Explicit refinements of Böcherer's conjecture for Siegel modular forms of squarefree level

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Abstract

We formulate an explicit refinement of Böcherer's conjecture for Siegel modular forms of degree 2 and squarefree level, relating weighted averages of Fourier coefficients with special values of L-functions. To achieve this, we compute the relevant local integrals that appear in the refined global Gan-Gross-Prasad conjecture for Bessel periods as proposed by Yifeng Liu. We note several consequences of our conjecture to arithmetic and analytic properties of L-functions and Fourier coefficients of Siegel modular forms.

Contents

1 Introduction

1.1 Böcherer's conjecture

Let f be a Siegel cusp form[1] of degree 2 and weight k for the group $\mathrm{Sp}(4, \mathbb{Z})$. Then f has a Fourier expansion

$$f(Z) = \sum_S a(f, S) e^{2\pi i \mathrm{tr}(SZ)},$$

where the Fourier coefficients $a(f, S)$ are indexed by matrices S of the form

$$S = \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix}, \qquad a, b, c \in \mathbb{Z}, \qquad a > 0, \qquad \mathrm{disc}(S) := b^2 - 4ac < 0. \tag{1}$$

For two matrices S_1, S_2 as above, we write $S_1 \sim S_2$ if there exists $A \in \mathrm{SL}(2, \mathbb{Z})$ such that $S_1 = {}^t A S_2 A$. Using the defining relation for Siegel cusp forms, we see that

$$a(f, S_1) = a(f, S_2) \qquad \text{if } S_1 \sim S_2, \tag{2}$$

thus showing that $a(f, S)$ only depends on the $\mathrm{SL}(2, \mathbb{Z})$-equivalence class of the matrix S, or equivalently, only on the proper equivalence class of the associated binary quadratic form.

Let $d < 0$ be a fundamental discriminant[2]. Put $K = \mathbb{Q}(\sqrt{d})$ and let Cl_K denote the ideal class group of K. It is well-known that the $\mathrm{SL}(2, \mathbb{Z})$-equivalence classes of binary quadratic forms of discriminant d are in natural bijective correspondence with the elements of Cl_K. In view of the comments above, it follows that the notation $a(f, c)$ makes sense for $c \in \mathrm{Cl}_K$. We define

$$R(f, K) = \sum_{c \in \mathrm{Cl}_K} a(f, c) = \sum_{\substack{S/\sim \\ \mathrm{disc}(S) = d}} a(F, S). \tag{3}$$

For odd k, it is easy to see that $R(f, K)$ equals 0. If k is even, Böcherer [3] made a remarkable conjecture that relates the quantity $R(f, K)$ to the central value of a certain L-function.

1.1 Conjecture. (Böcherer [3]) *Let k be even, $f \in S_k(\mathrm{Sp}(4, \mathbb{Z}))$ be a non-zero Hecke eigenform and π_f be the associated automorphic representation of $\mathrm{GSp}(4, \mathbb{A})$. Then there exists a constant c_f depending only on f such that for any imaginary quadratic field $K = \mathbb{Q}(\sqrt{d})$ with $d < 0$ a fundamental discriminant, we have*

$$|R(f, K)|^2 = c_f \cdot w(K)^2 \cdot |d|^{k-1} \cdot L_{\mathrm{f}}(1/2, \pi_f \otimes \chi_d).$$

Above, $\chi_d = \left(\frac{d}{\cdot}\right)$ is the Kronecker symbol (i.e., the quadratic Hecke character associated via class field theory to the field $\mathbb{Q}(\sqrt{d})$), $w(K)$ denotes the number of distinct roots of unity inside K, and $L_{\mathrm{f}}(s, \pi_f \otimes \chi_d)$ denotes the (finite part of the) associated degree 4 Langlands L-function.[3]

As far as we know, Böcherer did not speculate what the constant c_f is exactly (except for the case when f is a Saito-Kurokawa lift). For many applications, both arithmetic and analytic, it would be of interest to have a conjectural formula where the constant c_f is known precisely, and one of the original motivations of this paper was to do exactly that.

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[1] For definitions and background on Siegel modular forms, see [21].

[2] Recall that an integer n is a fundamental discriminant if *either* n is a squarefree integer congruent to 1 modulo 4 *or* $n = 4m$ where m is a squarefree integer congruent to 2 or 3 modulo 4.

[3] All global L-functions in this paper are normalized to have the functional equation taking $s \mapsto 1 - s$. We denote the completed L-functions by $L(s, \ )$ and reserve $L_{\mathrm{f}}(s, \ )$ for the finite part of the L-function, i.e., without the gamma factor $L_\infty(s, \ )$. Thus $L(s, \ ) = L_{\mathrm{f}}(s, \ ) L_\infty(s, \ )$.

This paper deals with refinements of Böcherer's conjecture for Siegel newforms with squarefree level. A special case of the results of this paper is the following *precise formula* for c_f, valid for all eigenforms f of full level and weight $k > 2$ that are not Saito-Kurokawa lifts:

$$c_f = \frac{2^{4k-4} \cdot \pi^{2k+1}}{(2k-2)!} \cdot \frac{L_f(1/2, \pi_f)}{L_f(1, \pi_f, \mathrm{Ad})} \cdot \langle f, f \rangle. \tag{4}$$

Above, $\langle f, f \rangle$ is the Petersson norm of f normalized as in (74), $L_f(s, \pi_f, \mathrm{Ad})$ is the finite part of the adjoint (degree 10) L-function for π_f, and $L_f(s, \pi_f)$ is the finite part of the spinor (degree 4) L-function for π_f. Indeed, combining the results of this paper with recent work of Furusawa and Morimoto [13], we get the following theorem.

1.2 Theorem. (c.f. Theorem 2 of [13]) *Let k be even, $f \in S_k(\mathrm{Sp}(4, \mathbb{Z}))$ be a non-zero Hecke eigenform that is not a Saito-Kurokawa lift and π_f be the associated automorphic representation of $\mathrm{GSp}(4, \mathbb{A})$. Then for any imaginary quadratic field $K = \mathbb{Q}(\sqrt{d})$ with $d < 0$ a fundamental discriminant, we have*

$$\frac{|R(f, K)|^2}{\langle f, f \rangle} = \frac{2^{4k-4} \, \pi^{2k+1}}{(2k-2)!} \, w(K)^2 \, |d|^{k-1} \frac{L_f(1/2, \pi_f)}{L_f(1, \pi_f, \mathrm{Ad})} \, L_f(1/2, \pi_f \otimes \chi_d).$$

A similar theorem is true for Saito-Kurokawa lifts and was proved in Böcherer in [3].

Conjecture 1.1 can be extended naturally as follows. For any character Λ of the finite group Cl_K, we make the definition

$$R(f, K, \Lambda) = \sum_{c \in \mathrm{Cl}_K} a(f, c) \Lambda^{-1}(c). \tag{5}$$

Let $\mathcal{AI}(\Lambda^{-1})$ be the automorphic representation of $\mathrm{GL}(2, \mathbb{A})$ given by the automorphic induction of Λ^{-1} from $\mathbb{A}_K^\times$; it is generated by (the adelization of) the theta-series of weight 1 and character χ_d given by

$$\theta_{\Lambda^{-1}}(z) = \sum_{0 \neq \mathfrak{a} \subset O_K} \Lambda^{-1}(\mathfrak{a}) e^{2\pi i N(\mathfrak{a}) z}.$$

We make the following refined version of Conjecture 1.1 (a further generalization of which to the case of squarefree levels is stated as Theorem 1.13 later in this introduction).

1.3 Conjecture. *Let $f \in S_k(\mathrm{Sp}(4, \mathbb{Z}))$ be a non-zero Hecke eigenform of weight $k \geq 2$ and π_f the associated automorphic representation of $\mathrm{GSp}(4, \mathbb{A})$. Suppose that f is not a Saito-Kurokawa lift. Then for any imaginary quadratic field $K = \mathbb{Q}(\sqrt{d})$ with $d < 0$ a fundamental discriminant and any character Λ of Cl_K, we have*

$$\frac{|R(f, K, \Lambda)|^2}{\langle f, f \rangle} = 2^{2k-4} \, w(K)^2 \, |d|^{k-1} \frac{L(1/2, \pi_f \times \mathcal{AI}(\Lambda^{-1}))}{L(1, \pi_f, \mathrm{Ad})}$$

$$= \frac{2^{4k-4} \, \pi^{2k+1}}{(2k-2)!} \, w(K)^2 \, |d|^{k-1} \frac{L_f(1/2, \pi_f \times \mathcal{AI}(\Lambda^{-1}))}{L_f(1, \pi_f, \mathrm{Ad})}$$

1.4 Remark. *For Saito-Kurokawa lifts, a similar equality is actually a theorem; see Theorem 3.13.*

1.5 Remark. *We note that $L_f(1, \pi_f, \mathrm{Ad})$ is not zero by Theorem 5.2.1 of [30].*

1.6 Remark. *The values of the relevant archimedean L-factors used above are as follows,*

$$L_\infty(1/2, \pi_f \times \mathcal{AI}(\Lambda^{-1})) = 2^4 (2\pi)^{-2k} (\Gamma(k-1))^2,$$

$$L_\infty(1, \pi_f, \mathrm{Ad}) = 2^5 (2\pi)^{-4k-1} (\Gamma(k-1))^2 \, \Gamma(2k-1).$$

1.7 Remark. *If $\Lambda = 1$, then $R(f, K, 1) = R(f, K)$ and*

$$L(1/2, \pi_f \otimes \mathcal{AI}(1)) = L(1/2, \pi_f)L(1/2, \pi_f \otimes \chi_d).$$

Hence, Conjecture 1.1 is a special case of Conjecture 1.3 with the particular value of c_f given in (4).

1.8 Remark. *Note that Conjecture 1.3 implies that whenever $R(f, K, \Lambda) \neq 0$, one also has $L(1/2, \pi_f \times \mathcal{AI}(\Lambda^{-1})) \neq 0$. This is a special case of the Gan-Gross-Prasad conjectures [14].*

As we explain in the next section, Conjecture 1.3 follows from the refined Gan-Gross-Prasad (henceforth GGP) conjectures as stated by Yifeng Liu [25]. These refined conjectures involve certain local factors which are essentially integrals of local matrix coefficients. The primary goal of the present paper is to compute these local factors exactly in several cases. Our calculation of the local factor at infinity for all $k > 2$ and at all places where π_f is unramified allows us to make Conjecture 1.3 (though we do our archimedean calculations only for $k > 2$, we speculate that the same answer holds for $k = 2$). We also calculate these local factors for certain Iwahori-spherical representations. As a consequence, we are able to generalize Conjecture 1.3 to all newforms of squarefree level. For the definition of the local factors and an overview of our results, we refer the reader to Sect. 1.3; for the resulting explicit refinement of Böcherer's conjecture, see Sect. 1.4 and in particular Theorem 1.13. The details and proofs of these appear in Sects. 2 and 3 respectively. We also prove some consequences of these refined conjectures in Sect. 3.6 (some of these consequences are summarized later in this introduction in Section 1.5).

1.2 Bessel periods and the refined Gan-Gross-Prasad conjecture

We now explain how Conjectures 1.1 and 1.3 fit into the framework of the GGP conjectures made in [14]. Let F be a number field and $H \subseteq G$ be reductive groups over F such that $Z_H(\mathbb{A}_F)H(F)\backslash H(\mathbb{A}_F)$ has finite volume (where Z_H denotes the center of H). Let π (resp. σ) be an irreducible, unitary, cuspidal, automorphic representation of $G(\mathbb{A}_F)$ (resp. $H(\mathbb{A}_F)$). Suppose that the central characters ω_π, ω_σ of π and σ are inverses of each other. Then if ϕ_π and ϕ_σ are automorphic forms in the space of π and σ respectively, one can form the global period

$$P(\phi_\pi, \phi_\sigma) = \int\limits_{Z_H(\mathbb{A}_F)H(F)\backslash H(\mathbb{A}_F)} \phi_\pi(h)\phi_\sigma(h)\, dh.$$

The basic philosophy is that, in certain cases, the quantity $|P(\phi_\pi, \phi_\sigma)|^2$ should be essentially equal to the value $L(1/2, \pi \otimes \sigma)$. The earliest prototype of this is due to Waldspurger [43], who dealt with the case $H = \mathrm{SO}(2)$, $G = \mathrm{SO}(2, 1) \simeq \mathrm{PGL}(2)$. The case $H = \mathrm{SO}(2, 1)$, $G = \mathrm{SO}(2, 2)$ is the famous triple product formula (note that $\mathrm{SL}(2) \times \mathrm{SL}(2)$ is a double cover for $\mathrm{SO}(2, 2)$), developed and refined by various people over the years, including Harris-Kudla [16], Watson [44], Ichino [18], and Nelson-Pitale-Saha [27]. Gross and Prasad made a series of conjectures in this vein for orthogonal groups; this was extended to other classical groups by Gan-Gross-Prasad [14]. In their original form, the Gross-Prasad and GGP conjectures do not predict an exact formula for $|P(\phi_\pi, \phi_\sigma)|^2$ but instead deal with the conditions for its non-vanishing in terms of the non-vanishing of $L(1/2, \pi \otimes \sigma)$. However, a recent work of Yifeng Liu [25], extending that of Ichino-Ikeda [19] and Prasad-Takloo-Bighash [33], makes a refined GGP conjecture by giving a precise conjectural formula for $|P(\phi_\pi, \phi_\sigma)|^2$ for a wide family of automorphic representations.

The case of Liu's conjecture that interests us in this paper is $G = \mathrm{PGSp}(4) \simeq \mathrm{SO}(3, 2)$, and $H = T_S N$ equal to the *Bessel subgroup*, which is an enlargement of $T_S \simeq \mathrm{SO}(2)$ with a unipotent subgroup N.

Let S be a symmetric 2×2 matrix that is anisotropic over F; in particular this implies that $d = \mathrm{disc}(S)$ is not a square in F. Define a subgroup T_S of $\mathrm{GL}(2)$ by

$$T_S(F) = \{g \in \mathrm{GL}(2, F) : {}^t g S g = \det(g)S\}. \tag{6}$$

It is easy to verify that $T_S(F) \simeq K^\times$ where $K = F(\sqrt{d})$. We consider T_S as a subgroup of GSp(4) via

$$T_S \ni g \longmapsto \begin{bmatrix} g & 0 \\ 0 & \det(g) \, {}^t g^{-1} \end{bmatrix} \in \mathrm{GSp}(4). \tag{7}$$

Let us denote by N the subgroup of GSp(4) defined by

$$N = \{ n(X) = \begin{bmatrix} 1 & X \\ 0 & 1 \end{bmatrix} \mid {}^t X = X \}.$$

Fix a non-trivial additive character ψ of $F \backslash \mathbb{A}_F$. We define the character θ_S on $N(\mathbb{A})$ by $\theta_S(n(X)) = \psi(\mathrm{tr}(SX))$. Let Λ be a character of $T_S(F) \backslash T_S(\mathbb{A}_F) \simeq K^\times \backslash \mathbb{A}_K^\times$ such that $\Lambda|_{\mathbb{A}_F^\times} = 1$. Let (π, V_π) be a cuspidal, automorphic representation of $\mathrm{GSp}(4, \mathbb{A})$ with trivial central character. For an automorphic form $\phi \in V_\pi$, we define the global Bessel period $B(\phi, \Lambda)$ on $\mathrm{GSp}(4, \mathbb{A})$ by

$$B(\phi, \Lambda) = \int\limits_{\mathbb{A}_F^\times T_S(F) \backslash T_S(\mathbb{A}_F)} \int\limits_{N(F) \backslash N(\mathbb{A}_F)} \phi(tn) \Lambda^{-1}(t) \theta_S^{-1}(n) \, dn \, dt. \tag{8}$$

The GGP conjecture in this case can be stated as follows.

1.9 Conjecture. (Special case of global GGP) *Let π, Λ be as above. Suppose that for almost all places v of F, the local representation π_v is generic. Suppose also that there exists an automorphic form ϕ in the space of π such that $B(\phi, \Lambda) \neq 0$. Then $L(1/2, \pi \times \mathcal{AI}(\Lambda^{-1})) \neq 0$.*

1.10 Remark. *The global GGP [14] actually predicts more in this case, namely that $L(1/2, \pi \times \mathcal{AI}(\Lambda^{-1})) \neq 0$ if and only if there exists a representation π' in the Vogan packet of π and an automorphic form ϕ' in the space of π' such that $B(\phi', \Lambda) \neq 0$.*

1.11 Remark. *In the special case $F = \mathbb{Q}$, if we take ϕ to be the adelization of a weight k Siegel cusp form, ψ the standard additive character, $d < 0$ a fundamental discriminant, and Λ corresponding to a character of Cl_K, then up to a constant, one has the relation (see [12])*

$$B(\phi, \Lambda) = R(f, K, \Lambda).$$

We now state Liu's refinement of Conjecture 1.9 in our case. For simplicity, we restrict to the case $F = \mathbb{Q}$ and write $\mathbb{A}$ for $\mathbb{A}_\mathbb{Q}$. Let the matrix S, the group $T = T_S$, the field $K = \mathbb{Q}(\sqrt{d})$, and the character χ_d be as above. Let $\pi = \otimes_v \pi_v$ be an irreducible, unitary, cuspidal, automorphic representation of $\mathrm{GSp}(4, \mathbb{A})$ with trivial central character and Λ be a unitary Hecke character of $K^\times \backslash \mathbb{A}_K^\times$ such that $\Lambda|_{\mathbb{A}^\times} = 1$. Fix the local Haar measures dn_v on $N(\mathbb{Q}_v)$ (resp. dt_v on $T(\mathbb{Q}_v)$) to be the standard additive measure at all places (resp. such that the maximal compact subgroup has volume one at almost all places). Fix the global measures dn and dt to be the global *Tamagawa measures*. Note that $dn = \prod_v dn_v$. Define the constant C_T by $dt = C_T \prod_v dt_v$. For each automorphic form ϕ in the space of π, define the global Bessel period $B(\phi, \Lambda)$ by (8) and define the Petersson norm $\langle \phi, \phi \rangle = \int_{Z(\mathbb{A})G(\mathbb{Q}) \backslash G(\mathbb{A})} |\phi(g)|^2 \, dg$ where we use the Tamagawa measure on $Z(\mathbb{A})G(\mathbb{Q}) \backslash G(\mathbb{A})$. For each place v, fix a $\mathrm{GSp}(4, \mathbb{Q}_v)$-invariant Hermitian inner product $\langle \, , \rangle_v$ on π_v. For $\phi_v \in V_{\pi_v}$, define

$$J_v(\phi_v) = \frac{L(1, \pi_v, \mathrm{Ad}) L(1, \chi_{d,v}) \int_{\mathbb{Q}_v^\times \backslash T(\mathbb{Q}_v)} \int_{N(\mathbb{Q}_v)} \frac{\langle \pi_v(t_v n_v) \phi_v, \phi_v \rangle}{\langle \phi_v, \phi_v \rangle} \Lambda_v^{-1}(t_v) \theta_S^{-1}(n_v) \, dn_v \, dt_v}{\zeta_{\mathbb{Q}_v}(2) \zeta_{\mathbb{Q}_v}(4) L(1/2, \pi_v \otimes \mathcal{AI}(\Lambda_v^{-1}))}.$$

Strictly speaking, the integral above may not converge absolutely, in which case one defines it via regularization (see [25, p. 6]). It can be shown that $J_v(\phi_v) = 1$ for almost all places. We now state the refined conjecture as phrased by Liu.

1.12 Conjecture. (Yifeng Liu [25]) *Let π, Λ be as above. Suppose that for almost all places v of $\mathbb{Q}$ the local representation π_v is generic. Let $\phi = \otimes_v \phi_v$ be an automorphic form in the space of π. Then*

$$\frac{|B(\phi, \Lambda)|^2}{\langle \phi, \phi \rangle} = \frac{C_T}{S_\pi} \frac{\zeta_{\mathbb{Q}}(2)\zeta_{\mathbb{Q}}(4)L(1/2, \pi \times \mathcal{AI}(\Lambda^{-1}))}{L(1, \pi, \mathrm{Ad})L(1, \chi_d)} \prod_v J_v(\phi_v), \tag{9}$$

where $\zeta_{\mathbb{Q}}(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ denotes the completed Riemann zeta function and S_π denotes a certain integral power of 2, related to the Arthur parameter of π. In particular,

$$S_\pi = \begin{cases} 4 & \text{if } L(s, \pi) = L(s, \pi_1)L(s, \pi_2) \text{ for irreducible cuspidal representations } \pi_i \text{ of } \mathrm{GL}(2, \mathbb{A}), \\ 2 & \text{if } L(s, \pi) = L(s, \Pi) \text{ for an irreducible cuspidal representation } \Pi \text{ of } \mathrm{GL}(4, \mathbb{A}). \end{cases}$$

Liu gave a proof of the above conjecture in the special case of Yoshida lifts. The proof in the case of the non-endoscopic analogue of Yoshida lifts has been very recently given by Corbett [9]. Recently, in a remarkable work [13], Furusawa and Morimoto have proved Conjecture 1.12 for $\Lambda = 1$ under some mild hypotheses.

