty} p_{s,k}(t) = \delta'(s,t) \frac{|\text{Aut}(s)|}{2},$$

where the limit is over even weights k, $\delta'(s,t)$ is the Kronecker delta for the $\text{GL}(g, \mathbf{Z})$-equivalence classes of s and t, and where $\text{Aut}(s) = O(s, \mathbf{Z})$ is the finite group of integral points of the orthogonal group of the quadratic form defined by s.

This result suggests to define the Poincaré series with an additional constant factor $2/|\text{Aut}(s)|$, in which case this theorem is exactly analogous to Proposition 1. And indeed, this is how Maass defined them [5].

Proof. We adapt the previous argument, writing first

$$p_{s,k}(t) = \int_{U_g} \mathcal{P}_{s,k}(z) e(-\text{Tr}(tz)) dz$$

where $U_g = U_g(y_0)$ will be taken to be the (compact) set of matrices

$$U_g(y_0) = \mathcal{U}_g + iy_0 \text{Id},$$

for some real number $y_0 > 1$ to be selected later, where

$$\mathcal{U}_g = \{x \in M(g, \mathbf{R}) \mid x \text{ symmetric and } |x_{i,j}| \leqslant 1/2 \text{ for all } 1 \leqslant i, j \leqslant g\}$$

the measure dz is again Lebesgue measure.

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[1] Forms of odd weight k do exist if g is even, but behave a little bit differently, and we restrict to k even for simplicity.

Before proceeding, we first recall that

$$(2) \qquad |e(\mathrm{Tr}(s\gamma \cdot z))| \leqslant 1$$

for all $s \in A_g$, $\gamma \in \Gamma_g$ and $z \in \mathbf{H}_g$. Indeed, since s is a real matrix, we have

$$|e(\mathrm{Tr}(s\gamma \cdot z))| = \exp(-2\pi \, \mathrm{Tr}(s \, \mathrm{Im}(\gamma \cdot z)))$$

and the result follows from the fact that

$$\mathrm{Tr}(sy) \geqslant 0$$

for any $s \in A_g$ and y positive definite. To see the latter, we write $y = {}^t qq$ for some matrix q, and we then have

$$sy = s^t qq = q^{-1} tq$$

with $t = qs^t q$; then t is still positive, while $\mathrm{Tr}(sy) = \mathrm{Tr}(t)$, so $\mathrm{Tr}(sy) \geqslant 0$.

We then have the following Lemma:[2]

Lemma 5. *For any integer $g \geqslant 1$, there exists a real number $y_0 > 1$, depending only on g, such that for any $\gamma \in \Gamma_g$ written*

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad (a,b,c,d) \in M(g, \mathbf{Z}),$$

with $c \neq 0$ and for all $z \in U_g(y_0)$, we have

$$(3) \qquad |\det(cz + d)| > 1$$

whereas if $c = 0$, we have $|\det(cz + d)| = 1$.

Assuming the truth of this lemma, we find that

$$(4) \quad |\det(cz+d)^{-k} e(\mathrm{Tr}(s\gamma \cdot z))| \leqslant |\det(cz+d)|^{-2g-1} \exp(-2\pi \, \mathrm{Tr}(s \, \mathrm{Im}(\gamma \cdot z)))$$

for any $k > 2g$, all $z \in U_g$ and $\gamma \in \Gamma_\infty \backslash \Gamma_g$, and also that

$$(5) \qquad \det(cz + d)^{-k} e(\mathrm{Tr}(s\gamma \cdot z)) \longrightarrow 0 \text{ as } k \to +\infty,$$

for all $z \in U_g$ and all γ with $c \neq 0$. On the other hand, if $c = 0$, we have

$$\gamma = \begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix},$$

up to Γ_∞-equivalence, where $a \in \mathrm{GL}(g, \mathbf{Z})$ and hence

$$\det(cz + d)^{-k} e(\mathrm{Tr}(s\gamma \cdot z)) = e(\mathrm{Tr}(saz^t a))) = e(\mathrm{Tr}(az^t as))$$
$$= e(\mathrm{Tr}({}^t asaz)) = e(\mathrm{Tr}((a \cdot s)z))$$

where $a \cdot s = {}^t asa$ (we use here that k is even).

Using (4), (5) and the absolute convergence of $\mathcal{M}_{s,2g+1}(z)$, we find that

$$\mathcal{P}_{s,k}(z) \longrightarrow \sum_{a \in \mathrm{GL}(g,\mathbf{Z})/\pm 1} e(\mathrm{Tr}((a \cdot s)z))$$

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[2] This statement is used to replace the inequality (1), which has no obvious analogue when $g \geqslant 2$.

as $k \to +\infty$, for all $z \in U_g$. (The series converges as a subseries of the Poincaré series.)

Then we multiply by $e(-\operatorname{Tr}(tz))$ and integrate over U_g, using (4) and the fact that $\mathcal{M}_{s,2g+1}$ is bounded on U_g to apply the dominated convergence theorem, and obtain

$$p_{s,k}(t) \longrightarrow \sum_{a \cdot s = t} 1,$$

a number which is either 0, if s and t are not equivalent, or has the same cardinality as $\operatorname{Aut}(s)/2$ if they are. This completes the proof of Theorem 4.
$\qquad\square$

We still need to prove Lemma 5. We are going to use the description of the Siegel fundamental domain $\mathcal{F}_g$ for the action of Γ_g on $\mathbf{H}_g$. Precisely, $\mathcal{F}_g$ is the set of $z \in \mathbf{H}_g$ satisfying all the following conditions:

(1) For all $\gamma \in \Gamma_g$, we have

$$|\det(cz+d)| \geqslant 1;$$

(2) The imaginary part $\operatorname{Im}(z)$ is Minkowski-reduced;
(3) The absolute value of all coefficients of $\operatorname{Re}(z)$ are $\leqslant 1/2$.