1.3   Computing local integrals

We now briefly describe our main local result. Fix the following purely local data (where for convenience we have dropped all subscripts).

  i) A non-archimedean local field F of characteristic 0. We let $\mathfrak{o}$ denote the ring of integers of F and q the cardinality of the residue class field.

  ii) A symmetric invertible matrix $S = \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix}$. Put $d = b^2 - 4ac$. Denote $K = F(\sqrt{d})$ if d is a non-square in F and $K = F \times F$ if d is a square in F. Assume that $a, b \in \mathfrak{o}$, $c \in \mathfrak{o}^\times$, and d generates the discriminant ideal of K/F; if $K = F \times F$, the discriminant ideal is just $\mathfrak{o}$. Let $\chi_{K/F}$ denote the quadratic character on $F^\times$ associated to the extension K/F.

  iii) Haar measures dn and dt on $N(F)$ and $F^\times \backslash T_S(F)$ respectively, normalized so that the subgroups $N(\mathfrak{o})$ and $\mathfrak{o}^\times \backslash T(F) \cap \mathrm{GL}(2, \mathfrak{o})$ have volume 1. Henceforth, we denote $T_S(\mathfrak{o}) = T(F) \cap \mathrm{GL}(2, \mathfrak{o})$.

  iv) An irreducible, admissible, unitary representation (π, V_π) of $\mathrm{GSp}(4, F)$ with trivial central character[4] and invariant inner product $\langle\,,\,\rangle$, and a choice of vector $\phi \in V_\pi$.

  v) A character Λ of $K^\times$, which we also think of as a character of $T_S(F)$, that satisfies $\Lambda|_{F^\times} = 1$.

  vi) An additive character ψ of F with conductor $\mathfrak{o}$. Define a character θ_S on $N(F)$ by $\theta(n(X)) = \psi(\mathrm{tr}(SX))$.

Given all the above data, we define the local quantity

$$J(\phi, \Lambda) := \frac{L(1, \pi, \mathrm{Ad})L(1, \chi_{K/F}) \int_{F^\times \backslash T_S(F)} \int_{N(F)} \frac{\langle \pi(tn)\phi, \phi \rangle}{\langle \phi, \phi \rangle} \Lambda^{-1}(t)\theta_S^{-1}(n)\,dn\,dt}{\zeta_F(2)\zeta_F(4)L(1/2, \pi \times \mathcal{AI}(\Lambda^{-1}))}, \tag{10}$$

where $\zeta_F(s) = (1 - q^{-s})^{-1}$. Note that the above quantity is exactly what appears in Liu's refined GGP conjecture (9) above. If all the data above is *unramified* (see Section 2.1), then one can show that $J(\phi, \Lambda) = 1$. Note also that $J(\phi, \Lambda)$ does not change if ϕ is multiplied by a constant.

In this paper, we explicitly evaluate the local quantity $J(\phi, \Lambda)$ for two types of ramified cases. First, we evaluate it in the case when π is unramified but the extension K/F is a ramified field extension. This

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[4]Our results immediately extend to the slightly more general case of unramified central character by considering a suitable unramified twist of π.

is achieved in Theorem 2.1. Secondly, we consider the case when π itself is (possibly) ramified, but has a vector fixed by the congruence subgroup P_1 of $\mathrm{GSp}(4, \mathfrak{o})$ defined by

$$P_1 = \left\{ A \in \mathrm{GSp}(4, \mathfrak{o}) : A \in \begin{bmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{o} & \mathfrak{o} \end{bmatrix} \right\}. \tag{11}$$

Using the standard classification (see [40] or [32] or the tables later in this paper) this implies that π is one of the types I, IIa, IIIa, Vb, Vc, VIa or VIb. For each π as above, we choose a certain orthogonal basis $\mathcal{B}$ of the P_1-fixed subspace of π. Then for each vector ϕ in $\mathcal{B}$, we *exactly compute* the local integrals $J(\phi, \Lambda)$, under the assumption that Λ is unramified, F has odd residual characteristic and K/F is an unramified field extension. For the complete results, see Theorem 2.12, which is the technical heart of this paper. Its proof relies on computations of various local integrals and matrix coefficients, which are performed in Section 2.

1.4 A refined conjecture in the classical language

We now use our results described in Section 1.3 to translate Conjecture 1.12 to the classical language of Fourier coefficients for Siegel modular forms of squarefree level.

For any congruence subgroup Γ of $\mathrm{Sp}(4, \mathbb{Z})$, and any integer k, let $S_k(\Gamma)$ denote the space of Siegel cusp forms of weight k with respect to Γ. For any positive integer N, let the congruence subgroup $\Gamma_0(N)$ be the usual congruence subgroup,

$$\Gamma_0(N) = \mathrm{Sp}(4, \mathbb{Z}) \cap \begin{bmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{bmatrix}. \tag{12}$$

Now, suppose that N is squarefree. By a newform of weight k for $\Gamma_0(N)$, we mean an element $f \in S_k(\Gamma_0(N))$ with the following properties.

i) f lies in the orthogonal complement of the space of oldforms $S_k(\Gamma_0(N))^{\mathrm{old}}$, as defined in [40].

ii) f is an eigenform for the local Hecke algebras for all primes p not dividing N and an eigenfunction of the $U(p)$ operator (see [39]) for all primes dividing N.

iii) The adelization of f generates an irreducible cuspidal representation π_f of $\mathrm{GSp}(4, \mathbb{A})$.

It is known that any newform f defined as above is of one of two types:

i) **CAP**: A CAP newform f is one for which the associated automorphic representation $\pi_f = \otimes_v \pi_{f,v}$ is nearly equivalent to a constituent of a global induced representation of a proper parabolic subgroup of $\mathrm{GSp}(4, \mathbb{A})$. These newforms f do not satisfy the Ramanujan conjecture. Furthermore, if $k \geq 3$, the CAP newforms are exactly those that are obtained via Saito-Kurokawa lifting from classical newforms of weight $2k-2$ and level N [41]. For $k = 1, 2$, one can potentially also have CAP newforms that are not of Saito-Kurokawa type; these are associated to the Borel or Klingen parabolics. Finally, if f is CAP, the representations $\pi_{f,v}$ are non-generic for all places v.

ii) **Non-CAP**: These are the newforms f in the complement of the above set. Weissauer proved [45] that the non-CAP newforms always satisfy the Ramanujan conjecture at all primes $p \nmid N$ whenever $k \geq 3$. Another key property of the non-CAP newforms is that the local representations $\pi_{f,p}$ are generic for all $p \nmid N$. In terms of the standard classification (see Section 2.2 of [36]), the representations $\pi_{f,p}$ are type I if $p \nmid N$, and one of types IIa, IIIa, Vb/c, VIa, VIb if $p|N$. (Conjecturally, they are not of type Vb/c, since these are non-tempered representations.)

Given any newform f in $S_k(\Gamma_0(N))$, any fundamental discriminant $d < 0$, and any character Λ of the ideal class group of $K = \mathbb{Q}(\sqrt{d})$, we can define the quantity $R(f, K, \Lambda)$ exactly as in (5). Then our local results lead to the following theorem.

1.13 Theorem. *Let N be squarefree and f be a non-CAP newform in $S_k(\Gamma_0(N))$. For $d < 0$ a funda-mental discriminant, and Λ a character of Cl_K, define the quantity $R(f, K, \Lambda)$ as in (5). Let $\pi_f = \otimes_v \pi_{f,v}$ be the automorphic representation of $\mathrm{GSp}(4, \mathbb{A})$ associated to f. Suppose that N is odd, $k > 2$, and $\left(\frac{d}{p}\right) = -1$ for all primes p dividing N. Then, assuming the truth of Conjecture 1.12, we have*

$$\frac{|R(f, K, \Lambda)|^2}{\langle f, f \rangle} = 2^{2k-s} \, w(K)^2 \, |d|^{k-1} \frac{L(1/2, \pi_f \times \mathcal{AI}(\Lambda^{-1}))}{L(1, \pi_f, \mathrm{Ad})} \prod_{p | N} J_p, \tag{13}$$

where $s = 5$ if f is a weak Yoshida lift[5] in the sense of [38] and $s = 4$ otherwise. The quantities J_p for $p | N$ are given as follows:

$$J_p = \begin{cases} (1 + p^{-2})(1 + p^{-1}) & \text{if } \pi_{f,p} \text{ is of type IIIa,} \\ 2(1 + p^{-2})(1 + p^{-1}) & \text{if } \pi_{f,p} \text{ is of type VIb,} \\ 0 & \text{otherwise.} \end{cases}$$

1.14 Remark. *The assumptions N odd and $\left(\frac{d}{p}\right) = -1$ arise due to the assumptions on our local non-archimedean computations. The assumption $k > 2$ is only needed because we use the formula for the matrix coefficients of holomorphic discrete series representations from [22, Proposition A.1]. It seems very plausible that this formula also holds for limit of discrete series representations, which would allow us to easily extend Theorem 1.13 to the case $k = 2$.*

1.15 Remark. *For CAP newforms of Saito-Kurokawa type, we prove a similar theorem to the above that does not rely on the truth of any conjecture. For the precise statement, see Theorem 3.13.*

As mentioned earlier, Conjecture 1.12 is now known in a few cases, and therefore the statement of Theorem 1.13 is unconditional in those cases. For ease of reference, we state this separately next.

1.16 Theorem. *Let N be squarefree and f be a non-CAP newform in $S_k(\Gamma_0(N))$. For $d < 0$ a funda-mental discriminant, and Λ a character of Cl_K, define the quantity $R(f, K, \Lambda)$ as in (5). Let $\pi_f = \otimes_v \pi_{f,v}$ be the automorphic representation of $\mathrm{GSp}(4, \mathbb{A})$ associated to f. Suppose that N is odd, $k > 2$, and $\left(\frac{d}{p}\right) = -1$ for all primes p dividing N. Suppose further that at least one of the following hold: a) $\Lambda = 1$ b) f is a Yoshida lift. Then we have*

$$\frac{|R(f, K, \Lambda)|^2}{\langle f, f \rangle} = 2^{2k-s} \, w(K)^2 \, |d|^{k-1} \frac{L(1/2, \pi_f \times \mathcal{AI}(\Lambda^{-1}))}{L(1, \pi_f, \mathrm{Ad})} \prod_{p | N} J_p, \tag{14}$$

where $s = 5$ if f is a Yoshida lift and $s = 4$ otherwise. The quantities J_p for $p | N$ are as in Theorem 1.13.

1.5 Some consequences of the refined conjecture

The formula (13) is the promised explicit refinement of Böcherer's conjecture for Siegel newforms of squarefree level. Our main global result, Theorem 3.10, is a mild extension of this formula that includes the case of oldforms. Historically, exact formulas relating periods and L-functions (such as Watson's triple product formula [44] or Kohnen's refinement of Waldspurger's formula [23]) have had various applications, ranging from quantum unique ergodicity in level aspect [27] to non-vanishing of central values modulo primes [7]. So one can expect Theorem 3.10 to have many interesting consequences as well. Towards the end of this paper (see Section 3.6) we work out some of these consequences. We note a few of them below.

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[5]In this case, one can show that a weak Yoshida lift is automatically a Yoshida lift; see Corollary 4.17 of [38].

Averages of *L*-functions. By combining Theorem 3.10 with some results proved in [24] and [10], we obtain some quantitative asymptotic formulas for averages of *L*-functions. For example, if $\mathcal{B}_k^T$ is an orthogonal Hecke basis for the space of Siegel cusp forms of full level and weight $k > 2$ that are not Saito-Kurokawa lifts, then we show that Conjecture 1.12 implies:

$$\sum_{f \in \mathcal{B}_k^T} \frac{L_{\mathrm{f}}(1/2, \pi_f \times \mathcal{AI}(\Lambda^{-1}))}{L_{\mathrm{f}}(1, \pi_f, \mathrm{Ad}) \ L_{\mathrm{f}}(1, \chi_d)} - \frac{k^3}{l\pi^6} \ll k^{7/3}, \tag{15}$$

where $l = 1$ if $\Lambda^2 = 1$ and $l = 2$ otherwise. In fact, we prove a version of (15) for forms of squarefree level; see Theorem 3.15 and Corollary 3.16.

Bounds for Fourier coefficients. Theorem 3.10 combined with the GRH allows us to predict the following best possible upper bound for the size of $R(f, K, \Lambda)$ for all non-CAP newforms f in $S_k(\Gamma_0(N))$,

$$|R(f, K, \Lambda)| \ll_\epsilon \langle f, f \rangle^{1/2} (2\pi e)^k \ k^{-k+\frac{3}{4}} |d|^{\frac{k-1}{2}} (Nkd)^\epsilon. \tag{16}$$

Arithmeticity of *L*-values. The refined Böcherer conjecture implies that for all f, Λ as in Theorem 1.13 such that f has algebraic integer Fourier coefficients, the quantity

$$\pi^{2k+1} \cdot \langle f, f \rangle \cdot \frac{L_{\mathrm{f}}(1/2, \pi_f \times \mathcal{AI}(\Lambda^{-1}))}{L_{\mathrm{f}}(1, \pi_f, \mathrm{Ad})} \tag{17}$$

is, up to an exactly specified rational number, an *algebraic integer* (see Proposition 3.18). We find this interesting because even the algebraicity of (17) appears to not have been conjectured previously.

The above results are of course conditional on Conjecture 1.12. However, there are important cases where Conjecture 1.12 is already proven. As mentioned earlier, Furusawa and Morimoto have recently proved this conjecture [13] for trivial ideal class character under some mild hypotheses. Furthermore, Furusawa and Morimoto write in [13, Remark 7] "*In the sequel, we shall pursue [...] extension to the case when the character of the ideal class group is not necessarily trivial.*" The conjecture is also known in full generality for Yoshida lifts, see [25], and for the non-endoscopic analogue of Yoshida lifts, see [9]. The statement of Theorem 1.16 above summarizes the types of Siegel newforms of squarefree levels where we can make unconditional statements at present.

Applying our formula to Yoshida lifts, we note the following unconditional *integrality* result about families of *L*-values of *classical modular forms*, which may be of independent interest:

1.17 Proposition. *Let p be an odd prime and let g_1, g_2 be distinct classical primitive forms of level p and weights ℓ, 2 respectively, where $\ell > 2$, ℓ even. Assume that the Atkin-Lehner eigenvalues of g_1 and g_2 at p are equal. Then there exists a positive real number Ω (depending only on g_1, g_2) such that for any imaginary quadratic field K with the property that p is inert in K, and any character Λ of Cl_K,*

$$\Omega \times \mathrm{disc}(K)^{\ell/2} \times L_{\mathrm{f}}(1/2, \pi_{g_1} \times \mathcal{AI}(\Lambda^{-1})) \ L_{\mathrm{f}}(1/2, \pi_{g_2} \times \mathcal{AI}(\Lambda^{-1}))$$

is a totally-positive algebraic integer in the field generated by Λ-values and the coefficients of g_1, g_2.

1.6 Acknowledgements

1.7 Notations

Throughout this paper, the letter G will denote the algebraic group $\mathrm{GSp}(4)$, defined as

$$\mathrm{GSp}(4, R) = \{g \in \mathrm{GL}(4, R) : {}^t g J g = \lambda(g)J, \ \lambda(g) \in R^\times\}, \qquad J = \begin{bmatrix} & & 1 \\ & & & 1 \\ -1 & & \\ & -1 & \end{bmatrix}$$

for a commutative ring R with identity. The subgroup for which $\lambda(g) = 1$ is denoted $\mathrm{Sp}(4, R)$. We denote by N the unipotent radical of the Siegel parabolic subgroup of $\mathrm{GSp}(4, R)$. Explicitly,

$$N = \{\begin{bmatrix} 1 & & x & y \\ & 1 & y & z \\ & & 1 & \\ & & & 1 \end{bmatrix} : \ x, y, z \in R\}. \tag{18}$$

The symbol $\mathbb{A}$ will denote the ring of adeles over $\mathbb{Q}$. For an additive group R, $\mathrm{Sym}(2, R)$ denotes the group of size 2 symmetric matrices over R. All (local and global) L-functions and ε-factors are defined via the local Langlands correspondence. The global L-functions denoted by $L(s, \ )$ include the archimedean factors. The finite part of an L-function (without the archimedean factors) is denoted by $L_{\mathrm{f}}(s, \ )$.

2 The local integral in the non-archimedean case

2.1 Assumptions and basic facts

The following notations will be used throughout Section 2. Let F be a non-archimedean local field of characteristic zero, $\mathfrak{o}$ the ring of integers of F, and $\mathfrak{p}$ the maximal ideal of $\mathfrak{o}$. Let $\varpi \in \mathfrak{o}$ be a fixed generator of $\mathfrak{p}$, and $q = \#(\mathfrak{o}/\mathfrak{p})$ be the cardinality of the residue class field. The normalized valuation on F is denoted by v, and the normalized absolute value by $|\cdot|$. The character of $F^\times$ obtained by restricting $|\cdot|$ to $F^\times$ is denoted by ν. We fix a non-trivial character ψ of F with conductor $\mathfrak{o}$ once and for all.

Table (19) lists all the irreducible, admissible, non-supercuspidal representations of $G(F)$ supported in the minimal parabolic subgroup. For further explanation of the notation, we refer to Sect. 2.2 of [36] (also see the comments in Section 2.4 of this paper).

type		representation	g	P_1	K
I		$\chi_1 \times \chi_2 \rtimes \sigma$ (irreducible)	•	4	1
II	a	$\chi \mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma$	•	1	0
	b	$\chi 1_{\mathrm{GL}(2)} \rtimes \sigma$		3	1
III	a	$\chi \rtimes \sigma \mathrm{St}_{\mathrm{GSp}(2)}$	•	2	0
	b	$\chi \rtimes \sigma 1_{\mathrm{GSp}(2)}$		2	1
IV	a	$\sigma \mathrm{St}_{\mathrm{GSp}(4)}$	•	0	0
	b	$L(\nu^2, \nu^{-1}\sigma \mathrm{St}_{\mathrm{GSp}(2)})$		2	0
	c	$L(\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-3/2}\sigma)$		1	0
	d	$\sigma 1_{\mathrm{GSp}(4)}$		1	1
V	a	$\delta([\xi, \nu\xi], \nu^{-1/2}\sigma)$	•	0	0
	b	$L(\nu^{1/2}\xi \mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$		1	0
	c	$L(\nu^{1/2}\xi \mathrm{St}_{\mathrm{GL}(2)}, \xi\nu^{-1/2}\sigma)$		1	0
	d	$L(\nu\xi, \xi \rtimes \nu^{-1/2}\sigma)$		2	1
VI	a	$\tau(S, \nu^{-1/2}\sigma)$	•	1	0
	b	$\tau(T, \nu^{-1/2}\sigma)$		1	0
	c	$L(\nu^{1/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$		0	0
	d	$L(\nu, 1_{F^\times} \rtimes \nu^{-1/2}\sigma)$		2	1

$$(19)$$

The "g" column indicates which representations are generic. The last two columns P_1 and K give the dimensions of the subspace of vectors fixed by the Siegel congruence subgroups P_1 (see (43)) and the maximal compact subgroup $K = G(\mathfrak{o})$ respectively, assuming that the inducing characters are unramified. This data will only be useful in Section 2. The representations IVb, IVc are never unitary and hence are not relevant for our global applications. The representations that contain a K-fixed vector are known as spherical representations; we see from the table that these are of types I, IIb, IIIb, IVd, Vd, or VId.

We introduce the following notations regarding Bessel models for GSp_4. For $a, b, c \in F$ let

$$S = \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix}. \tag{20}$$

Assume that $d = b^2 - 4ac$ is not zero and put $D = d/4 = -\det(S)$. If $D \notin F^{\times 2}$, then let $\Delta = \sqrt{D}$ be a square root of D in an algebraic closure $\bar{F}$, and $K = F(\Delta)$. If $D \in F^{\times 2}$, then let $\sqrt{D}$ be a square root of D in $F^\times$, $K = F \times F$, and $\Delta = (-\sqrt{D}, \sqrt{D}) \in K$. In both cases K is a two-dimensional F-algebra containing F, $K = F + F\Delta$, and $\Delta^2 = D$. Let

$$A = A_S = \left\{ \begin{bmatrix} x+yb/2 & yc \\ -ya & x-yb/2 \end{bmatrix} : x, y \in F \right\}. \tag{21}$$

Then A is a two-dimensional F-algebra naturally containing F. One can verify that

$$A = \{g \in \mathrm{M}_2(F) : {}^t g S g = \det(g) S\}. \tag{22}$$

We write $T = T_S = A^\times$. We define an isomorphism of F-algebras,

$$A \xrightarrow{\sim} K, \qquad \begin{bmatrix} x+yb/2 & yc \\ -ya & x-yb/2 \end{bmatrix} \longmapsto x + y\Delta. \tag{23}$$

The restriction of this isomorphism to T is an isomorphism $T \xrightarrow{\sim} K^{\times}$, and we identify characters of T and characters of $K^{\times}$ via this isomorphism. We embed T into $G(F)$ via the map

$$t \longmapsto \left[\begin{smallmatrix} t & \\ & \det(t)t' \end{smallmatrix} \right], \qquad t \in T, \quad t' = {}^t t^{-1}. \tag{24}$$

For S as in (20), we define a character $\theta = \theta_{a,b,c} = \theta_S$ of N by

$$\theta\left(\left[\begin{smallmatrix} 1 & x & y \\ & 1 & y & z \\ & & 1 & \\ & & & 1 \end{smallmatrix} \right] \right) = \psi(ax + by + cz) = \psi(\mathrm{tr}(S[\begin{smallmatrix} x & y \\ y & z \end{smallmatrix}])) \tag{25}$$

for $x, y, z \in F$. Every character of N is of this form for uniquely determined a, b, c in F, or, alternatively, for a uniquely determined symmetric 2×2 matrix S. It is easily verified that $\theta(tnt^{-1}) = \theta(n)$ for $n \in N$ and $t \in T$. We refer to $T \ltimes N$ as the *Bessel subgroup* defined by the character θ (or the matrix S). Given a character Λ of T (identified with a character of $K^{\times}$ as explained above), we can define a character $\Lambda \otimes \theta$ of $T \ltimes N$ by

$$(\Lambda \otimes \theta)(tn) = \Lambda(t)\theta(n) \qquad \text{for } n \in N \text{ and } t \in T.$$

Every character of $T \ltimes N$ whose restriction to N coincides with θ is of this form for an appropriate Λ.

Let (π, V) be an admissible representation of $G(F)$. Let S and θ be as above, and let Λ be a character of the associated group T. By a (Λ, θ)-*Bessel functional* for π, we mean a non-zero functional $\beta : V \to \mathbb{C}$ such that $\beta(\pi(tn)v) = \Lambda(t)\theta(n)\beta(v)$, for all $t \in T, n \in N$ and $v \in V$. We note that if π admits a central character ω_{π} and a (Λ, θ)-Bessel functional, then $\Lambda|_{F^{\times}} = \omega_{\pi}$.

We now make the following assumptions, which will be in force through all of Section 2.

i) $S = \left[\begin{smallmatrix} a & b/2 \\ b/2 & c \end{smallmatrix} \right]$ with $a, b \in \mathfrak{o}$, $c \in \mathfrak{o}^{\times}$, and $d = b^2 - 4ac \neq 0$.

ii) The element d has the property that it generates the discriminant ideal of K/F, where $K := F(\sqrt{d})$ or $K := F \times F$ depending on whether $d \notin F^{\times 2}$ or $d \in F^{\times 2}$. In particular, d is a unit if and only if either a) $K = F \times F$ or b) K is the unramified quadratic field extension of F.

iii) The additive character ψ of F has conductor $\mathfrak{o}$ and the character Λ on $T(F) \cong K^{\times}$ satisfies $\Lambda|_{F^{\times}} = 1$.

iv) (π, V) is a unitary, irreducible, admissible representation of $G(F)$ with trivial central character and invariant hermitian inner product $\langle \cdot, \cdot \rangle$.

v) The relevant Haar measures are normalized so that the groups $N(\mathfrak{o})$ and $\mathfrak{o}^{\times} \backslash T_S(\mathfrak{o})$ both have volume 1.

As explained in [32, Section 2.3], there is no loss of generality in making these assumptions. For any vector $\phi \in V$, define Φ_{ϕ} to be the function on $G(F)$ defined by

$$\Phi_{\phi}(g) = \langle \pi(g)\phi, \phi \rangle / \langle \phi, \phi \rangle. \tag{26}$$

For each $t \in T = T_S \simeq K^{\times}$, consider the integrals

$$J_0^{(k)}(\phi, t) := \int_{N(\mathfrak{p}^{-k})} \Phi_{\phi}(nt)\theta_S^{-1}(n) \, dn, \tag{27}$$

where k is a positive integer. It was proved by Liu [25] that for each t the values $J_0^{(k)}(\phi, t)$ stabilize as $k \to \infty$, and hence the definition

$$J_0(\phi, t) = \lim_{k \to \infty} J_0^{(k)}(\phi, t) \tag{28}$$

is meaningful. In fact, Liu showed there is some open compact subgroup U of N such that

$$\int\limits_{U} \Phi_\phi(nt)\theta_S^{-1}(n)\,dn = \int\limits_{U'} \Phi_\phi(nt)\theta_S^{-1}(n)\,dn,$$

for *every* open compact subgroup U' containing U. Define the quantity

$$J_0(\phi,\Lambda) = \int\limits_{F^\times\backslash T} J_0(\phi,t)\Lambda^{-1}(t)\,dt \tag{29}$$

whenever the integral on the right side above converges absolutely and define the normalized quantity

$$J(\phi,\Lambda) = \frac{L(1,\pi,\mathrm{Ad})L(1,\chi_{K/F})}{\zeta_F(2)\zeta_F(4)L(1/2,\pi\times\mathcal{AI}(\Lambda^{-1}))}\,J_0(\phi,\Lambda) \tag{30}$$

whenever $L(1,\pi,\mathrm{Ad})\neq\infty$ (this happens whenever π is a generic L-packet).

A slightly different local integral (relying on a different regularization) was defined by Qiu in [34], which is useful for dealing with non-tempered representations, including ones for which Liu's integral does not converge, or for which $L(s,\pi,\mathrm{Ad})$ has a pole at $s=1$. We now describe this.

Let $\mathrm{Sym}(2,F)$ be the set of symmetric matrices of size 2 over F, and $\mathrm{Sym}_0(2,F) = \mathrm{Sym}(2,F)\cap \mathrm{GL}(2,F)$. For each $t\in T$, Qiu shows [34, Prop. 2] that there exists an integrable, locally constant function $W_{\phi,t}$ on $\mathrm{Sym}_0(2,F)$ with the property that

$$\int\limits_{N(F)}\int\limits_{\mathrm{Sym}(2,F)} \Phi_\phi(nt)\theta_Y^{-1}(n)f(Y)\,dY\,dn = \int\limits_{\mathrm{Sym}_0(2,F)} W_{\phi,t}(Y)f(Y)\,dY \tag{31}$$

for all Schwartz-Bruhat (locally constant, compactly supported) functions f on $\mathrm{Sym}(2,F)$. Observe here that the left side of the integral is well-defined because the Fourier transform of a Schwartz-Bruhat function is also Schwartz-Bruhat; observe also that the function $W_{\phi,t}$ (once we know that it exists) is clearly unique. Furthermore, Qiu defines a height function Δ on the group T, which has the property of being equal to the constant function 1 whenever K/F is a field. Now, following Qiu, let

$$J_0^*(\phi,t):=W_{\phi,t}(S),\quad J_0^*(s,\phi)=\int\limits_{F^\times\backslash T} J_0^*(\phi,t)\Delta(t)^s\,dt, \tag{32}$$

and finally

$$J^*(\phi) = \frac{L(s+1,\pi,\mathrm{Ad})L(1,\chi_{K/F})}{\zeta_F(2)\zeta_F(4)L(s+1/2,\pi)L(s+1/2,\pi\times\chi_{K/F}))}\,J_0^*(s,\phi)\bigg|_{s=0}. \tag{33}$$

Note that in the integrals $J^*(\phi)$, $J_0^*(s,\phi)$, there is no presence of Λ (they correspond to $\Lambda=1$).

2.2   The local integral in the spherical case

We remind the reader that the assumptions stated at the beginning of §2.1 continue to be in force for this and all the remaining local sections. In this subsection, we will prove the following theorem.

2.1 Theorem. *Suppose that π has a (unique up to multiples) $G(\mathfrak{o})$-fixed vector and let ϕ be a non-zero such vector. If the residue field characteristic q is even, assume moreover that $F=\mathbb{Q}_2$.*

   i) *If π is a type I representation and Λ is an unramified character, then $J(\phi,\Lambda)=1$.*

   ii) *If π is a type IIb representation of the form $\chi 1_{\mathrm{GL}(2)}\rtimes\chi^{-1}$, then $J^*(\phi)=2$.*

The above theorem is already known when $K = F \times F$ or if K/F is an *unramified* field extension. Indeed, the first assertion of the theorem in these cases is due to Liu [25] while the second assertion is due to Qiu [34].[6] So we will only need to take care of the case when K/F is a ramified field extension. Therefore, throughout this subsection, we make the following assumption:

Assumption: *The vector $\phi \in V_\pi$ is $G(\mathfrak{o})$-fixed. K/F is a ramified field extension (equivalently, $d \notin F^{\times 2}$ and $d \notin \mathfrak{o}^\times$). Moreover, if the residue field characteristic q is even, then $F = \mathbb{Q}_2$.*

The above assumption and the fact that d generates the discriminant ideal of K/F implies that $b/2 \in \mathfrak{o}$, and moreover $d/4 \in \mathfrak{o}^\times \cup \varpi\mathfrak{o}^\times$ with $d/4 \in \mathfrak{o}^\times$ possible only if $q = 2$. Moreover it is clear (using the well-known description of discriminants of quadratic extensions of $\mathbb{Q}_2$) that if $d/4 \in \mathfrak{o}^\times$, then $1 - d/4$ is in $\varpi\mathfrak{o}^\times$ and therefore is a uniformizer in F.

Next, observe that the matrix $S' = \left[\begin{smallmatrix} & -\frac{d}{4c} \\ c & \end{smallmatrix}\right]$ satisfies the first assumption of §2.1 and moreover, the matrix $A = \left[\begin{smallmatrix} 1 & \\ x & 1 \end{smallmatrix}\right]$ with $x = -c^{-1}b/2$ has the property that $S' = {}^t A S A$. As $A \in \mathrm{GL}(2, \mathfrak{o})$, a simple computation shows that $J_0^{(k)}(\phi, t) = \int_{N(\mathfrak{p}^{-k})} \Phi_\phi(nt')\theta_{S'}^{-1}(n)\, dn$, where $t' = A^{-1}tA \in T_{S'}$. Hence in the sequel we can (and will) replace S by S', *and hence assume for the rest of this subsection that $b = 0$.*

2.2 Lemma. *Let $J_0(\phi, t)$, $J_0^*(\phi, t)$ be as defined in the previous subsection. Then $J_0(\phi, t) = J_0^*(\phi, t)$.*

Proof. Let k be a positive integer, and put $f = f_k$ equal to $\mathrm{vol}(\mathfrak{p}^k \mathrm{Sym}(2, \mathfrak{o}))^{-1}$ times the characteristic function of $S + \mathfrak{p}^k \mathrm{Sym}(2, \mathfrak{o})$. Then, as $k \to \infty$, the right hand side of (31) approaches $W_{\phi,t}(S) = J_0^*(\phi, t)$. On the other hand, the left hand side is equal to

$$\int_{N(F)} \Phi_\phi(nt)\psi(-\mathrm{tr}(Sn))\left(\int_{\mathrm{Sym}(2,\mathfrak{o})} \psi(-\mathrm{tr}(\varpi^k Y n))dY \right) dn.$$

Using the fact that ψ has conductor $\mathfrak{o}$, we see that the inner integral equals 0 unless $n = \left[\begin{smallmatrix} n_1 & n_2/2 \\ n_2/2 & n_3 \end{smallmatrix}\right]$ with $n_1, n_2, n_3 \in \varpi^{-k}\mathfrak{o}$, in which case the inner integral equals 1. It follows that the left hand side equals $\int_{n \in \left[\begin{smallmatrix} \mathfrak{p}^{-k} & \mathfrak{p}^{-k}/2 \\ \mathfrak{p}^{-k}/2 & \mathfrak{p}^{-k} \end{smallmatrix}\right]} \Phi_\phi(nt)\psi(-\mathrm{tr}(Sn))\, dn$, which equals $J_0(\phi, t)$ for all k sufficiently large. The proof is complete. ∎

2.3 Remark. *The content of the above lemma is that for the part of the local integral that is over the unipotent subgroup N, the regularization employed by Liu and the regularization using the Fourier transform (as implemented by Qiu) are equivalent. This is also alluded to in [25, Remark 3.17].*

2.4 Lemma. *Suppose that $\Lambda = 1$. Then the integrals defining $J_0(\phi, \Lambda)$ and $J_0^*(0, \phi)$ are absolutely convergent, and $J_0(\phi, \Lambda) = J_0^*(0, \phi)$.*

Proof. This is immediate from the previous lemma, as the hypothesis $d \notin F^{\times 2}$ implies that $F^\times \backslash T$ is a compact set. ∎

Let $t_K \in T$ be an element such that the element ϖ_K of $K^\times$ corresponding to t_K (under the fixed isomorphism $T \cong K^\times$) is a uniformizer for $\mathfrak{o}_K$. It follows that the entries of t_K belong to $\mathfrak{o}$ and $\det(t_K) \in \varpi\mathfrak{o}^\times$. In fact, we can and will fix t_K as follows:

$$t_K = \begin{bmatrix} b'/2 & c \\ -a & b'/2 \end{bmatrix}, \quad \text{where } b' = \begin{cases} 0 & \text{if } d/4 \notin \mathfrak{o}^\times, \\ 2 & \text{if } d/4 \in \mathfrak{o}^\times. \end{cases} \tag{34}$$

---

[6]Note that the conditions for the evaluation of the unramified local integrals $J(\phi, \Lambda)$, $J^*(\phi)$ in these cases, set out explicitly in [25, Sect. 2.1], are met in our case due to the above assumptions on S, π, Λ. ψ, ϕ.

2.5 Lemma. *Suppose that Λ is an unramified character. Then $J_0(\phi, \Lambda) = J_0(\phi, 1) + \Lambda^{-1}(\varpi_K) J_0(\phi, t_K)$.*

Proof. This follows immediately from (29) and the decomposition $F^\times \backslash T = \mathfrak{o}^\times \backslash T(\mathfrak{o}) \sqcup \mathfrak{o}^\times \backslash (t_K T(\mathfrak{o}))$.   ∎

2.6 Proposition. *Let b' be as in (34). Then there exist $u_1, u_2 \in \mathfrak{o}^\times$ such that*

$$J_0^{(k)}(\phi, 1) = \int\limits_{x,y,z \in \mathfrak{p}^{-k}} \Phi_0\left(\begin{bmatrix} 1 & & x & y \\ & 1 & y & z \\ & & 1 & \\ & & & 1 \end{bmatrix}\right)\psi^{-1}(u_1 \varpi x - b'y + u_2 z)\, dx\, dy\, dz,$$

$$J_0^{(k)}(\phi, t_K) = \int\limits_{x,y,z \in \mathfrak{p}^{-k}} \Phi_0\left(\begin{bmatrix} 1 & & x & y \\ & 1 & y & z \\ & & 1 & \\ & & & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & \varpi & & \\ & & \varpi & \\ & & & 1 \end{bmatrix}\right)\psi^{-1}(u_1 \varpi x - b'y + u_2 z)\, dx\, dy\, dz.$$

Proof. In order to treat both equalities simultaneously, put $t = 1$ or t_K, and think of t as an element of $G(F)$ via the embedding defined earlier. Put $\kappa = \begin{bmatrix} 1 & \\ c^{-1} & 1 \end{bmatrix}$ if $d/4 \in \mathfrak{o}^\times$ and $\kappa = 1$ otherwise. Put $w = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $A = \begin{bmatrix} \kappa & \\ & {}^t\kappa^{-1} \end{bmatrix}$, $A' = \begin{bmatrix} {}^t\kappa^{-1} & \\ & \kappa \end{bmatrix}$, $B = \begin{bmatrix} w & \\ & w \end{bmatrix}$. Note that A, B, A' are elements of $\mathrm{Sp}(4, \mathfrak{o})$. Put $X = \begin{bmatrix} x & y \\ y & z \end{bmatrix}$, $X' = \kappa X {}^t\kappa$ and $S' = {}^t\kappa^{-1}\begin{bmatrix} a & \\ & c \end{bmatrix}\kappa^{-1} = \begin{bmatrix} \varpi u_1 & -b'/2 \\ -b'/2 & u_2 \end{bmatrix}$ where u_1, u_2 are in $\mathfrak{o}^\times$. Now we have

$$J_0^{(k)}(\phi, t) = \int\limits_{x,y,z \in \mathfrak{p}^{-k}} \Phi_0\left(\begin{bmatrix} I_2 & X \\ & I_2 \end{bmatrix} t\right)\psi^{-1}(ax + cz)\, dx\, dy\, dz,$$

$$= \int\limits_{x,y,z \in \mathfrak{p}^{-k}} \Phi_0\left(\begin{bmatrix} I_2 & X' \\ & I_2 \end{bmatrix} AtBA'\right)\psi^{-1}(\mathrm{tr}(SX))\, dx\, dy\, dz,$$

$$= \int\limits_{x,y,z \in \mathfrak{p}^{-k}} \Phi_0\left(\begin{bmatrix} I_2 & X \\ & I_2 \end{bmatrix} AtBA'\right)\psi^{-1}(\mathrm{tr}(S'X))\, dx\, dy\, dz,$$

$$= \int\limits_{x,y,z \in \mathfrak{p}^{-k}} \Phi_0\left(\begin{bmatrix} I_2 & X \\ & I_2 \end{bmatrix} AtBA'\right)\psi^{-1}(u_1 \varpi x - b'y + u_2 z)\, dx\, dy\, dz.$$

The result now follows from the observation that $AtBA'$ is a diagonal matrix of the desired form up to right multiplication by an element of $G(\mathfrak{o})$.   ∎

Henceforth, for convenience, denote $\Phi_0 = \Phi_\phi$. In particular, Φ_0 is bi-$Z(F)G(\mathfrak{o})$ invariant and satisfies $\Phi_0(1) = 1$. For non-negative integers ℓ, m, let

$$h(\ell, m) = \begin{bmatrix} \varpi^{\ell+2m} & & & \\ & \varpi^{\ell+m} & & \\ & & 1 & \\ & & & \varpi^m \end{bmatrix}.$$

Due to the disjoint double coset decomposition,

$$G = \bigsqcup_{\ell \geq 0, m \geq 0} Z(F)G(\mathfrak{o})h(\ell, m)G(\mathfrak{o})$$

and the fact that Φ_0 is bi-$G(\mathfrak{o})$ invariant, it follows that Φ_0 is determined by its values at the matrices $h(\ell, m)$. The following special case of [22, Proposition 3.2] will be useful for us.

2.7 Lemma. *Let $g = \begin{bmatrix} 1 & & x & y \\ & 1 & y & z \\ & & 1 & \\ & & & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & u & & \\ & & u & \\ & & & 1 \end{bmatrix}$ for some $x, y, z \in F$, $u \in \mathfrak{o}$. Define*

$$\ell = v(u) + 2\left(\min\left(0, v(ux), v(y), v(z)\right) - \min\left(0, v(u) + v(xz - y^2), v(ux), v(uy), v(z)\right)\right),$$

$$m = \min\left(0, v(u) + v(xz - y^2), v(ux), v(uy), v(z)\right) - 2\min\left(0, v(ux), v(y), v(z)\right).$$

Then $g \in Z(F)G(\mathfrak{o})h(\ell, m)G(\mathfrak{o})$. In particular, $\Phi_0(g) = \Phi_0(h(\ell, m))$.

We now come to a key result.

2.8 Proposition. *We have*

- $J_0(\phi, 1) = 1 - \Phi_0(h(0,1)) - q^2\Phi_0(h(0,2)) + q^2\Phi_0(h(2,1))$.

- $J_0(\phi, t_K) = q\Phi_0(h(1,0)) - (q^2 + q)\Phi_0(h(1,1)) + q^2\Phi_0(h(3,0))$.

Proof. It will be convenient for us to define the sets, for $m \geq 0$,

$$\Pi_m = \begin{cases} \varpi^{-m}\mathfrak{o}^\times & \text{if } m > 0, \\ \mathfrak{o} & \text{if } m = 0. \end{cases}$$

We assume throughout the proof that $k \geq 2$.

  Computing $J_0^{(k)}(\phi, 1)$. Let m, n, r be non-negative integers. First assume $m + r \neq 2n$. If $(x, y, z) \in \Pi_m \times \Pi_n \times \Pi_r$ then, using the formula of Proposition 2.6 and then applying Lemma 2.7 (which implies in particular that in this case, the value of Φ_0 depends only on m, n, r) we are reduced to computing a character integral

$$\int\limits_{x \in \Pi_m} \int\limits_{y \in \Pi_n} \int\limits_{z \in \Pi_r} \psi^{-1}(u_1\varpi x - b'y + u_2 z) \, dz \, dy \, dx.$$

The above integral is zero if $m > 2$ or $r > 1$. Thus, these contributions vanish. If $m = 2$ and $n \geq 2$, the same argument shows that the two integrals for $r = 0$ and $r = 1$ will sum to give zero. Similarly, we dispose of the case $m = 1$, $n \geq 2$, and $m = 0$, $n \geq 1$.