Siegel showed that the first condition can be weakened to a finite list of inequalities (see, e.g., [3, Prop. 3.3, p. 33]): there exists a finite subset $C_g \subset \Gamma_g$, such that $z \in \mathbf{H}_g$ belongs to $\mathcal{F}_g$ if and only if z satisfies (2), (3) and

(6) $\qquad\qquad |\det(cz+d)| \geqslant 1$ for all $\gamma \in C_g$ with $c \neq 0$.

Moreover, if (6) holds with equality sign for some $\gamma \in C_g$, then z is in the boundary of $\mathcal{F}_g$; if this is not the case, then $|\det(cz+d)| > 1$ for all $\gamma \in \Gamma_g$ with $c \neq 0$.

Proof of Lemma 5. First, we show that if $y_0 > 1$ is chosen large enough, the matrix $iy_0\operatorname{Id}$ is in $\mathcal{F}_g$. The only condition that must be checked is (6) when $\gamma \in C_g$ satisfies $c \neq 0$, since the other two are immediate (once the definition of Minkowski-reduced is known; it holds for $y_0\operatorname{Id}$ when $y_0 \geqslant 1$). For this, we use the following fact, due to Siegel [7, Lemma 9] (see also [3, Lemma 3.3, p. 34]): for any fixed $z = x + iy \in \mathbf{H}_g$ and any $\gamma \in \Gamma_g$ with $c \neq 0$, the function

$$\alpha \mapsto |\det(c(x+i\alpha)+d)|^2$$

is strictly increasing on $[0, +\infty[$ and has limit $+\infty$ as $\alpha \to +\infty$. Taking $z = i$, we find that

$$\lim_{y \to +\infty} |\det(iyc+d)| = +\infty$$

for every $\gamma \in C_g$. In particular, since C_g is finite, there exists $y_0 > 1$ such that

$$|\det(cz_0+d)| > 1$$

for $z_0 = iy_0$ and $\gamma \in C_g$, which is (6) for iy_0.

Because $\mathcal{U}_g$ is compact, it is now easy to also extend this to $z = x + iy_0$ with $x \in \mathcal{U}_g$. Precisely, for fixed $\gamma \in \Gamma_g$ with $c \neq 0$, the function

$$\begin{cases} \mathcal{U}_g \times]0, +\infty[\to \mathbf{R} \\ (x, \alpha) \mapsto |\det(c(x + i\alpha) + d)|^2 \end{cases}$$

is a polynomial in the variables (x, α). As a polynomial in α, as observed by Siegel, it is in fact a polynomial in α^2 with non-negative coefficients, and it is non-constant because $c \neq 0$. (It is not difficult to check that the degree, as polynomial in α, is $2\operatorname{rank}(c)$). This explains the limit

$$\lim_{y \to +\infty} |\det(c(x + iy) + d)|^2 = +\infty,$$

but it shows also that it is uniform over the compact set $\mathcal{U}_g$, and over the $\gamma \in C_g$ with $c \neq 0$. Therefore we can find y_0 large enough so that (6) holds for all $z \in U_g$, and indeed holds with the strict condition $|\det(cz + d)| > 1$ on the right-hand side. By the remark after (6), this means that z is not in the boundary of $\mathcal{F}_g$, and hence (3) holds for all γ with $c \neq 0$. $\qquad\square$

Remark 6. The argument is very clear when $\det(c) \neq 0$: we write

$$\det(c(x + iy) + d) = \det(iyc)\det(1 - iy^{-1}c^{-1}(cx + d))$$
$$= (iy)^g \det(c)(1 + O(y^{-1}))$$

for fixed (c, d), uniformly for $x \in \mathcal{U}_g$.

Remark 7. It would be interesting to know the optimal value of y_0 in Lemma 5. For $g = 1$, any $y_0 > 1$ is suitable. For $g = 2$, Gottschling [1, Satz 1] has determined a finite set C_2 which determines as above the Siegel fundamental domain, consisting of 19 pairs of matrices (c, d); there are 4 in which c has rank 1, c is the identity for the others. Precisely: for c of rank 1, (c, d) belongs to

$$\left\{ \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right), \left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right), \left(\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \text{ or } 1 \\ -2 & 1 \end{pmatrix} \right) \right\},$$

and for c of rank 2, we have $c = 1$ and d belongs to

$$\left\{ 0, \begin{pmatrix} s & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & s \end{pmatrix}, \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix}, \begin{pmatrix} s & 0 \\ 0 & -s \end{pmatrix}, \begin{pmatrix} 0 & s \\ s & 0 \end{pmatrix}, \begin{pmatrix} s & s \\ s & 0 \end{pmatrix}, \begin{pmatrix} 0 & s \\ s & s \end{pmatrix} \right\}$$

where $s \in \{-1, 1\}$. It should be possible to deduce a value of y_0 using this information. Indeed, quick numerical experiments suggest that, as in the case $g = 1$, any $y_0 > 1$ would be suitable.

Remark 8. Analogues of Corollary 3 can not be derived immediately in the setting of Siegel modular forms because the link between Fourier coefficients and Satake parameters is much more involved; the case $g = 2$ is considered, together with further applications and quantitative formulations, in [4].

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