  Next, assume $m + r = 2n$. Conjugating the argument of Φ_0 by $\mathrm{diag}(1, 1, \lambda, \lambda)$ with $\lambda \in \mathfrak{o}^\times$ and making a change of variables the integral in question becomes

$$\int\limits_{x \in \Pi_m} \int\limits_{y \in \Pi_n} \int\limits_{z \in \Pi_r} \Phi_0\left(\begin{bmatrix} 1 & & x & y \\ & 1 & y & z \\ & & 1 & \\ & & & 1 \end{bmatrix}\right)\psi^{-1}(\lambda(u_1\varpi x - b'y + u_2 z)) \, dz \, dy \, dx.$$

Choosing $\lambda = 1 + pw$ with $w \in \mathfrak{o}$ and integrating both sides of this equality over $w \in \mathfrak{o}$ we get zero whenever $n \geq 2$. From the above, we get

$$J_0^{(k)}(\phi, 1) = \sum_{(m,n,r) \in S} \int\limits_{x \in \Pi_m} \int\limits_{y \in \Pi_n} \int\limits_{z \in \Pi_r} \Phi_0\left(\begin{bmatrix} 1 & & x & y \\ & 1 & y & z \\ & & 1 & \\ & & & 1 \end{bmatrix}\right)\psi^{-1}(u_1\varpi x - b'y + u_2 z) \, dz \, dy \, dx,$$

where

$$S = \{(2,1,1), (2,0,1), (2,0,0), (1,1,0), (1,0,1), (1,0,0), (0,0,1), (2,1,0), (1,1,1), (0,1,2), (0,0,0,)\}.$$

The values of the integrals for tuples in S are easily computed using Lemma 2.7; summarized in the following table:[7]

(m,n,r)	$h(\ell,m)$	Coefficient		(m,n,r)	$v(z - y^2 x^{-1})$	$h(\ell,m)$	Coefficient
$(2,1,1)$	$h(2,1)$	$q^2 - q$		$(2,1,0)$		$h(0,2)$	$-q^2 + q$
$(2,0,1)$	$h(2,1)$	q		$(1,1,1)$	-1	$h(2,0)$	$-q^2 + 3q - 2$
$(2,0,0)$	$h(0,2)$	$-q$			≥ 0	$h(0,1)$	$-q + 1$
$(1,1,0)$	$h(2,0)$	$q^2 - 2q + 1$		$(0,1,2)$		$h(0,2)$	0
$(1,0,1)$	$h(2,0)$	$-q + 1$		$(0,0,0)$		$h(0,0)$	1
$(1,0,0)$	$h(0,1)$	$q - 1$		$(0,0,1)$		$h(0,1)$	-1

---

[7]The "Coefficient" column computes the value of the integral $\int_{x \in \Pi_m} \int_{y \in \Pi_n} \int_{z \in \Pi_r} \psi^{-1}(u_1\varpi x - b'y + u_2 z)$.

which gives

$$J_0^{(k)}(\phi, 1) = \Phi_0(1_4) - \Phi_0(h(0,1)) - q^2\Phi_0(h(0,2)) + q^2\Phi_0(h(2,1)).$$

Computing $J_0^{(k)}(\phi, t_K)$. The argument is essentially the same. Once again we get

$$J_0^{(k)}(\phi, t_K) = \sum_{(m,n,r)\in S} \int_{x\in\Pi_m} \int_{y\in\Pi_n} \int_{z\in\Pi_r} \Phi_0\left(\begin{bmatrix} 1 & x & y \\ & 1 & y & z \\ & & 1 \\ & & & 1 \end{bmatrix}\begin{bmatrix} 1 \\ & \varpi \\ & & \varpi \\ & & & 1 \end{bmatrix}\right)\psi^{-1}(u_1\varpi x - b'y + u_2 z)\, dz\, dy\, dx,$$

where

$$S = \{(2,1,1),(2,0,1),(2,0,0),(1,1,0),(1,0,1),(1,0,0),(0,0,1),(2,1,0),(1,1,1),(0,1,2),(0,0,0,)\}.$$

The values of the integrals for tuples in S are summarized in the following table:

(m,n,r)	$h(l,m)$	Coefficient	(m,n,r)	$v(z - y^2x^{-1})$	$h(l,m)$	Coefficient
$(2,1,1)$	$h(3,0)$	$q^2 - q$	$(2,1,0)$		$h(1,1)$	$-q^2 + q$
$(2,0,1)$	$h(3,0)$	q	$(1,1,1)$	-1	$h(1,1)$	$-q^2 + 3q - 2$
$(2,0,0)$	$h(1,1)$	$-q$		≥ 0	$h(1,1)$	$-q + 1$
$(1,1,0)$	$h(1,1)$	$q^2 - 2q + 1$	$(0,1,2)$		$h(1,2)$	0
$(1,0,1)$	$h(1,1)$	$-q + 1$	$(0,0,0)$		$h(1,0)$	1
$(1,0,0)$	$h(1,0)$	$q - 1$	$(0,0,1)$		$h(1,1)$	-1

which gives $J_0^{(k)}(\phi, t_K) = q^2 h(3,0) - (q^2 + q)h(1,1) + qh(1,0)$. ∎

Henceforth, assume that π is either a type I or a type IIb representation. We fix the following notation for Satake parameters. If $\pi = \chi_1 \times \chi_2 \rtimes \sigma$ is a type I representation, put $\alpha = \chi_1(\varpi), \beta = \chi_2(\varpi), \gamma = \sigma(\varpi)$. If $\pi = \chi 1_{\mathrm{GL}(2)} \rtimes \sigma$ is a type IIb representation, put $\alpha = q^{-\frac{1}{2}}\chi(\varpi), \beta = q^{\frac{1}{2}}\chi(\varpi), \gamma = \sigma(\varpi)$. In both cases we have that $\alpha\beta\gamma^2 = 1$ and the degree 5 L-function for π is given by

$$L(s, \pi, \mathrm{Std}) = \left((1 - q^{-s})(1 - \alpha q^{-s})(1 - \alpha^{-1}q^{-s})(1 - \beta q^{-s})(1 - \beta^{-1}q^{-s})\right)^{-1}. \tag{35}$$

In the IIb case, we will assume in addition that $\chi\sigma$ is trivial (note that $\chi^2\sigma^2$ is trivial by the central character condition), i.e., that π is of the form $\chi 1_{\mathrm{GL}(2)} \rtimes \chi^{-1}$. It is only under this condition that π admits a Bessel model for trivial Λ.

2.9 Lemma. *Let π be a type I representation, or a type IIb representation of the form $\chi 1_{\mathrm{GL}(2)} \rtimes \chi^{-1}$. In the former case, let Λ be an unramified character and put $l = \Lambda(\varpi_K) \in \{1, -1\}$; in the latter case define $l = 1$. Define the quantity C to be equal to*

$$\frac{(1 + q^{-1})^2(1 + q^{-2})}{(1 + l\gamma\alpha q^{-\frac{1}{2}})(1 + l\gamma q^{-\frac{1}{2}})(1 + l\gamma^{-1}q^{-\frac{1}{2}})(1 + l\gamma\beta q^{-\frac{1}{2}})(1 - \alpha q^{-1})(1 - \alpha^{-1}q^{-1})(1 - \beta q^{-1})(1 - \beta^{-1}q^{-1})}.$$

Then, if π is a type I representation then $J(\phi, \Lambda) = CJ_0(\phi, \Lambda)$ and if π is a type IIb representation, then $J^*(\phi) = 2CJ_0^*(0, \phi)$.

Proof. This follows from the definitions (30), (33) and well-known formulas for the L-functions involved in terms of the Satake parameters. ∎

Next we write down explicitly a special case of Macdonald's formula (see [8]); the proof follows by a routine computation and is therefore omitted.

2.10 Proposition. *For all non-negative integers ℓ, m, we have*

$$\Phi_0(h(\ell,m)) = \frac{q^{-(4m+3\ell)/2}}{1 + 2q^{-1} + 2q^{-2} + 2q^{-3} + q^{-4}} \sum_{i=1}^{8} A_i B_i, \tag{36}$$

where the quantities A_i, B_i are given as follows:

i	A_i	B_i
1	$\frac{1-q^{-1}\alpha^{-1}\beta}{1-\alpha^{-1}\beta} \frac{1-q^{-1}\beta^{-1}}{1-\beta^{-1}} \frac{1-q^{-1}\alpha^{-1}\beta^{-1}}{1-\alpha^{-1}\beta^{-1}} \frac{1-q^{-1}\alpha^{-1}}{1-\alpha^{-1}}$	$\alpha^{2m+\ell}\beta^{m+\ell}\gamma^{2m+\ell}$
2	$\frac{1-q^{-1}\alpha\beta^{-1}}{1-\alpha\beta^{-1}} \frac{1-q^{-1}\alpha^{-1}}{1-\alpha^{-1}} \frac{1-q^{-1}\alpha^{-1}\beta^{-1}}{1-\alpha^{-1}\beta^{-1}} \frac{1-q^{-1}\beta^{-1}}{1-\beta^{-1}}$	$\alpha^{m+\ell}\beta^{2m+\ell}\gamma^{2m+\ell}$
3	$\frac{1-q^{-1}\alpha^{-1}\beta^{-1}}{1-\alpha^{-1}\beta^{-1}} \frac{1-q^{-1}\beta}{1-\beta} \frac{1-q^{-1}\alpha^{-1}\beta}{1-\alpha^{-1}\beta} \frac{1-q^{-1}\alpha^{-1}}{1-\alpha^{-1}}$	$\alpha^{2m+\ell}\beta^{m}\gamma^{2m+\ell}$
4	$\frac{1-q^{-1}\alpha\beta}{1-\alpha\beta} \frac{1-q^{-1}\alpha^{-1}}{1-\alpha^{-1}} \frac{1-q^{-1}\alpha^{-1}\beta}{1-\alpha^{-1}\beta} \frac{1-q^{-1}\beta}{1-\beta}$	$\alpha^{m+\ell}\gamma^{2m+\ell}$
5	$\frac{1-q^{-1}\alpha^{-1}\beta^{-1}}{1-\alpha^{-1}\beta^{-1}} \frac{1-q^{-1}\alpha}{1-\alpha} \frac{1-q^{-1}\alpha\beta^{-1}}{1-\alpha\beta^{-1}} \frac{1-q^{-1}\beta^{-1}}{1-\beta^{-1}}$	$\alpha^{m}\beta^{2m+\ell}\gamma^{2m+\ell}$
6	$\frac{1-q^{-1}\alpha\beta}{1-\alpha\beta} \frac{1-q^{-1}\beta^{-1}}{1-\beta^{-1}} \frac{1-q^{-1}\alpha\beta^{-1}}{1-\alpha\beta^{-1}} \frac{1-q^{-1}\alpha}{1-\alpha}$	$\beta^{m+\ell}\gamma^{2m+\ell}$
7	$\frac{1-q^{-1}\alpha^{-1}\beta}{1-\alpha^{-1}\beta} \frac{1-q^{-1}\alpha}{1-\alpha} \frac{1-q^{-1}\alpha\beta}{1-\alpha\beta} \frac{1-q^{-1}\beta}{1-\beta}$	$\alpha^{m}\gamma^{2m+\ell}$
8	$\frac{1-q^{-1}\alpha\beta^{-1}}{1-\alpha\beta^{-1}} \frac{1-q^{-1}\beta}{1-\beta} \frac{1-q^{-1}\alpha\beta}{1-\alpha\beta} \frac{1-q^{-1}\alpha}{1-\alpha}$	$\beta^{m}\gamma^{2m+\ell}$

Proof of Theorem 2.1. This follows by combining Lemma 2.4, Lemma 2.5, Proposition 2.8, Lemma 2.9 and Proposition 2.10.

2.3   Basic structure theory

In this subsection we recall some of the structure theory of $G(F)$ concerning Weyl groups, parahoric subgroups, and the Iwahori-Hecke algebra. Many of the stated facts can be verified directly. For others, we refer to [5] and the references therein.

Weyl groups

Let D be the diagonal torus of $\mathrm{Sp}(4)$. Let $\tilde{N}$ be the normalizer of D in $\mathrm{Sp}(4)$. Let

$$\tilde{W} = \tilde{N}(F)/D(\mathfrak{o}) \qquad \text{and} \qquad W = \tilde{N}(F)/D(F). \tag{37}$$

The Weyl group W has 8 elements and is generated by the images of

$$s_1 = \begin{bmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{bmatrix}, \quad s_2 = \begin{bmatrix} 1 & & & \\ & & 1 & \\ & -1 & & \\ & & & 1 \end{bmatrix}.$$

There is an exact sequence

$$1 \longrightarrow D(F)/D(\mathfrak{o}) \longrightarrow \tilde{W} \longrightarrow W \longrightarrow 1, \tag{38}$$

where $D(F)/D(\mathfrak{o}) \cong \mathbb{Z}^2$ via the map $\mathrm{diag}(a,b,a^{-1},b^{-1}) \mapsto (v(a), v(b))$. The *Atkin-Lehner element* of $G(F)$ is defined by

$$\eta = \begin{bmatrix} & & 1 & \\ & & & -1 \\ -\varpi & & & \\ & \varpi & & \end{bmatrix} = s_2 s_1 s_2 \begin{bmatrix} \varpi & & & \\ & -\varpi & & \\ & & 1 & \\ & & & -1 \end{bmatrix} = \begin{bmatrix} -1 & & & \\ & & 1 & \\ & -\varpi & & \\ & & & \varpi \end{bmatrix} s_2 s_1 s_2. \tag{39}$$

Set

$$s_0 = \eta s_2 \eta^{-1} = \begin{bmatrix} 1 & & & \varpi^{-1} \\ & 1 & & \\ & & 1 & \\ & -\varpi & & \end{bmatrix}.$$ (40)

Then $\tilde{W}$ is generated by the images of s_0, s_1, s_2. (We will not distinguish in notation between the matrices s_0, s_1, s_2 and their images in the Weyl groups.) There is a length function on the Coxeter group $\tilde{W}$, which we denote by ℓ.

Parahoric subgroups

The Iwahori subgroup is defined as

$$I = G(\mathfrak{o}) \cap \begin{bmatrix} \mathfrak{o} & \mathfrak{p} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{o} \end{bmatrix}.$$ (41)

If $J \subsetneq \{0, 1, 2\}$, then the corresponding standard parahoric subgroup P_J is generated by I and s_j, where j runs through J. More precisely,

$$P_J = \bigsqcup_{w \in \langle s_j \colon j \in J \rangle} IwI.$$ (42)

The parahoric corresponding to $\{1\}$ is called the *Siegel congruence subgroup* and denoted by P_1. The parahoric corresponding to $\{2\}$ is called the *Klingen congruence subgroup* and denoted by P_2. Explicitly,

$$P_1 = G(\mathfrak{o}) \cap \begin{bmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{o} & \mathfrak{o} \end{bmatrix}, \qquad P_2 = G(\mathfrak{o}) \cap \begin{bmatrix} \mathfrak{o} & \mathfrak{p} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{o} & \mathfrak{p} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{o} \end{bmatrix}.$$ (43)

Note that P_1 is normalized by η, but P_2 is not. We let $P_0 := \eta P_2 \eta^{-1}$; this is the parahoric corresponding to $\{0\}$. The parahoric corresponding to $\{1, 2\}$ is $K := G(\mathfrak{o})$.

The Iwahori-Hecke algebra

We recall the structure of the Iwahori-Hecke algebra $\mathcal{I}$, which is the convolution algebra of compactly supported left and right I-invariant functions on $G(F)$. Explicitly, for T and T' in $\mathcal{I}$, their product is given by

$$(T \cdot T')(x) = \int\limits_{G(F)} T(xy^{-1}) T'(y) \, d^I y.$$

Here, $d^I y$ is the Haar measure on $G(F)$ which gives I volume 1. The characteristic function of I is the identity element of $\mathcal{I}$; we denote it by e. The characteristic function of ηI is an element of $\mathcal{I}$, which we denote again by η. For $j = 0, 1, 2$ let e_i be the characteristic function of Is_iI. Then $\mathcal{I}$ is generated by e_0, e_1, e_2 and η. For $w \in \tilde{W}$, let $q_w = \#IwI/I$. It is easy to verify that

$$q_{s_i} = q \qquad \text{for } i = 0, 1, 2.$$ (44)

It is known that

$$q_{w_1 w_2} = q_{w_1} q_{w_2} \qquad \text{if} \quad \ell(w_1 w_2) = \ell(w_1) + \ell(w_2).$$ (45)

For $w \in \tilde{W}$, let $T_w \in \mathcal{I}$ be the characteristic function of IwI. It is known that

$$T_{w_1 w_2} = T_{w_1} \cdot T_{w_2} \qquad \text{if} \quad \ell(w_1 w_2) = \ell(w_1) + \ell(w_2).$$ (46)

Action of the Iwahori-Hecke algebra on smooth representations

The Iwahori-Hecke algebra $\mathcal{I}$ acts on our representation (π, V) by

$$Tv = \int\limits_{G(F)} T(g)\pi(g)v\, d^I g, \qquad T \in \mathcal{I},\ v \in V.$$

We denote by V^I the subspace of I-invariant vectors. The action of $\mathcal{I}$ induces an endomorphism of V^I. If V is irreducible, then so is V^I.

Set

$$d_i = \frac{1}{q+1}(e + e_i), \qquad i \in \{0, 1, 2\}. \tag{47}$$

Then $d_i^2 = d_i$, and as operator on V, the element d_i acts as a projection onto the space of fixed vectors V^{P_i}. We refer to the operator d_1 as *Siegelization*, and to d_2 as *Klingenization*.

Our goal

Our goal for the rest of §2 is to compute the quantities $J(\phi, \Lambda)$ for suitable vectors $\phi \in \pi$ in certain cases where ϕ is not the spherical vector in a type I representation. Specifically, we will cover the cases where π has a P_1-fixed vector, and we will take ϕ to be such a vector. If π is spherical (i.e., has a K-fixed vector), we will assume it is generic; no such assumption will be made for the remaining representations.

Let us look more closely at the relevant representations π. Looking at the last two columns of Table (19), we see that such a π must be one of types I, IIa, IIIa, Vb/c, VIa or VIb. (Note that the representations IVb, IVc are not unitary). For each of these representations π, we will choose ϕ to be any member of a suitable orthogonal basis for the space of P_1-fixed vectors. We will evaluate $J(\phi, \Lambda)$ exactly in each of these cases under certain additional assumptions which are listed later.

2.4 Calculation of matrix coefficients

We consider the endomorphisms of the space V^{P_1} induced by the elements

$$d_1 e_1 e_0 e_1 \qquad \text{and} \qquad d_1 e_0 e_1 e_0 \tag{48}$$

of the Iwahori-Hecke algebra $\mathcal{I}$. (Since the element $e_0 e_1 e_0$ commutes with e_1, we could have omitted the projection d_1 on the second operator, but we will carry it along for symmetry.) In this section we will calculate the matrix coefficients

$$\lambda(\phi) := \frac{\langle d_1 e_1 e_0 e_1 \phi, \phi \rangle}{\langle \phi, \phi \rangle} \qquad \text{and} \qquad \mu(\phi) := \frac{\langle d_1 e_0 e_1 e_0 \phi, \phi \rangle}{\langle \phi, \phi \rangle} \tag{49}$$

for certain π and certain $\phi \in V^{P_1}$. The results will be needed as input for the calculation of the quantity $J_0(\phi, \Lambda)$ in the next subsection. Evidently, the matrix coefficients (49) depend neither on the normalization of the inner product, nor on the normalization of the vector ϕ.

All irreducible, admissible (π, V) for which V^{P_1} is not zero can be realized as subrepresentations of a full induced representation $\chi_1 \times \chi_2 \rtimes \sigma$ with unramified χ_1, χ_2, σ. Since we are working with a version of GSp(4) which is different from the one in [36], it is necessary to clarify the notation. Let

$$G' = \{g \in \text{GL}(4) : {}^t g J' g = \lambda J',\ \lambda \in \text{GL}(1)\}, \qquad J' = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{bmatrix}.$$

Then G' is the *symmetric* version of GSp(4); this is the one used in [36]. There is an isomorphism between $G(F)$ and $G'(F)$ given by switching the first two rows and the first two columns. For example, the Iwahori subgroup I defined in (41) corresponds to the subgroup I' of $G'(\mathfrak{o})$ consisting of matrices

which are upper triangular mod $\mathfrak{p}$. If (π, V) is a representation of $G(F)$, let (π', V) be the representation of $G'(F)$ obtained by composing π with the isomorphism $G(F) \cong G'(F)$. This establishes a one-one correspondence between representations of $G(F)$ and representations of $G'(F)$. When we write $\chi_1 \times \chi_2 \rtimes \sigma$, we mean the representation of $G(F)$ that corresponds to the representation of $G'(F)$ which in [36] was denoted by the same symbol.

In the following we will rely, sometimes without mentioning it, on Table A.15 of [36], which lists the dimensions of the spaces of fixed vectors under all parahoric subgroups for all Iwahori-spherical representations of $G(F)$.

Type I

Let $\pi = \chi_1 \times \chi_2 \rtimes \sigma$ with unramified χ_1, χ_2, σ. Let V be the standard model of π, and let V^I be the subspace of I-invariant vectors. Then V^I has the basis f_w, $w \in W$, where f_w is the unique I-invariant element of V with $f_w(w) = 1$ and $f_w(w') = 0$ for $w' \in W$, $w' \neq w$. It is convenient to order the basis as follows:

$$f_e, \quad f_1, \quad f_2, \quad f_{21}, \quad f_{121}, \quad f_{12}, \quad f_{1212}, \quad f_{212}, \tag{50}$$

where we have abbreviated $f_1 = f_{s_1}$ and so on. A basis for the four-dimensional space V^{P_1} is given by

$$f_e + f_1, \quad f_2 + f_{21}, \quad f_{121} + f_{12}, \quad f_{1212} + f_{212}. \tag{51}$$

Having fixed the basis (50), the operators e_0, e_1, e_2 and η on V^I become 8×8–matrices. They depend on the Satake parameters

$$\alpha = \chi_1(\varpi), \qquad \beta = \chi_2(\varpi), \qquad \gamma = \sigma(\varpi) \tag{52}$$

and are given[8] in Lemma 2.1.1 of [40].

Now assume that χ_1, χ_2, σ are unitary characters, or equivalently, that α, β, γ have absolute value 1. In this case $\chi_1 \times \chi_2 \rtimes \sigma$ is an irreducible representation of type I. It is unitary and tempered, with hermitian inner product on V given by

$$\langle f, f' \rangle = \int_K f(g)\, \overline{f'(g)}\, dg. \tag{53}$$

The vectors (50) are orthogonal, since they are supported on disjoint cosets. Let $\phi_1, \ldots, \phi_4$ be the vectors in (51), in this order. Then $\phi_1, \ldots, \phi_4$ is an orthogonal basis of V^{P_1}. The vector $\phi = \phi_1 + \phi_2 + \phi_3 + \phi_4$ is the K-fixed vector. If we write $d_1 e_1 e_0 e_1 \phi_i = \sum_{j=1}^4 c_{ij} \phi_j$, $d_1 e_0 e_1 e_0 \phi_i = \sum_{j=1}^4 c'_{ij} \phi_j$, then the quantities defined in (49) are given by $\lambda(\phi_i) = c_{ii}$ and $\mu(\phi_i) = c'_{ii}$. Working out the linear algebra, we find that

$$\lambda(\phi_1) = (q-1)q^2, \qquad \lambda(\phi_2) = \frac{q-1}{q+1}q^2, \qquad \lambda(\phi_3) = \frac{q-1}{q+1}q^2, \qquad \lambda(\phi_4) = 0, \tag{54}$$

$$\mu(\phi_1) = (q-1)q^2, \qquad \mu(\phi_2) = (q-1)q, \qquad \mu(\phi_3) = 0, \qquad \mu(\phi_4) = 0. \tag{55}$$

Type IIIa

Let $\pi = \chi \rtimes \sigma \mathrm{St}_{\mathrm{GSp}(2)}$ with $\chi \notin \{1, \nu^{\pm 2}\}$. Then π is a representation of type IIIa. We assume $\chi \sigma^2 = 1$, so that π has trivial central character. We realize π as a subrepresentation of $\chi \times \nu \rtimes \nu^{-1/2}\sigma$, and set

$$\alpha = \chi(\varpi), \qquad \beta = q^{-1}, \qquad \gamma = q^{1/2}\delta,$$

where $\delta = \sigma(\varpi)$. Let V be the standard space of the full induced representation $\chi \times \nu \rtimes \nu^{-1/2}\sigma$, and let U be the subspace of V realizing π. To determine the two-dimensional space U^{P_1}, we observe that

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[8]The matrix for e_1 in Lemma 2.1.1 of [40] contains a typo: The two lower right 2×2-blocks need to be conjugated by $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

$\dim U^{P_2} = 1$, so that the Klingenization map $d_2 : U^{P_1} \to U^{P_2}$ has a non-trivial kernel. The condition $d_2\phi = 0$ characterizes a unique vector in V^{P_1}, which must thus lie in U^{P_1}. A second, linearly independent vector can then be obtained by applying η. Thus we find that U^{P_1} is spanned by

$$\phi_1 = {}^t[q, q, -1, -1, 0, 0, 0, 0] \qquad \text{and} \qquad \phi_2 = {}^t[0, 0, 0, 0, -q, -q, 1, 1]; \tag{56}$$

these are column vectors using the basis (50). These vectors are eigenvectors for the endomorphism $T_{1,0} = e_2 e_1 e_2 \eta$:

$$T_{1,0}\phi_1 = \alpha\delta q\phi_1, \qquad T_{1,0}\phi_2 = \delta q\phi_2. \tag{57}$$

Now assume that χ and σ are unitary. In this case π is unitary and tempered. Since the invariant inner product $\langle \cdot, \cdot \rangle$ is not a priori given by formula (53), the following lemma is not obvious:

2.11 Lemma. *The vectors ϕ_1 and ϕ_2 are orthogonal.*

Proof.   First note that

$$\langle \pi(\eta)v, w \rangle = \langle v, \pi(\eta)w \rangle \qquad \text{for all } v, w \in U; \tag{58}$$

this is immediate, since η^2 acts as the identity on V. By Proposition 2.1.2 of [40], we have

$$\langle e_i v, w \rangle = \langle v, e_i w \rangle \qquad \text{for all } v, w \in U^I \text{ and } i \in \{1, 2\}. \tag{59}$$

We will now calculate $\langle T_{1,0}\phi_1, \phi_2 \rangle$ in two different ways. On the one hand, by (57),

$$\langle T_{1,0}\phi_1, \phi_2 \rangle = \alpha\delta q\langle \phi_1, \phi_2 \rangle. \tag{60}$$

On the other hand, by (58) and (59), $\langle T_{1,0}\phi_1, \phi_2 \rangle = \langle \phi_1, \eta e_2 e_1 e_2 \phi_2 \rangle$. A calculation using the explicit form (56) shows that $\eta e_2 e_1 e_2 \phi_2 = \delta^{-1} q\phi_2$. Hence

$$\langle T_{1,0}\phi_1, \phi_2 \rangle = \delta q\langle \phi_1, \phi_2 \rangle. \tag{61}$$

Our assertion follows from (60) and (61), since $\alpha \neq 1$ for representations of type IIIa.   ∎

In view of Lemma 2.11, we can now calculate the quantities (49) for the vectors ϕ_1 and ϕ_2 similar to the type I case. The result is

$$\lambda(\phi_i) = -\frac{q^2}{q+1}, \qquad \mu(\phi_i) = 0, \tag{62}$$

for $i = 1, 2$.

Type VIb

Consider the representation $\pi = \tau(T, \nu^{-1/2}\sigma)$ of type VIb. We will assume $\sigma^2 = 1$, so that π has trivial central character. There is a unique P_1-fixed element (up to scalars) ϕ_1 in the space of π. A similar calculation to the above shows that

$$\lambda(\phi_1) = -q^2 \qquad \text{and} \qquad \mu(\phi_1) = q. \tag{63}$$

Types IIa, Vb/c and VIa

Let (π, V) be an Iwahori-spherical representation of type IIa, Vb, Vc or VIa. Then V^{P_1} is one-dimensional by Table (19). Assume in addition that π is unitary and has trivial central character. If ϕ_1 is a vector spanning V^{P_1}, then calculations similar to the above show that

$$\lambda(\phi_1) = \frac{q-1}{q+1}q^2 \qquad \text{and} \qquad \mu(\phi_1) = -q. \tag{64}$$

We mention these representations just for completeness; they are not relevant for our global applications, since, by Theorem 6.2.2 of [37], they do not admit a special, non-split Bessel model.

2.5 The local integral for certain ramified representations

Our goal in this section is to prove the following analogue of Theorem 2.1.

2.12 Theorem. *Suppose that π is an irreducible, admissible, unitary, Iwahori-spherical representation of $\mathrm{GSp}(4, F)$ with trivial central character. We assume that π is one of the following types: I, IIa, IIIa, Vb/c, VIa or VIb. For type I, let the vectors $\phi_1, \ldots, \phi_4$ be the vectors in (51), in this order. For type IIIa, let ϕ_1 and ϕ_2 be as given in (56); in all other cases let ϕ_1 be the essentially unique P_1-invariant vector. Suppose that the residual characteristic of F is odd, $K = F(\sqrt{d})$ is the unramified quadratic extension of F, and Λ is the trivial character on $K^\times$.[9] Then the quantities $J_0(\phi, \Lambda)$, $J(\phi, \Lambda)$, $J^*(\phi)$ are as given by the last two columns of the table below.*

type	ϕ	$J_0(\phi, \Lambda)$	$J(\phi, \Lambda) = J^*(\phi)$
I	ϕ_1	q^{-1}	$q^{-1} L(1, \pi, \mathrm{Std})(1 - q^{-4})$
	ϕ_2	1	$L(1, \pi, \mathrm{Std})(1 - q^{-4})$
	ϕ_3	q^{-1}	$q^{-1} L(1, \pi, \mathrm{Std})(1 - q^{-4})$
	ϕ_4	1	$L(1, \pi, \mathrm{Std})(1 - q^{-4})$
IIIa	ϕ_1	$1 + q^{-1}$	$(1 + q^{-2})(1 + q^{-1})$
	ϕ_2	$1 + q^{-1}$	$(1 + q^{-2})(1 + q^{-1})$
VIb	ϕ_1	$2(1 + q^{-1})$	$2(1 + q^{-2})(1 + q^{-1})$
IIa, Vb/c, VIa	ϕ_1	0	0

2.13 Remark. *Above, $L(1, \pi, \mathrm{Std})$ denotes the special value of the degree 5 L-function, defined in (35).*

Throughout the rest of this subsection we assume that:

  i) The residual characteristic of F is odd.

  ii) $K = F(\sqrt{d})$ is the unramified quadratic extension of F. (Equivalently, $d := b^2 - 4ac$ is in $\mathfrak{o}^\times$ but not in $\mathfrak{o}^{\times 2}$. Since $1 + \mathfrak{p} \subset \mathfrak{o}^{\times 2}$, this implies $a \in \mathfrak{o}^\times$.)

  iii) $\Lambda = 1$.

  iv) ϕ is a P_1-invariant vector.

Note that the assumptions imply that

$$J_0(\phi, \Lambda) = J_0(\phi, 1). \tag{65}$$

As in the proof of Theorem 2.1, we may assume that $b = 0$. Also, as before, we use Φ_0 to denote Φ_ϕ. Note that Φ_0 is a ZP_1-bi-invariant function. The following result is the analogue of Proposition 2.8 in this case.

2.14 Proposition. *We have*

$$J_0(\phi, \Lambda) = 1 - (q + 1)\Phi_0(g_1) + q\Phi_0(g_2), \tag{66}$$

where

$$g_1 = \begin{bmatrix} \varpi^{-1} & & & \\ & 1 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} s_2, \qquad g_2 = \begin{bmatrix} \varpi^{-1} & & & \\ & \varpi^{-1} & & \\ & & \varpi & \\ & & & \varpi \end{bmatrix} s_2 s_1 s_2. \tag{67}$$

[9]This is equivalent to saying that Λ is an unramified character on $K^\times$, since $\Lambda|_{F^\times} = 1$ and K/F is an unramified field extension.

Proof. By (65), $J_0(\phi, \Lambda) = J_0(\phi, 1) = \lim_{k \to \infty} J_0^{(k)}(\phi, 1)$. So we need to compute

$$J_0^{(k)}(\phi, 1) = \int_{x,y,z \in \mathfrak{p}^{-k}} \Phi_0\left(\begin{bmatrix} 1 & x & y \\ & 1 & y & z \\ & & 1 \\ & & & 1 \end{bmatrix}\right) \psi^{-1}(ax + cz) \, dxdydz.$$

We assume $k \geq 1$. A similar argument to that in Proposition 2.8 shows that

$$J_0^{(k)}(\phi, 1) = \int_{\mathfrak{p}^{-1}} \int_{\mathfrak{p}^{-k}} \int_{\mathfrak{p}^{-1}} \Phi_0\left(\begin{bmatrix} 1 & x & y \\ & 1 & y & z \\ & & 1 \\ & & & 1 \end{bmatrix}\right) \psi^{-1}(ax + cz) \, dz \, dy \, dx.$$

Using the left invariance properties of Φ_0, and a simple calculation using the identity

$$\begin{bmatrix} 1 & y \\ & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ y^{-1} & 1 \end{bmatrix} \begin{bmatrix} y \\ & y^{-1} \end{bmatrix} \begin{bmatrix} & 1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 \\ y^{-1} & 1 \end{bmatrix}, \tag{68}$$

one can check that

$$\int_{\mathfrak{p}^{-1}} \int_{\varpi^{-\ell}\mathfrak{o}^\times} \int_{\mathfrak{p}^{-1}} \Phi_0\left(\begin{bmatrix} 1 & x & y \\ & 1 & y & z \\ & & 1 \\ & & & 1 \end{bmatrix}\right) \psi^{-1}(ax + cz) \, dz \, dy \, dx = 0 \qquad \text{for } \ell \geq 2. \tag{69}$$

We conclude that

$$J_0^{(k)}(\phi, 1) = \int_{\mathfrak{p}^{-1}} \int_{\mathfrak{p}^{-1}} \int_{\mathfrak{p}^{-1}} \Phi_0\left(\begin{bmatrix} 1 & x & y \\ & 1 & y & z \\ & & 1 \\ & & & 1 \end{bmatrix}\right) \psi^{-1}(ax + cz) \, dz \, dy \, dx.$$

Write $J_0^{(k)}(\phi, 1) = J_1 + J_2$, where J_1 is the part where $x \in \mathfrak{o}$, and J_2 is the part where $x \in \varpi^{-1}\mathfrak{o}^\times$. Evidently,

$$J_1 = \int_{\mathfrak{p}^{-1}} \int_{\mathfrak{p}^{-1}} \Phi_0\left(\begin{bmatrix} 1 & & y \\ & 1 & y & z \\ & & 1 \\ & & & 1 \end{bmatrix}\right) \psi^{-1}(cz) \, dz \, dy.$$

Let J_{11} be the part where $y \in \mathfrak{o}$, and let J_{12} be the part where $y \notin \mathfrak{o}$. Using (68), we have

$$J_{11} = \Phi_0(1) - \Phi_0\left(\begin{bmatrix} 1 \\ & \varpi^{-1} \\ & & 1 \\ & & & \varpi \end{bmatrix} s_1 s_2 s_1\right).$$

By a conjugation of the argument of Φ_0 and the use of bi-I-invariance we see that, for any $w \in \mathfrak{o}$,

$$J_{12} = \int_{\varpi^{-1}\mathfrak{o}^\times} \int_{\mathfrak{p}^{-1}} \Phi_0\left(\begin{bmatrix} 1 & & y \\ & 1 & y & z \\ & & 1 \\ & & & 1 \end{bmatrix}\right) \psi^{-1}(cz) \, dz \, dy$$

$$= \int_{\varpi^{-1}\mathfrak{o}^\times} \int_{\mathfrak{p}^{-1}} \Phi_0\left(\begin{bmatrix} 1 & & y \\ & 1 & y & z+2wy \\ & & 1 \\ & & & 1 \end{bmatrix}\right) \psi^{-1}(cz) \, dz \, dy.$$

The element w may depend on y and z. Choosing $w = -y^{-1}z/2$, we see $J_{12} = 0$. Thus

$$J_1 = J_{11} = \Phi_0(1) - \Phi_0\left(\begin{bmatrix} 1 \\ & \varpi^{-1} \\ & & 1 \\ & & & \varpi \end{bmatrix} s_1 s_2 s_1\right).$$

A similar (but slightly more involved) computation gives

$$J_2 = J_{21} + J_{22} = -q\Phi_0\left(\begin{bmatrix} \varpi^{-1} \\ & 1 \\ & & \varpi \\ & & & 1 \end{bmatrix} s_2\right) + q\Phi_0\left(\begin{bmatrix} \varpi^{-1} \\ & \varpi^{-1} \\ & & \varpi \\ & & & \varpi \end{bmatrix} s_2 s_1 s_2 s_1\right).$$

This concludes the proof, using the P_1-invariance of Φ_0. ∎

2.15 Remark. *The proof of the above proposition used only the bi-ZP_1-invariance of the function Φ_0, just as the proof of Proposition 2.8 relied only on the bi-$ZG(\mathfrak{o})$-invariance.*

2.16 Remark. *Suppose that ϕ is no longer assumed to be P_1-invariant, but merely I-invariant. Then with some additional work one can show*

$$J_0(\phi, \Lambda) = \frac{1}{1+q}\left[1 + q\Phi_\phi(s_1) - q^2\Phi_\phi(\begin{bmatrix} \varpi^{-1} & & \\ & 1 & \\ & & \varpi & \\ & & & 1 \end{bmatrix} s_2) - q\Phi_\phi(\begin{bmatrix} 1 & & \\ & \varpi^{-1} & \\ & & 1 & \\ & & & \varpi \end{bmatrix} s_1 s_2)\right.$$

$$-q\Phi_\phi(\begin{bmatrix} \varpi^{-1} & & \\ & 1 & \\ & & \varpi & \\ & & & 1 \end{bmatrix} s_2 s_1) - \Phi_\phi(\begin{bmatrix} 1 & & \\ & \varpi^{-1} & \\ & & 1 & \\ & & & \varpi \end{bmatrix} s_1 s_2 s_1)$$

$$\left.+q\Phi_\phi(\begin{bmatrix} \varpi^{-1} & & \\ & \varpi^{-1} & \\ & & \varpi & \\ & & & \varpi \end{bmatrix} s_2 s_1 s_2) + q^2\Phi_\phi(\begin{bmatrix} \varpi^{-1} & & \\ & \varpi^{-1} & \\ & & \varpi & \\ & & & \varpi \end{bmatrix} s_1 s_2 s_1 s_2)\right].$$

The values $\Phi_0(g_i)$ appearing in Proposition 2.14 depend on the representation π. We can convert an operator $\pi(g)$ on V^I into an element of the Iwahori-Hecke algebra $\mathcal{I}$. Recall that e, the characteristic function of I, is the identity element of $\mathcal{I}$. For $g \in G(F)$, let T_g be the characteristic function of IgI. We have the following easy lemma, whose proof we omit.

2.17 Lemma. *Let (π, V) be a smooth representation of $G(F)$. Let g be any element of $G(F)$. Then $e \circ \pi(g) = \frac{1}{\#IgI/I} T_g$ as operators on V^I.*

Proof of Theorem 2.12. Applying Lemma 2.17 to the elements g_1 and g_2 in (67), we see that

$$e \circ \pi(g_i) = \frac{1}{\#Ig_iI/I} \operatorname{char}(Ig_iI) \tag{70}$$

as operators on V^{P_1}. Note that, as elements of $\tilde{W}$,

$$g_1 = s_1 s_0 s_1, \qquad g_2 = s_0 s_1 s_0. \tag{71}$$

From (44), (45), (46) and (70), we therefore get

$$e \circ \pi(g_1) = q^{-3} e_1 e_0 e_1, \qquad e \circ \pi(g_2) = q^{-3} e_0 e_1 e_0. \tag{72}$$

Substituting into (66), we obtain the formula

$$J_0(\phi, \Lambda) = 1 - (q+1)q^{-3}\lambda(\phi) + q^{-2}\mu(\phi), \tag{73}$$

with $\lambda(\phi)$ and $\mu(\phi)$ as defined in (49). (We can insert d_1 into the inner product because ϕ is assumed to be P_1-invariant.)

The quantities $\lambda(\phi)$ and $\mu(\phi)$ have been calculated in Sect. 2.4 for various vectors in a number of representations. Using these, we can now compute $J_0(\phi, \Lambda)$ in each case using (73). Furthermore, writing

$$M(\pi) = \frac{L(1, \pi, \operatorname{Ad})L(1, \chi_{K/F})}{\zeta_F(2)\zeta_F(4)L(1/2, \pi \times \mathcal{AI}(\Lambda^{-1}))},$$

we have $J(\phi, \Lambda) = M(\pi)J_0(\phi, \Lambda)$. We can write down $M(\pi)$ in each case using the relevant data from [40] and [2]. As a result, we can compute $J(\phi, \Lambda)$ in each case as well. The result is exactly the values in Theorem 2.12. The proof of the fact that $J_0(\phi, \Lambda)$ equals $J^*(\phi)$ follows by an identical argument to Lemma 2.4 (recall that we are assuming that Λ is the trivial character).

2.18 Remark. *Note that for representations of type IIa, Vb/c and VIa we obtain $J_0(\phi_1) = 0$ for the essentially unique P_1-invariant vector ϕ_1. This is consistent with the fact that these representations do not admit a non-split Bessel functional with trivial character Λ on the torus $T(F)$; see Theorem 6.2.2 of [37]. Hence, such representations will not be relevant for our global applications to Siegel modular forms.*

2.19 Remark. *Note that whenever $J(\phi, \Lambda) \neq 0$, we have $\sum_i J(\phi_i) - 2 \ll q^{-1}$.*

3   Global results

In this final section, we return to a global setup. We begin with some basic results on the correspondence between Siegel modular forms and automorphic representations on $G(\mathbb{A})$. Then we move on to the relation between Bessel periods and Fourier coefficients and the classical reformulation of Conjecture 1.12. Finally, we work out several consequences of our explicit refinement of Böcherer's conjecture. We remind the reader that all our measures on adelic groups are equal to the respective Tamagawa measures.

3.1   Siegel modular forms and representations

We use the definitions and some basic properties of Siegel modular forms of degree 2 without proof; we refer the reader to [40] or [38] for details. Let k and N be positive integers with N square-free. Let $\Gamma_0(N) \subseteq \mathrm{Sp}(4, \mathbb{Z})$ denote the Siegel congruence subgroup of squarefree level N, defined in (12). If $p|N$ is prime, set $P_{1,p}(N) = P_1$, the local analog of $\Gamma_0(N)$ defined in (11); if $p \nmid N$ is prime, set $P_{1,p}(N) = \mathrm{GSp}(4, \mathbb{Z}_p)$. We let $S_k(\Gamma_0(N))$ denote the space of Siegel cusp forms of weight k with respect to the group $\Gamma_0(N)$. For any $f \in S_k(\Gamma_0(N))$, define the Petersson inner product

$$\langle f, f \rangle = \frac{1}{[\mathrm{Sp}(4, \mathbb{Z}) : \Gamma_0(N)]} \int_{\Gamma_0(N) \backslash \mathbb{H}_2} |f(Z)|^2 (\det Y)^{k-3} \, dX \, dY. \tag{74}$$

The space $S_k(\Gamma_0(N))$ has a natural orthogonal (with respect to the Petersson inner product) decomposition

$$S_k(\Gamma_0(N)) = S_k(\Gamma_0(N))^{\mathrm{old}} \oplus S_k(\Gamma_0(N))^{\mathrm{new}}$$

into the newspace and the oldspace in the sense of [40].

For any $\phi \in L^2(Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A}))$, let $\langle \phi, \phi \rangle = \int_{Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})} |\phi(g)|^2 \, dg$.

3.1 Proposition. *Suppose that N is a squarefree integer and k a positive integer. Let π be an irreducible cuspidal automorphic representation of $G(\mathbb{A})$ with trivial central character. Suppose that[10] $\pi_\infty \simeq L(k, k)$ and π_p has a non-zero $P_{1,p}(N)$-fixed vector for all primes $p|N$. Let $\phi \in V_\pi \subset L^2(Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A}))$ be a non-zero function such that[11] $\phi(gk_N k_\infty) = \det(J(k_\infty, iI_2))^{-k}\phi(g)$ for all $k_\infty \in K_\infty$, $k_N \in \prod_{p<\infty} P_{1,p}(N)$. Define the function f on $\mathbb{H}_2$ by $f(g_\infty i) = \det(J(g_\infty, iI_2))^k \phi(g_\infty)$. Then f is a non-zero element of $S_k(\Gamma_0(N))$ and is an eigenfunction for the local Hecke algebras at all $p \nmid N$. The function f belongs to $S_k(\Gamma_0(N))^{\mathrm{old}}$ if and only if π_p is spherical for some $p|N$; else it belongs to $S_k(\Gamma_0(N))^{\mathrm{new}}$. Furthermore,*

$$\frac{\langle f, f \rangle}{\mathrm{vol}(\mathrm{Sp}(4, \mathbb{Z})\backslash \mathbb{H}_2)} = \frac{\langle \phi, \phi \rangle}{\mathrm{vol}(Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A}))}.$$

Proof. This is routine; see Sect. 3.2 of [40] and Sect. 4.1 of [1].   ∎

Given any $f \in S_k(\Gamma_0(N))$, we can define its adelization as in [40] or [38]. A non-zero $f \in S_k(\Gamma_0(N))$ arises via Proposition 3.1 if and only if its adelization generates an *irreducible* automorphic representation, in which case the adelization precisely equals the function ϕ in the proposition. It is easy to see that the set of such f spans the space $S_k(\Gamma_0(N))$, a fact that follows immediately from the decomposition of the cuspidal automorphic spectrum into a direct sum of irreducible representations.

Let π, ϕ, f be as in Proposition 3.1. Then using the results of [40] (or see table (19) earlier) we see that the local representations π_p at finite primes have the following properties.

- If $p \nmid N$, then π_p is spherical, i.e., one of types I, IIb, IIIb, IVd, Vd, VId. Of these, types IIb, IIIb, IVd, Vd, VId are non-tempered (as well as non-generic). Type I is generic; moreover it is tempered provided the inducing characters are unitary.

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[10]See the paper [29] for the definition of $L(k, \ell)$; this was also called $\mathcal{E}(k, \ell)$ previously by us.
[11]Our conditions on π imply that such a function always exists.

- If $p|N$, then π_p is either spherical (in which case it is one of those from the above list) or non-spherical, in which case it is one of types IIa, IIIa, Vb/c, VIa or VIb. Of these, Vb/c are non-generic and non-tempered, VIb is non-generic and tempered, IIIa/VIa are generic and tempered, and IIa is generic and can be either tempered or non-tempered. Recall that conjecturally, π_p is always tempered.

A representation π of $G(\mathbb{A})$ is said to be CAP if it is nearly equivalent to a constituent of a global induced representation of a proper parabolic subgroup of $G(\mathbb{A})$. Otherwise we say that it is non-CAP. A CAP representation can be either P-CAP (associated to the Siegel parabolic) or Q-CAP (associated to the Klingen parabolic) or B-CAP (associated to the Borel parabolic). If π, f are as in Proposition 3.1 with π a P-CAP representation, then f is called a *Saito-Kurokawa* lift.

A non-CAP representation is type I at all primes where π_p is spherical, and hence is generic almost everywhere. A CAP representation *is not generic at any place*, and hence cannot have a type I component anywhere. Further, a non-CAP representation is *expected* to satisfy the generalized Ramanujan conjecture, which postulates that it must be tempered at all places and hence conjecturally can never equal one of the representations IIb, IIIb, IVd, Vb/c, Vd or VId.

If $k \geq 3$, and π, f are as in Proposition 3.1, then the following additional facts are known to be true:

- If π is CAP then π must be P-CAP (see Corollary 4.5 of [31]) and therefore π_p is type IIb whenever it is spherical and type VIb whenever it is not spherical. In particular, f is a Saito-Kurokawa lift.

- If π is non-CAP, then π_p is *tempered* type I whenever it is spherical. This is due to Weissauer [45].

We let $S_k(\Gamma_0(N))^{\mathrm{CAP}}$, (resp. $S_k(\Gamma_0(N))^{\mathrm{T}}$) denote the subspace spanned by all the f as in Proposition 3.1 for which the associated π are CAP (resp. non-CAP). The letter T is chosen because the non-CAP forms are (conjecturally) precisely the ones that are tempered everywhere. Recall that if $k \geq 3$, then it is known that the space $S_k(\Gamma_0(N))^{\mathrm{CAP}}$ is spanned precisely by the Saito-Kurokawa lifts, and that the representations attached to the space $S_k(\Gamma_0(N))^{\mathrm{T}}$ are tempered at all unramified places.

It is clear that we have orthogonal (with respect to Petersson inner product) direct sum decompositions

$$S_k(\Gamma_0(N)) = S_k(\Gamma_0(N))^{\mathrm{CAP}} \oplus S_k(\Gamma_0(N))^{\mathrm{T}},$$

$$S_k(\Gamma_0(N))^{\mathrm{old}} = S_k(\Gamma_0(N))^{\mathrm{old,CAP}} \oplus S_k(\Gamma_0(N))^{\mathrm{old,T}},$$

$$S_k(\Gamma_0(N))^{\mathrm{new}} = S_k(\Gamma_0(N))^{\mathrm{new,CAP}} \oplus S_k(\Gamma_0(N))^{\mathrm{new,T}},$$

$$S_k(\Gamma_0(N))^{\mathrm{CAP}} = S_k(\Gamma_0(N))^{\mathrm{new,CAP}} \oplus S_k(\Gamma_0(N))^{\mathrm{old,CAP}},$$

$$S_k(\Gamma_0(N))^{\mathrm{T}} = S_k(\Gamma_0(N))^{\mathrm{new,T}} \oplus S_k(\Gamma_0(N))^{\mathrm{old,T}}.$$

Recall that Conjecture 1.12 only applies to those π that are generic almost everywhere, i.e., only to non-CAP π. However the CAP case is actually much easier, and Qiu [34] has recently proved a theorem that asserts that an analog of Conjecture 1.12 holds for all CAP representations; see Theorem 3.11.

3.2 Newforms and orthogonal Hecke bases

Let N be a squarefree integer. We say that a non-zero f in $S_k(\Gamma_0(N))$ is a **newform** if

i) f belongs to the newspace $S_k(\Gamma_0(N))^{\mathrm{new}}$.

ii) For each prime $p|N$, f is an eigenfunction of the $U(p)$ operator (see [39, Sect. 2.3] for the definition).

iii) The adelization ϕ of f generates an irreducible representation π (in other words, π, f are in the situation of Proposition 3.1).

The conditions imply that the automorphic form ϕ corresponds to a *factorizable* vector $\phi = \otimes_v \phi_v$ in π. Indeed, for any $p|N$, the local representation π_p has a unique $P_{1,p}(N)$-invariant vector, except if π_p is of type IIIa, in which case the space is two dimensional but has (up to multiples) exactly two vectors that are eigenfunctions of the local analog of $U(p)$, which is the $T_{1,0}$ operator considered earlier.

Using the result from [26], it can be shown that if $f \in S_k(\Gamma_0(N))$ is an eigenfunction of the local Hecke algebras at all places, then it is a newform. However the converse is not true (newforms are not necessarily an eigenfunction of the local Hecke algebra at all primes, in fact they fail to be so precisely at the primes p where the corresponding π_p is a type IIIa representation).

3.2 Lemma. *The space $S_k(\Gamma_0(N))^{\mathrm{new}}$ has an orthogonal basis consisting of newforms.*

Proof. This is immediate by decomposing the space $S_k(\Gamma_0(N))^{\mathrm{new}}$ into mutually orthogonal subspaces corresponding to irreducible automorphic representations, and then using the fact that two linearly independent local vectors in the type IIIa representation can be chosen to be orthogonal. ∎

Next, for any four positive, mutually coprime, squarefree integers a, b, c, d, we will define a linear map $\delta_{a,b,c,d}$. This map will take $S_k(\Gamma_0(e))^{\mathrm{T}}$ to $S_k(\Gamma_0(abcde))^{\mathrm{T}}$ for each positive squarefree integer e coprime to $abcd$. It is defined as follows.

Let $f \in S_k(\Gamma_0(e))^{\mathrm{T}}$ be such that its adelization ϕ generates an irreducible representation π. (It suffices to define the map on such f because these span the full space $S_k(\Gamma_0(e))^{\mathrm{T}}$.) The automorphic form ϕ corresponding to f factors as $\phi = \phi_S \otimes_{p \nmid e} \phi_p$ where S denotes the set of places dividing e. Let p be any prime dividing $abcd$. Then the local vector ϕ_p is a spherical vector in a type I representation. Using the notation of Sect. 2.4, we have an orthogonal decomposition $\phi_p = \phi_{1,p} + \phi_{2,p} + \phi_{3,p} + \phi_{4,p}$. Define $\delta_{a,b,c,d}(\phi) = \phi_S \otimes_{p \nmid e} \phi_p'$ where $\phi_p' = \phi_p$ if $p \nmid abcd$, $\phi_p' = \phi_{1,p}$ if $p|a$, $\phi_p' = \phi_{2,p}$ if $p|b$, $\phi_p' = \phi_{3,p}$ if $p|c$, $\phi_p' = \phi_{4,p}$ if $p|d$. Using Proposition 3.1, we see that this takes f to an element $\delta_{a,b,c,d}(f)$ of $S_k(\Gamma_0(abcde))^{\mathrm{T}}$. Note that $\delta_{1,1,1,1}$ is just the identity map.

3.3 Proposition. *For any five coprime squarefree positive integers a, b, c, d, e, the map $\delta_{a,b,c,d}$ is an injective linear map that takes $S_k(\Gamma_0(e))^{\mathrm{T}}$ to $S_k(\Gamma_0(abcde))^{\mathrm{T}}$. Furthermore, for any positive squarefree integer N, we have an orthogonal direct sum decomposition*

$$S_k(\Gamma_0(N))^{\mathrm{T}} = \bigoplus_{\substack{a,b,c,d,e \\ abcde=N}} \delta_{a,b,c,d}\left(S_k(\Gamma_0(e))^{\mathrm{new,T}}\right).$$

Proof. By construction, for each a, b, c, d, e, the space $\delta_{a,b,c,d}\left(S_k(\Gamma_0(e))^{\mathrm{new,T}}\right)$ is the span of all those f in $S_k(\Gamma_0(N))^{\mathrm{T}}$ that generate an irreducible representation $\pi_f = \otimes_v' \pi_v$ with the property that π_p is spherical if and only if $p \nmid e$, and moreover, $\phi_p = \phi_{1,p}$ for all primes dividing a, ..., $\phi_p = \phi_{4,p}$ for all primes dividing d. Clearly, as a, b, c, d, e vary, this takes care of all representations $\pi = \pi_f$ that come from Proposition 3.1. The orthogonality and injectivity properties come from the corresponding properties of the local maps which were proved earlier. ∎

3.4 Corollary. *Let $N > 0$ be squarefree. For all positive integers $e|N$, let $\mathcal{B}_{k,e}^{\mathrm{new,T}}$ be an orthogonal basis of $S_k(\Gamma_0(e))^{\mathrm{new,T}}$ consisting of newforms (such a basis is known to exist by Lemma 3.2). Then the set $\{\delta_{a,b,c,d}(f)\}_{\substack{a,b,c,d,e \\ abcde=N \\ f \in \mathcal{B}_{k,e}^{\mathrm{new,T}}}}$ gives an orthogonal basis for $S_k(\Gamma_0(N))^{\mathrm{T}}$.*

Proof. This follows immediately from the above proposition. ∎

Thus, we have constructed a nice orthogonal Hecke basis for $S_k(\Gamma_0(N))^{\mathrm{T}}$. The adelization ϕ of any element of our basis generates an irreducible representation. Moreover, ϕ is a factorizable vector and its local components ϕ_p are *precisely* the local vectors for which we have computed $J(\phi_p)$ previously.

3.3   Bessel periods and Fourier coefficients

Let $f \in S_k(\Gamma_0(N))$ have the Fourier expansion as in (1),

$$f(Z) = \sum_T a(f,T)e^{2\pi i \operatorname{Tr}(TZ)}.$$

Let ϕ_f be the function associated to f via strong approximation, as in [40] or [38]. Let $d < 0$ be a fundamental discriminant and put

$$S = S(d) = \begin{cases} \begin{bmatrix} \frac{-d}{4} & 0 \\ 0 & 1 \end{bmatrix} & \text{if } d \equiv 0 \pmod 4, \\[2ex] \begin{bmatrix} \frac{1-d}{4} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} & \text{if } d \equiv 1 \pmod 4. \end{cases} \tag{75}$$

Given S as above, let the group T_S be defined as before. So $T_S \simeq K^\times$ where $K = \mathbb{Q}(\sqrt{d})$. Define

$$\operatorname{Cl}_K = T_S(\mathbb{A})/T_S(\mathbb{Q})T_S(\mathbb{R})K_0, \qquad \text{where } K_0 = \prod_{p<\infty}(T_S(\mathbb{Q}_p) \cap \operatorname{GL}(2,\mathbb{Z}_p)).$$

Then Cl_K can be naturally identified with the ideal class group of K (see Proposition VI.1.3 of [28]). We set $h_K = |\operatorname{Cl}_K|$. Let t_c, $c \in \operatorname{Cl}_K$, be coset representatives of Cl_K with $t_c \in \prod_{p<\infty} T_S(\mathbb{Q}_p)$. We can write $t_c = \gamma_c m_c \kappa_c$ with $\gamma_c \in \operatorname{GL}(2,\mathbb{Q})$, $m_c \in \operatorname{GL}(2,\mathbb{R})^+$, and $\kappa_c \in \prod_{p<\infty} \operatorname{GL}(2,\mathbb{Z}_p)$; note that $(\gamma_c)_\infty = m_c^{-1}$. The matrices $S_c = \det(\gamma_c)^{-1} \, ({}^t\gamma_c)S\gamma_c$ have discriminant d, and form a set of representatives of the $\operatorname{SL}(2,\mathbb{Z})$-equivalence classes of primitive semi-integral positive definite matrices of discriminant d.

Let $\psi : \mathbb{Q}\backslash\mathbb{A} \to \mathbb{C}^\times$ be the character such that $\psi(x) = e^{2\pi i x}$ if $x \in \mathbb{R}$ and $\psi(x) = 1$ for $x \in \mathbb{Z}_p$. One obtains a character θ_S of $N(\mathbb{Q})\backslash N(\mathbb{A})$ by $\theta_S(\begin{bmatrix} 1 & X \\ & 1 \end{bmatrix}) = \psi(\operatorname{Tr}(SX))$. Let Λ be a character of Cl_K. Then, as before, we can define the Bessel period

$$B(\phi_f,\Lambda) = \int\limits_{\mathbb{A}^\times T_S(\mathbb{Q})\backslash T_S(\mathbb{A})} \int\limits_{N(\mathbb{Q})\backslash N(\mathbb{A})} \phi_f(tn)\Lambda^{-1}(t)\theta_S^{-1}(n)\, dn\, dt. \tag{76}$$

Then we have the following result.

3.5 Proposition. *We have*

$$\frac{B(\phi_f,\Lambda)}{\operatorname{vol}(\mathbb{A}^\times T_S(\mathbb{Q})\backslash T_S(\mathbb{A}))} = e^{-2\pi\operatorname{tr}(S)}\frac{1}{h_K}\sum_{c\in\operatorname{Cl}_K}\Lambda(c)^{-1}a(f,S_c), \tag{77}$$

where we take any measure on $\mathbb{A}^\times T_S(\mathbb{Q})\backslash T_S(\mathbb{A})$, but normalize the measure on $N(\mathbb{Q})\backslash N(\mathbb{A})$ to have total volume 1.

Proof. For a function ϕ on $T_S(\mathbb{A})$, which is invariant under $\mathbb{A}^\times T_S(\mathbb{Q})K_0$, we have

$$\int\limits_{\mathbb{A}^\times T_S(\mathbb{Q})\backslash T_S(\mathbb{A})} \phi(t)\, dt = \sum_{c\in\operatorname{Cl}_K} \int\limits_{\mathbb{A}^\times T_S(\mathbb{Q})\backslash T_S(\mathbb{Q})T_S(\mathbb{R})K_0} \phi(t_c t)\, dt$$

$$= \frac{\operatorname{vol}(\mathbb{A}^\times T_S(\mathbb{Q})\backslash T_S(\mathbb{Q})T_S(\mathbb{R})K_0)}{\operatorname{vol}(\mathbb{R}^\times\backslash T_S(\mathbb{R}))} \sum_{c\in\operatorname{Cl}_K} \int\limits_{\mathbb{R}^\times\backslash T_S(\mathbb{R})} \phi(t_c t)\, dt.$$

Note that $\operatorname{vol}(\mathbb{A}^\times T_S(\mathbb{Q})\backslash T_S(\mathbb{A})) = h_K \operatorname{vol}(\mathbb{A}^\times T_S(\mathbb{Q})\backslash T_S(\mathbb{Q})T_S(\mathbb{R})K_0)$. Hence

$$B(\phi_f,\Lambda) = \frac{\operatorname{vol}(\mathbb{A}^\times T_S(\mathbb{Q})\backslash T_S(\mathbb{A}))}{h_K} \sum_{c\in\operatorname{Cl}_K}\Lambda(c)^{-1}\left(\frac{1}{\operatorname{vol}(\mathbb{R}^\times\backslash T_S(\mathbb{R}))}\int_{\mathbb{R}^\times\backslash T_S(\mathbb{R})}(\phi_f)_S(t_c t_\infty)\, dt_\infty\right),$$

where $(\phi_f)_S$ is as defined in Lemma 4.1 of [39].

A standard calculation, (see Lemma 4.1 and Prop 4.3 of [39]), gives

$$\frac{1}{\mathrm{vol}(\mathbb{R}^\times \backslash T_S(\mathbb{R}))} \int_{\mathbb{R}^\times \backslash T_S(\mathbb{R})} (\phi_f)_S(t_c t_\infty)\, dt_\infty = e^{-2\pi \mathrm{tr}(S)} a(f, S_c).$$

The result follows. ■

3.4 A translation of Conjecture 1.12

Let π, f and $\phi = \phi_f$ be as in Proposition 3.1 and Λ be a character of Cl_K as in the previous subsection. We will use the notation

$$R(f, K, \Lambda) = \sum_{c \in \mathrm{Cl}_K} \Lambda(c)^{-1} a(f, c). \tag{78}$$

Here, $a(f, c) = a(f, S_c)$. If Conjecture 1.12 holds for the representation π, then Proposition 3.1 and Proposition 3.5 together imply the following: *Assuming $f \in S_k(\Gamma_0(N))^T$, we have*

$$
\begin{aligned}
\frac{|R(f, K, \Lambda)|^2}{\langle f, f \rangle} =& e^{4\pi \mathrm{Tr}(S)} \left(\frac{\mathrm{vol}(Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A}))}{\mathrm{vol}(\mathrm{Sp}(4, \mathbb{Z})\backslash \mathbb{H}_2)} \right) \cdot \left(\frac{h_K}{\mathrm{vol}(\mathbb{A}^\times T_S(\mathbb{Q})\backslash T_S(\mathbb{A}))} \right)^2 \\
& \times \frac{C_T}{S_\pi} \cdot \zeta_\mathbb{Q}(2)\zeta_\mathbb{Q}(4) \cdot \frac{L(1/2, \pi \times \mathcal{AI}(\Lambda^{-1}))}{L(1, \pi, \mathrm{Ad})L(1, \chi_d)} \cdot J_\infty \cdot \prod_{p|N} J(\phi_p).
\end{aligned}
\tag{79}
$$

The above equality holds for the Tamagawa measure on the global adelic groups. The constant C_T relates the global Tamagawa measure and the product measure used by us. Recall that we defined local measures dt_p on $\mathbb{Q}_p^\times \backslash T_S(\mathbb{Q}_p)$ for each non-archimedean place p in Section 2. This measure, which was normalized to give total volume 1 on the subgroup $T_S(\mathbb{Z}_p) := T_S(\mathbb{Q}_p) \cap \mathrm{GL}_2(\mathbb{Z}_p)$, was used to compute the local integrals $J_p = J(\phi_p)$. (Note that this measure differs from the measure used in [13] at the places dividing d). Let us now fix some Haar measure on $\mathbb{R}^\times \backslash T(\mathbb{R})$. Then the volume of $\mathbb{A}^\times T(\mathbb{Q})\backslash T(\mathbb{A})$ with respect to the product measure equals $2h_K \mathrm{vol}(\mathbb{R}^\times \backslash T(\mathbb{R}))/w(K)$, a fact that follows from our definition of Cl_K. This gives us

$$C_T = \frac{w(K)}{h_K \mathrm{vol}(\mathbb{R}^\times \backslash T(\mathbb{R}))}.$$

3.6 Proposition. *Let π, f be as in Proposition 3.1 and suppose that $f \in S_k(\Gamma_0(N))^T$. Let Λ be a character of Cl_K and suppose that (9) holds for our setup. Then*

$$\frac{|R(f, K, \Lambda)|^2}{\langle f, f \rangle} = e^{4\pi \mathrm{Tr}(S)} \left(\frac{|d|^{1/2} w(K)^2}{2^3} \right) \cdot \frac{1}{S_\pi} \cdot \frac{L(1/2, \pi \times \mathcal{AI}(\Lambda^{-1}))}{L(1, \pi, \mathrm{Ad})} \cdot \frac{J_\infty}{\mathrm{vol}(\mathbb{R}^\times \backslash T_S(\mathbb{R}))} \cdot \prod_{p|N} J(\phi_p). \tag{80}$$

Proof. With the choice of global Tamagawa measures, we get

$$\mathrm{vol}(Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})) = \mathrm{vol}(\mathbb{A}^\times T(\mathbb{Q})\backslash T(\mathbb{A})) = 2,$$

and with our choice of measure $(\det Y)^{-3} dX dY$ on $\mathbb{H}_2$, we have $\mathrm{vol}(\mathrm{Sp}(4, \mathbb{Z})\backslash \mathbb{H}_2) = 2\pi^{-3}\zeta(2)\zeta(4) = 2\zeta_\mathbb{Q}(2)\zeta_\mathbb{Q}(4)$. Finally the class number formula gives $2h_K = w(K)|d|^{1/2} L(1, \chi_d)$. Substituting these into (79), we get (80). ■

3.5   The computation of J_∞

In this section, we compute the archimedean factor $\frac{J_\infty}{\mathrm{vol}(\mathbb{R}^\times \backslash T_S(\mathbb{R}))}$ (which does not depend on the choice of measure on $\mathbb{R}^\times \backslash T_S(\mathbb{R})$). We briefly recall the setup. Let π_∞ be the discrete series representation of $\mathrm{PGSp}_4(\mathbb{R})$ with scalar minimal K-type (k, k). This is where we need to assume $k > 2$ since the case $k = 2$ is not discrete series. Let v be a vector in the space of π_∞ spanning this one-dimensional K-type, and let

$$\Phi(g) = \frac{\langle \pi_\infty(g)v, v \rangle}{\langle v, v \rangle}$$

be the corresponding matrix coefficient. By the formula given in Proposition A.1 of [22], for $g = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$,

$$\Phi(g) = \begin{cases} \dfrac{\lambda(g)^k 4^k}{\det(A + D + i(C - B))^k} & \text{if } \lambda(g) > 0, \\[2mm] 0 & \text{if } \lambda(g) < 0. \end{cases}$$

Let the matrix $S = S(d)$ be given by (75). Define

$$J_0 = \int\limits_{\mathbb{R}^\times \backslash T_S(\mathbb{R})} \int\limits_{M_2^{\mathrm{sym}}(\mathbb{R})} \Phi(t[\begin{smallmatrix} 1 & X \\ & 1 \end{smallmatrix}]) e^{-2\pi i \,\mathrm{Tr}(SX)}\, dX\, dt$$

The quantity J_∞ is given by

$$J_\infty = \frac{L(1, \pi_\infty, \mathrm{Ad})L(1, \chi_{d,\infty})}{\Gamma_\mathbb{R}(2)\Gamma_\mathbb{R}(4)L(1/2, \pi_\infty \otimes \mathcal{AI}(1))} J_0 = \frac{\Gamma(2k - 1)\pi}{(2\pi)^{2k}} J_0, \tag{81}$$

where we use the archimedean L-factors given explicitly in Remark 1.6. We now compute J_0. We have $T_S(\mathbb{R}) = \mathbb{R}_{>0} \times T_S^1(\mathbb{R})$, where $T_S^1(\mathbb{R}) = T_S(\mathbb{R}) \cap \mathrm{SL}_2(\mathbb{R}) = \{g \in \mathrm{SL}_2(\mathbb{R}) : {}^t gSg = S\}$. (This uses the fact that all elements in $T_S(\mathbb{R})$ have positive determinant in our case.) Using the formula for $\Phi(g)$ we have

$$J_0 = \int\limits_{\{\pm 1\} \backslash T_S^1(\mathbb{R})} \int\limits_{M_2^{\mathrm{sym}}(\mathbb{R})} \Phi(t[\begin{smallmatrix} 1 & X \\ & 1 \end{smallmatrix}]) e^{-2\pi i \,\mathrm{Tr}(SX)}\, dX\, dt$$

$$= \int\limits_{\{\pm 1\} \backslash T_S^1(\mathbb{R})} \int\limits_{M_2^{\mathrm{sym}}(\mathbb{R})} \frac{4^k}{\det(A + {}^t A^{-1} - iAX)^k} e^{-2\pi i \,\mathrm{Tr}(SX)}\, dX\, dA. \tag{82}$$

Special case $S = 1$

Assume that $d = -4$, so that $S = 1$. The computation of J_0 in this special case was kindly supplied to us by Kazuki Morimoto and we thank him for allowing us to include this here. In this case

$$\mathbb{R}^\times \backslash T_S(\mathbb{R}) \simeq \{\pm 1\} \backslash \mathrm{SO}(2)$$

and (82) becomes

$$J_0 = \int\limits_{\{\pm 1\} \backslash \mathrm{SO}(2)} \int\limits_{M_2^{\mathrm{sym}}(\mathbb{R})} \frac{4^k}{\det(A + {}^t A^{-1} - iAX)^k} e^{-2\pi i \,\mathrm{Tr}(X)}\, dX\, dA \tag{83}$$

$$= \int\limits_{\{\pm 1\} \backslash \mathrm{SO}(2)} \int\limits_{M_2^{\mathrm{sym}}(\mathbb{R})} \frac{4^k}{\det(2 - iX)^k} e^{-2\pi i \,\mathrm{Tr}(X)}\, dX\, dA$$

$$= \mathrm{vol}(\{\pm 1\} \backslash \mathrm{SO}(2)) \int\limits_{\mathbb{R}^3} \frac{4^k}{((2 - ix)(2 - iz) + y^2)^k} e^{-2\pi i(x+z)}\, dx\, dy\, dz. \tag{84}$$

By 3.249.1 of [15],

$$\int\limits_0^\infty \frac{1}{(y^2+a)^k}\, dy = \frac{(2k-3)!!}{2(2k-2)!!}\frac{\pi}{a^{k-1/2}}.$$

Hence

$$J_0 = \mathrm{vol}(\{\pm 1\}\backslash \mathrm{SO}(2))\frac{(2k-3)!!}{(2k-2)!!}4^k\pi\left(\int\limits_\mathbb{R}\frac{1}{(2-ix)^{k-1/2}}\,e^{-2\pi ix}\,dx\right)^2. \tag{85}$$

By 3.382.7 of [15],

$$\int\limits_\mathbb{R}\frac{1}{(\beta-ix)^\nu}e^{-ipx}\,dx = \frac{2\pi p^{\nu-1}e^{-\beta p}}{\Gamma(\nu)} \qquad \text{for } p>0.$$

Hence, with $\beta=2$, $p=2\pi$ and $\nu=k-1/2$,

$$\int\limits_\mathbb{R}\frac{1}{(2-ix)^{k-1/2}}e^{-2\pi ix}\,dx = \frac{(2\pi)^{k-1/2}e^{-4\pi}}{\Gamma(k-1/2)}.$$

It follows that

$$J_0 = \mathrm{vol}(\{\pm 1\}\backslash \mathrm{SO}(2))\frac{(2k-3)!!}{(2k-2)!!}4^k\pi\frac{(2\pi)^{2k-1}e^{-8\pi}}{\Gamma(k-1/2)^2}. \tag{86}$$

$$= \mathrm{vol}(\{\pm 1\}\backslash \mathrm{SO}(2))\,2^{4k-2}\,\frac{(2\pi)^{2k-1}e^{-8\pi}}{\Gamma(2k-1)}. \tag{87}$$

where we have used standard properties of the Gamma function. Note that

$$\mathrm{vol}(\{\pm 1\}\backslash \mathrm{SO}(2)) = \mathrm{vol}(\mathbb{R}^\times\backslash T_S(\mathbb{R})). \tag{88}$$

Hence

$$J_0 = \mathrm{vol}(\mathbb{R}^\times\backslash T_S(\mathbb{R}))\,2^{4k-2}\,\frac{(2\pi)^{2k-1}e^{-8\pi}}{\Gamma(2k-1)}. \tag{89}$$

The general case

We return to arbitrary S. Let V be a fixed model for our representation π_∞. Let $v_0 \in V$ be a fixed non-zero vector of weight (k,k). Let $\theta_S([\begin{smallmatrix}1&X\\&1\end{smallmatrix}]) = e^{2\pi i\,\mathrm{Tr}(SX)}$, $[\begin{smallmatrix}1&X\\&1\end{smallmatrix}] \in N(\mathbb{R})$. Recall that we are trying to calculate

$$J_0 = \int\limits_{\{\pm 1\}\backslash T_S^1(\mathbb{R})}\int\limits_{N(\mathbb{R})}\frac{\langle \pi_\infty(tn)v_0, v_0\rangle}{\langle v_0, v_0\rangle}\theta_S^{-1}(n)\,dn\,dt.$$

For $v, w \in V$, we define

$$A(v,w) = \int\limits_{\{\pm 1\}\backslash \mathrm{SO}(2)}\int\limits_{N(\mathbb{R})}\frac{\langle \pi_\infty(tn)v, w\rangle}{\langle v_0, v_0\rangle}\,\theta_1(n)^{-1}\,dn\,dt. \tag{90}$$

Note that A is linear in the first variable, and anti-linear in the second variable. The value $A(v_0, v_0)$ is independent of the choice of v_0 and the normalization of the hermitian inner product. By (89), we have

$$A(v_0, v_0) = \mathrm{vol}(\mathbb{R}^\times\backslash T_1(\mathbb{R}))\,2^{4k-2}\,\frac{(2\pi)^{2k-1}e^{-8\pi}}{\Gamma(2k-1)}. \tag{91}$$

3.7 Lemma. *Let $t_0 \in \mathrm{SL}(2,\mathbb{R})$ and $\alpha > 0$ be such that $\alpha\, {}^t t_0 S t_0 = 1$. Set*

$$M_d = \begin{bmatrix} t_0^{-1} & \\ & \alpha\, {}^t t_0 \end{bmatrix}. \tag{92}$$

Then

$$J_0 = \alpha^3 A(\pi_\infty(M_d)v_0, \pi_\infty(M_d)v_0). \tag{93}$$

Proof. It is easy to see that $\mathrm{SO}(2) = t_0^{-1} T_S^1(\mathbb{R}) t_0$. We calculate

$$
\begin{aligned}
A(\pi_\infty(M_d)v_0, \pi_\infty(M_d)v_0) &= \int\limits_{\{\pm 1\}\backslash \mathrm{SO}(2)} \int\limits_{N(\mathbb{R})} \frac{\langle \pi_\infty(tn)\pi_\infty(M_d)v_0, \pi_\infty(M_d)v_0\rangle}{\langle v_0, v_0\rangle} \theta_1(n)^{-1}\, dn\, dt \\
&= \int\limits_{\{\pm 1\}\backslash \mathrm{SO}(2)} \int\limits_{N(\mathbb{R})} \frac{\langle \pi_\infty(t_0 t t_0^{-1} M_d^{-1} n M_d)v_0, v_0\rangle}{\langle v_0, v_0\rangle} \theta_1(n)^{-1}\, dn\, dt \\
&= \int\limits_{\{\pm 1\}\backslash T_S^1(\mathbb{R})} \int\limits_{N(\mathbb{R})} \frac{\langle \pi_\infty(t M_d^{-1} n M_d)v_0, v_0\rangle}{\langle v_0, v_0\rangle} \theta_1(n)^{-1}\, dn\, dt \\
&= \int\limits_{\{\pm 1\}\backslash T_S^1(\mathbb{R})} \int\limits_{M_2^{\mathrm{sym}}(\mathbb{R})} \frac{\langle \pi_\infty(t\begin{bmatrix} 1 & \alpha t_0 X\, {}^t t_0 \\ & 1 \end{bmatrix})v_0, v_0\rangle}{\langle v_0, v_0\rangle} e^{-2\pi i \,\mathrm{Tr}(X)}\, dX\, dt \\
&= \alpha^{-3} \int\limits_{\{\pm 1\}\backslash T_S^1(\mathbb{R})} \int\limits_{M_2^{\mathrm{sym}}(\mathbb{R})} \frac{\langle \pi_\infty(t\begin{bmatrix} 1 & X \\ & 1 \end{bmatrix})v_0, v_0\rangle}{\langle v_0, v_0\rangle} e^{-2\pi i \,\mathrm{Tr}(SX)}\, dX\, dt \\
&= \alpha^{-3} J_0.
\end{aligned}
$$

This concludes the proof. ∎

We recall the function B_0 defined in (60) of [32] (where we set $m = 0$ and $l = l' = k$). The function B_0 is the weight (k,k)-vector in the special $(1, \theta_1)$-Bessel model $\mathcal{B}_{1,\theta_1}(\pi_\infty)$ of π_∞, normalized such that

$$B_0(1) = e^{-4\pi}. \tag{94}$$

On certain diagonal elements, this function has the formula

$$B_0(h(\lambda, \zeta, 0, 0)) = \begin{cases} \lambda^k e^{-2\pi\lambda(\zeta^2 + \zeta^{-2})} & \text{if } \lambda > 0, \\ 0 & \text{if } \lambda < 0, \end{cases} \tag{95}$$

where we denote

$$h(\lambda, \zeta, 0, 0) := \begin{bmatrix} \lambda\begin{bmatrix} \zeta & \\ & \zeta^{-1} \end{bmatrix} & \\ & \begin{bmatrix} \zeta^{-1} & \\ & \zeta \end{bmatrix} \end{bmatrix}. \tag{96}$$

We define a Bessel functional $\ell : V \to \mathbb{C}$ as the composition

$$V \xrightarrow{\sim} \mathcal{B}_{1,\theta_1}(\pi_\infty) \longrightarrow \mathbb{C},$$

where the first map identifies v_0 with B_0, and the second map is given by evaluation at 1. Note that

$$\ell(v_0) = B_0(1) = e^{-4\pi} \neq 0. \tag{97}$$

3.8 Lemma. *We have*

$$A(v, w) = c\,\ell(v)\overline{\ell(w)} \tag{98}$$

for all $v, w \in V$, where

$$c = \text{vol}(\mathbb{R}^\times \backslash T_S(\mathbb{R}))\, 2^{4k-2} \frac{(2\pi)^{2k-1}}{\Gamma(2k-1)}. \tag{99}$$

Proof. It is easy to see that for fixed $w \in V$ the map $v \mapsto A(v, w)$ is a $(1, \theta_1)$-Bessel functional on V. Hence there exists a constant c_w such that

$$A(v, w) = c_w \ell(v) \qquad \text{for all } v \in V.$$

It is further easy to see that the map $w \mapsto \overline{c_w}$ is also a $(1, \theta_1)$-Bessel functional on V. Hence there exists a constant c such that

$$\overline{c_w} = c\ell(w) \qquad \text{for all } w \in V.$$

Thus we get

$$A(v, w) = c\,\ell(v)\overline{\ell(w)} \tag{100}$$

for all $v, w \in V$. Substituting $v = w = v_0$ and observing (91) and (97), we see that

$$c = \text{vol}(\mathbb{R}^\times \backslash T_S(\mathbb{R}))\, 2^{4k-2} \frac{(2\pi)^{2k-1}}{\Gamma(2k-1)}.$$

This concludes the proof. ∎

From Lemma 3.8 and (93) we now get

$$\begin{aligned}
J_0 &= \alpha^3 A(\pi_\infty(M_d)v_0, \pi_\infty(M_d)v_0) \\
&= \alpha^3 c\,|B_0(M_d)|^2. \tag{101}
\end{aligned}$$

First assume that $S = \begin{bmatrix} \frac{|d|}{4} & \\ & 1 \end{bmatrix}$. Then we may take

$$t_0 = \begin{bmatrix} 2^{1/2}|d|^{-1/4} & \\ & 2^{-1/2}|d|^{1/4} \end{bmatrix}, \qquad \alpha = 2|d|^{-1/2},$$

so that

$$\begin{aligned}
M_d &= \begin{bmatrix} \begin{bmatrix} 2^{-1/2}|d|^{1/4} & \\ & 2^{1/2}|d|^{-1/4} \end{bmatrix} & \\ & \alpha \begin{bmatrix} 2^{1/2}|d|^{-1/4} & \\ & 2^{-1/2}|d|^{1/4} \end{bmatrix} \end{bmatrix} \\
&= \alpha \begin{bmatrix} \alpha^{-1}\begin{bmatrix} 2^{-1/2}|d|^{1/4} & \\ & 2^{1/2}|d|^{-1/4} \end{bmatrix} & \\ & \begin{bmatrix} 2^{1/2}|d|^{-1/4} & \\ & 2^{-1/2}|d|^{1/4} \end{bmatrix} \end{bmatrix} \\
&= \alpha h(2^{-1}|d|^{1/2}, 2^{-1/2}|d|^{1/4}, 0, 0).
\end{aligned}$$

By (95), $B_0(h(\lambda, \zeta, 0, 0)) = \lambda^k e^{-2\pi\lambda(\zeta^2 + \zeta^{-2})}$, which gives

$$B_0(M_d) = \left(\frac{|d|}{4}\right)^{k/2} e^{-2\pi(|d|/4 + 1)}.$$

Substituting into (101), we get

$$J_0 = \text{vol}(\mathbb{R}^\times \backslash T_S(\mathbb{R}))\, 2^{4k-2} \frac{(2\pi)^{2k-1}}{\Gamma(2k-1)} \left(\frac{|d|}{4}\right)^{k-3/2} e^{-4\pi(|d|/4 + 1)}. \tag{102}$$

Now assume that $S = \begin{bmatrix} \frac{1-d}{4} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}$. Then we may take

$$t_0 = \begin{bmatrix} 1 & \\ -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 2^{1/2}|d|^{-1/4} & \\ & 2^{-1/2}|d|^{1/4} \end{bmatrix}, \qquad \alpha = 2|d|^{-1/2},$$

so that

$$M_d = \begin{bmatrix} \begin{bmatrix} 2^{-1/2}|d|^{1/4} & \\ & 2^{1/2}|d|^{-1/4} \end{bmatrix} \begin{bmatrix} 1 & \\ \frac{1}{2} & 1 \end{bmatrix} & \\ & \alpha \begin{bmatrix} 2^{1/2}|d|^{-1/4} & \\ & 2^{-1/2}|d|^{1/4} \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} \\ & 1 \end{bmatrix} \end{bmatrix}.$$

Lemma 5.1 of [20] shows that

$$\begin{bmatrix} 2^{1/2}|d|^{-1/4} & \\ & 2^{-1/2}|d|^{1/4} \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} \\ & 1 \end{bmatrix} = k_1 \begin{bmatrix} \zeta^{-1} & \\ & \zeta \end{bmatrix} k_2$$

with $k_1, k_2 \in SO(2)$ and

$$\zeta^2 = \frac{|d| + 5 - \sqrt{|d|^2 - 6|d| + 25}}{4|d|^{1/2}}.$$

Hence

$$M_d = \alpha \begin{bmatrix} k_1 & \\ & k_1 \end{bmatrix} \begin{bmatrix} \alpha^{-1}\zeta & & & \\ & \alpha^{-1}\zeta^{-1} & & \\ & & \zeta^{-1} & \\ & & & \zeta \end{bmatrix} \begin{bmatrix} k_2 & \\ & k_2 \end{bmatrix}.$$

In this case, a similar calculation to before gives

$$B_0(M_d) = \left(\frac{|d|}{4} \right)^{k/2} e^{-2\pi \frac{|d|+5}{4}}.$$

Substituting into (101) gives

$$J_0 = \mathrm{vol}(\mathbb{R}^\times \backslash T_S(\mathbb{R})) \, 2^{4k-2} \frac{(2\pi)^{2k-1}}{\Gamma(2k-1)} \left(\frac{|d|}{4} \right)^{k-3/2} e^{-4\pi \frac{|d|+5}{4}}. \tag{103}$$

Summarizing (102) and (103), we get

$$J_0 = \mathrm{vol}(\mathbb{R}^\times \backslash T_S(\mathbb{R})) \, 2^{4k-2} \frac{(2\pi)^{2k-1}}{\Gamma(2k-1)} \left(\frac{|d|}{4} \right)^{k-3/2} e^{-4\pi \mathrm{Tr}(S)}. \tag{104}$$

As a consequence, for any $k > 2$,

$$J_\infty = \frac{\Gamma(2k-1)\pi}{(2\pi)^{2k}} J_0$$

$$= \mathrm{vol}(\mathbb{R}^\times \backslash T_S(\mathbb{R})) \, 2^{4k-3} \left(\frac{|d|}{4} \right)^{k-3/2} e^{-4\pi \, \mathrm{Tr}(S)}. \tag{105}$$

This formula will be used in the proof of Theorem 3.10 below.

3.9 Remark. *Hsieh and Yamana [17, Prop. 5.7] have recently computed the factor J_∞ using a different method based on Shimura's work on confluent hypergeometric functions.*

3.6   The main result and some consequences

Throughout this subsection, N denotes a positive squarefree odd integer and $k > 2$ an integer. We begin by stating the main theorem of this paper.

3.10 Theorem. *Let $f \in S_k(\Gamma_0(N))^{\mathrm{T}}$. Assume that $f = \delta_{a,b,c,d}(g)$ where $abcde = N$ and g is a newform in $S_k(\Gamma_0(e))^{\mathrm{T}}$; note that f itself is a newform if and only if $a = b = c = d = 1$ (whence $f = g$). Let π be the representation attached to g (or equivalently, to f). Let $d < 0$ be a fundamental discriminant such that $\left(\frac{d}{p}\right) = -1$ for all $p|N$ and let Λ be an ideal class character of $K = \mathbb{Q}(\sqrt{d})$. Furthermore, assume that at least one of the following statements is true:*

   i) Conjecture 1.12 is true for π,

   ii) $\Lambda = 1$,

   iii) f is a weak Yoshida lift (in the sense of [38]).

Then we have

$$\frac{|R(f,K,\Lambda)|^2}{\langle f,f \rangle} = 2^{2k-s} \cdot w(K)^2 \, |d|^{k-1} \frac{L(1/2, \pi \times \mathcal{AI}(\Lambda^{-1}))}{L(1,\pi,\mathrm{Ad})} \prod_{p|N} J_p$$

$$= \frac{2^{4k-s} \cdot \pi^{2k+1}}{(2k-2)!} \, w(K)^2 \, |d|^{k-1} \frac{L_{\mathrm{f}}(1/2, \pi \times \mathcal{AI}(\Lambda^{-1}))}{L_{\mathrm{f}}(1,\pi,\mathrm{Ad})} \prod_{p|N} J_p,$$

where $s = 5$ if f is a weak Yoshida lift and $s = 4$ otherwise. The quantities J_p are given as follows:

$$J_p = \begin{cases} L(1,\pi_p,\mathrm{Std})(1-p^{-4}) & \text{if } p|bd, \text{ (cannot occur if } f \text{ is a newform)}, \\ L(1,\pi_p,\mathrm{Std})(1-p^{-4})p^{-1} & \text{if } p|ac, \text{ (cannot occur if } f \text{ is a newform)}, \\ (1+p^{-2})(1+p^{-1}) & \text{if } p|e \text{ and } \pi_p \text{ is of type IIIa}, \\ 2(1+p^{-2})(1+p^{-1}) & \text{if } p|e \text{ and } \pi_p \text{ is of type VIb}, \\ 0 & \text{otherwise}. \end{cases}$$

Proof. This follows by combining (80) with Theorem 2.1, Theorem 2.12 and (105). Note that $S_\pi = 4$ if f is a weak Yoshida lift and $S_\pi = 2$ otherwise. We note that Conjecture 1.12 is currently known in the cases where $\Lambda = 1$ or f is a weak Yoshida lift. ∎

Theorem 1.13 is valid for forms that are not Saito-Kurokawa lifts. We now prove an unconditional theorem for Saito-Kurokawa lifts. This relies on a result due to Qiu [34]. We quote Qiu's result in a very special case:

3.11 Theorem. (Qiu) *Let π, f, ϕ be as in Propostion 3.1. Assume further that f is a Saito-Kurokawa lift (i.e., the corresponding representation π is of P-CAP type) and that $\phi = \otimes_v \phi_v$ corresponds to a factorizable vector. Let π_0 be the automorphic representation of $\mathrm{GL}(2, \mathbb{A})$ that π_f is associated to, i.e., $L_{\mathrm{f}}(s,\pi_f) = L_{\mathrm{f}}(s,\pi_0)\zeta(s+1/2)\zeta(s-1/2)$. Let $d < 0$ be a fundamental discriminant, $K = \mathbb{Q}(\sqrt{d})$, and Λ an ideal class character of K. Then $R(f,K,\Lambda) = 0$ if Λ is a non-trivial character, and*

$$\frac{|R(f,K,1)|^2}{\langle f,f \rangle} = \left(\frac{3|d|w(K)^2}{4} \right) e^{4\pi \mathrm{Tr}(S))} \frac{L(1/2, \pi_0 \otimes \chi_d)L(1,\chi_d)^2}{L(3/2, \pi_0)L(1,\pi_0,\mathrm{Ad})} \cdot \frac{J_\infty}{\mathrm{vol}(\mathbb{R}^\times \backslash T_S(\mathbb{R}))} \cdot \prod_{p|N} \left(2^{-1} J^*(\phi_p) \right).$$

$$(106)$$

3.12 Remark. *Equation* (106) *follows from the first formula in Theorem 2 of* [34]. *The second equality stated therein is not correct: in the case F is totally real of degree d, one must multiply the right hand side of* [34, (4.13)] *by* $(2\pi)^d$, *as can be checked by computing $L(0, \Pi_v, \chi_v)$ for v a real place. Of course, the product over v should also be of* $2^{-1}\mathcal{P}_v^\#(f_{1,v}, f_{2,v})$.

Now using (106) instead of (80) and the same argument as before, we get

3.13 Theorem. *Let $f \in S_k(\Gamma_0(N))^{\mathrm{new,CAP}}$. Assume that f is a newform and is a Saito-Kurokawa lift. Let π_0 be the automorphic representation of $\mathrm{GL}(2, \mathbb{A})$ that π_f is associated to, i.e., $L_f(s, \pi_f) = L_f(s, \pi_0)\zeta(s + 1/2)\zeta(s - 1/2)$. Let $d < 0$ be a fundamental discriminant such that $\left(\frac{d}{p}\right) = -1$ for all $p|N$ and let Λ be an ideal class character of $K = \mathbb{Q}(\sqrt{d})$. Then $R(f, K, \Lambda) = 0$ if Λ is non-trivial, and*

$$\frac{|R(f, K, 1)|^2}{\langle f, f \rangle} = (3 \cdot 2^{2k-2})w(K)^2 \, |d|^{k-\frac{1}{2}} \frac{L(1/2, \pi_0 \times \chi_d)L(1, \chi_d)^2}{L(3/2, \pi_0)L(1, \pi_0, \mathrm{Ad})} \prod_{p|N}(1 + p^{-2})(1 + p^{-1}).$$

A sanity check

Consider the special case $N = 1$. Let f be a Saito-Kurokawa lift of weight k as above. Let g be the classical cusp form of half-integral weight $k - 1/2$ that gives rise to f. We have the following classical results:

$$|R(f, K, 1)|^2 = |d| \left(\frac{w(K)L(1, \chi_d)}{2} \right)^2 |c(g, d)|^2, \tag{107}$$

which can be derived by combining the equation on page 74 of [11] with the class number formula. Also, from [6], Corollary 6.3,[12]

$$\frac{|c(g, d)|^2}{\langle f, f \rangle} = \frac{3\pi}{k - 1}|d|^{k-3/2}2^{2k+1} \frac{L_f(1/2, \pi_0 \otimes \chi_d)}{L_f(3/2, \pi_0)L(1, \pi_0, \mathrm{Ad})}. \tag{108}$$

Combining these formulas, we recover Theorem 3.13 exactly for $N = 1$. This gives an independent verification of all our computations.

We now work out several consequences of Theorem 3.10. For each f as in Theorem 3.10, let us use the notation

$$J_{f,N} = \prod_{p|N} J_p. \tag{109}$$

We have the following bound, which follows immediately from the formulas for J_p,

$$J_{f,N} \ll_\epsilon N^\epsilon. \tag{110}$$

Yoshida lifts

We recall the representation theoretic construction of scalar valued Yoshida lifts described in [39]. Let N_1, N_2 be two positive, squarefree integers such that $M = \gcd(N_1, N_2) > 1$. Let g_1 be a classical newform of weight $2k - 2$ and level N_1 and g_2 be a classical newform of weight 2 and level N_2, such that g_1 and g_2 are not multiples of each other. Assume that for all primes p dividing M the Atkin-Lehner eigenvalues of g_1 and g_2 coincide. Put $N = \mathrm{lcm}(N_1, N_2)$. For any $p|M$, let δ_p be the common Atkin-Lehner eigenvalue of g_1 and g_2. Then for any divisor M_1 of M with an *odd* number of prime factors, there exists a non-zero holomorphic Siegel cusp form $f = f_{g_1,g_2;M_1}$ such that f is a newform in $S_k(\Gamma_0(N))^{\mathrm{T}}$. Furthermore, the automorphic representation π generated by the adelization of f has the following properties.

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[12]Note that the formula stated in [6] differs from ours by a factor of $6 = [\mathrm{SL}(2, \mathbb{Z}) : \Gamma_0(4)]$. This is because the Kohnen–Zagier formula is quoted incorrectly in Lemma 6.1 of [6], owing to a different normalization of the Petersson inner product.

    i) $L(s, \pi_f) = L(s, \pi_{g_1})L(s, \pi_{g_2})$.

   ii) If $p|N$, $p \nmid M$, then π_p is of type IIa.

  iii) If $p|M$, $p \nmid M_1$, then π_p is of type VIa. The associated character σ_p is trivial if $\delta_p = -1$ and unramified quadratic if $\delta_p = 1$.

  iv) If $p|M_1$, then π_p is of type VIb. The associated character σ_p is trivial if $\delta_p = -1$ and unramified quadratic if $\delta_p = 1$.

Let $f = f_{g_1,g_2;M_1}$ be a Yoshida lift, as above. Then $S_\pi = 4$. It is of interest to see what Theorem 3.10 gives us in this case. Indeed, in this case Conjecture 1.12 has been proved already, so we get an *unconditional statement*.

3.14 Proposition. *Let* N_1, N_2, g_1, g_2, M, M_1 *be as above. Assume that N_1 and N_2 are odd and $k > 2$. Let $f = f_{g_1,g_2;M_1}$ be a Yoshida lift. Then, for any $d < 0$ a fundamental discriminant such that $\left(\frac{d}{p}\right) = -1$ for all $p|N$, and any ideal class character Λ of $K = \mathbb{Q}(\sqrt{d})$, we have*

$$\frac{|R(f,K,\Lambda)|^2}{\langle f, f \rangle} = \begin{cases} \dfrac{2^{4k-5}\pi^{2k+1}w(K)^2|d|^{k-1}}{(2k-2)!} \times \displaystyle\prod_{p|N}\left(2(1+p^{-2})(1+p^{-1})\right) \\[4mm] \qquad \times \dfrac{L_{\mathrm{f}}(1/2, \pi_{g_1} \times \mathcal{AI}(\Lambda^{-1}))L_{\mathrm{f}}(1/2, \pi_{g_2} \times \mathcal{AI}(\Lambda^{-1}))}{L_{\mathrm{f}}(1, \pi_{g_1}, \mathrm{Ad})L_{\mathrm{f}}(1, \pi_{g_2}, \mathrm{Ad})L_{\mathrm{f}}(1, \pi_{g_1} \times \pi_{g_2})} \quad \textit{if } N_1 = N_2 = M = M_1, \\[4mm] 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \textit{otherwise.} \end{cases}$$

$$(111)$$

In the special case $\Lambda = 1$, the above result can probably also be derived from the formulas in [4].

Averages of L-functions

For each $e|N$, let $\mathcal{B}_{k,e}^{\mathrm{new,T}}$ be an orthogonal basis of $S_k(\Gamma_0(e))^{\mathrm{new,T}}$ consisting of newforms. Put

$$\mathcal{B}_{k,N}^{\mathrm{T}} = \bigcup_{abcde=N} \{\delta_{a,b,c,d}(g) : g \in \mathcal{B}_{k,e}^{\mathrm{new,T}}\}.$$

Then by Corollary 3.4, $\mathcal{B}_{k,N}^{\mathrm{T}}$ is an orthogonal basis for $S_k(\Gamma_0(N))^{\mathrm{T}}$. Each $f \in \mathcal{B}_{k,N}^{\mathrm{T}}$ is a Hecke eigenform at places not dividing N, and gives rise to an irreducible cuspidal automorphic representation π_f.

3.15 Theorem. *Let $k \geq 6$ be even and N squarefree and odd. Let $\mathcal{B}_{k,N}^{\mathrm{T}}$ be as above. Suppose that Conjecture 1.12 holds. Fix a fundamental discriminant $d < 0$ and an ideal class character Λ of $K = \mathbb{Q}(\sqrt{d})$. Suppose that $\left(\frac{d}{p}\right) = -1$ for all $p|N$. Put $l = 1$ if $\Lambda^2 = 1$ and $l = 2$ otherwise. Then*

$$\sum_{f \in \mathcal{B}_{k,N}^{\mathrm{T}}} \frac{L_{\mathrm{f}}(1/2, \pi_f \times \mathcal{AI}(\Lambda^{-1}))}{L_{\mathrm{f}}(1, \pi_f, \mathrm{Ad})} \cdot J_{f,N} \cdot u_f = \frac{k^3\ L_{\mathrm{f}}(1, \chi_d)\ [\mathrm{Sp}(4,\mathbb{Z}):\Gamma_0(N)]}{l\ \pi^6}\left(1 + O(k^{-2/3}N^{-1})\right),\quad (112)$$

where $J_{f,N}$ is as defined in (109) *and $u_f = 2$ if f is a weak Yoshida lift, and $u_f = 2^2$ otherwise. The implied constant in O depends only on d.*

Proof. Let $t_{k,d} = \frac{2^{4k-4}\cdot\pi^{2k+1}}{(2k-2)!}\ w(K)^2\ |d|^{k-1}$. Assuming Conjecture 1.12, our main theorem above says

$$\frac{|R(f,K,\Lambda)|^2}{\langle f, f \rangle} = t_{k,d} \cdot u_f \cdot \frac{L_{\mathrm{f}}(1/2, \pi_f \times \mathcal{AI}(\Lambda^{-1}))}{L_{\mathrm{f}}(1, \pi_f, \mathrm{Ad})} J_{f,N}.$$

On the other hand, by Theorem 7.3 and Corollary 9.4 of [10], we know that

$$\sum_{f \in \mathcal{B}_{k,N}^{\mathrm{T}}} \frac{\pi^6 l}{k^3 t_{k,d} \, [\mathrm{Sp}(4,\mathbb{Z}):\Gamma_0(N)] L_{\mathrm{f}}(1,\chi_d)} \frac{|R(f,K,\Lambda)|^2}{\langle f, f \rangle} = 1 + O_d(N^{-1} k^{-2/3}).$$

This completes the proof.   ∎

If we put $N = 1$ in (112), we get the result (15) stated in the introduction. Furthermore, we can use the above theorem and sieve for various divisors of N to get a version of (112) that only sums over newforms, i.e., elements of $\mathcal{B}_{k,N}^{\mathrm{new,T}}$. We state and prove such a result in the simplest case of prime modulus.

3.16 Corollary. *Let $k \geq 6$ be even, let p be an odd prime and let $\mathcal{B}_{k,p}^{\mathrm{new,T}}$ be an orthogonal basis of $S_k(\Gamma_0(p))$ consisting of newforms. Let $\Phi_{k,p}^{\mathrm{new,T}}$ be the set of distinct irreducible subspaces*[13] *of $L^2(Z(\mathbb{Q})G(\mathbb{Q})\backslash G(\mathbb{A}))$ generated by the adelizations of elements of $\mathcal{B}_{k,p}^{\mathrm{new,T}}$. Suppose that Conjecture 1.12 holds. Fix a fundamental discriminant $d < 0$ such that $\left(\frac{d}{p}\right) = -1$, and an ideal class character Λ of $K = \mathbb{Q}(\sqrt{d})$. Put $l = 1$ if $\Lambda^2 = 1$ and $l = 2$ otherwise. Then*

$$\sum_{\substack{\pi \in \Phi_{k,p}^{\mathrm{new,T}} \\ \pi_p \in \{\mathrm{IIIa, \, VIb}\}}} \frac{L_{\mathrm{f}}(1/2, \pi \times \mathcal{AI}(\Lambda^{-1}))}{L_{\mathrm{f}}(1, \pi, \mathrm{Ad})} \, u_\pi = \frac{k^3 p^3 L_{\mathrm{f}}(1,\chi_d)}{l \, \pi^6} + O_d(k^{7/3} p^2 + k^3),$$

where $u_\pi = 2$ if π is a weak endoscopic lift, and $u_\pi = 2^2$ otherwise.

Proof. This follows from Theorem 3.15 for $N = 1$ and $N = p$, combined with the observation that for any $\pi \in \Phi_{k,p}^{\mathrm{new,T}}$ with π_p of type IIIa or VIb, $\sum_{\substack{f \in \mathcal{B}_{k,p}^{\mathrm{new,T}} \\ \pi_f = \pi}} J_{f,p} = 2 + O(p^{-1})$; see Remark 2.19.   ∎

Size of Fourier coefficients

As an interesting consequence of Theorem 3.10, we *predict* the best possible upper bound for $|R(f,K,\Lambda)|$.

3.17 Proposition. *Let f, π, d, K, Λ be as in Theorem 3.10. Assume the truth of both Conjecture 1.12 and the Generalized Riemann Hypothesis for π. Then*

$$|R(f,K,\Lambda)| \ll_\epsilon \langle f, f \rangle^{1/2} (2\pi e)^k \, k^{-k+\frac{3}{4}} |d|^{\frac{k-1}{2}} (Nkd)^\epsilon.$$

Proof. This follows from Theorem 3.10, the bound (110), and Stirling's formula. Note that the GRH for π implies that the quotient of the various finite parts of L-functions appearing in 3.10 is bounded by $(Nkd)^\epsilon$.   ∎

Unfortunately, the above proposition does not seem to predict the strongest expected bound on an *individual* Fourier coefficient $|a(F,S)|$. That would require knowledge of the distribution of the size and arguments of the complex numbers $R(f,K,\Lambda)$ as Λ varies over characters of Cl_K, which appears to be a difficult problem. We note here that a conjecture of Resnikoff and Saldana [35] predicts that as d varies, we have $|a(F,S)| \ll_F d^{k/2-3/4+\epsilon}$ for any S with $\mathrm{disc}(S) = d$. The bound of Proposition 3.17 and the conjectured bound of Resnikoff and Saldana do not appear to imply each other in any direction without assuming equidistribution of the arguments of the Fourier coefficients.

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[13]The "multiplicity one" conjecture predicts that two such subspaces are distinct if and only if the corresponding automorphic representations are non-isomorphic; however we do not assume the truth of this conjecture here.

Integrality of L-values

If $L(s, \mathcal{M})$ is an L-series associated to an arithmetic object $\mathcal{M}$, it is of interest to study its values at certain critical points $s = m$. For these critical points, conjectures due to Deligne predict that $L(m, \mathcal{M})$ is the product of a suitable transcendental number Ω and an algebraic number. One can go even further and try to predict what primes divide the numerator and denominator of the algebraic number above (once the period is suitably normalized). Our main theorem implies the following result.

3.18 Proposition. *Let $k > 2$, N squarefree and odd, and $f \in S_k(\Gamma_0(N))^{\mathrm{T}}$ be a newform such that all its Fourier coefficients are algebraic integers.*[14] *Let π be the representation attached to f, and assume that π_p is of type IIIa or VIb for all $p|N$. Assume the truth of Conjecture 1.12 for π. Then, for all fundamental discriminants $d < 0$ such that $\left(\frac{d}{p}\right) = -1$ for all $p|N$ and all ideal class characters Λ of $K = \mathbb{Q}(\sqrt{d})$, we have*

$$\langle f, f \rangle \times \frac{L_{\mathrm{f}}(1/2, \pi_f \times \mathcal{AI}(\Lambda^{-1}))}{L_{\mathrm{f}}(1, \pi_f, \mathrm{Ad})} \times \frac{3^2 \cdot 2^{4k} \pi^{2k+1} |d|^{k-1} \sigma_0(N)\sigma_1(N)\sigma_2(N)}{(2k-2)! \; N^3}$$

is an algebraic integer, where $\sigma_i(N) = \sum_{1 \le d | N} d^i$.

Proof. This follows from Theorem 3.10 as our assumptions imply that $R(f, K, \Lambda)$ is an algebraic integer. ∎

Proof of Proposition 1.17. We first claim that there exists a Yoshida lift f of g_1, g_2 whose Fourier coefficients are algebraic *integers* in the CM-field generated by g_1, g_2. Indeed, by Theorem 6 of [38], it follows that such a lift exists whose Fourier coefficients are algebraic *numbers* in the field generated by g_1, g_2. Now, the main result of [42] assures us that one can always multiply any Siegel modular form (of any level) with algebraic Fourier coefficients with a large enough integer such that the Fourier coefficients are algebraic integers. This proves the claim. Given such a lift f, we now apply Proposition 3.14. We can absorb all the remaining terms appearing in Proposition 3.14 that depend only on f, into the constant Ω. This immediately the proof.

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