Structure and arithmeticity for nearly holomorphic modular forms

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The Riemann zeta function

Let $\zeta(s)$ be the Riemann zeta function defined on $\mathrm{Re}(s) > 1$ by the *absolutely convergent series*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

or equivalently, by the *Euler product*

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Some properties of $\zeta(s)$:

- $\zeta(s)$ can be analytically continued to a meromorphic function on the entire complex plane whose only pole is simple and at $s = 1$.
- $\zeta(s)$ is bounded in vertical strips (excluding $s = 1$).
- $\zeta(s)$ satisfies a functional equation sending $s \mapsto 1 - s$.
- $\zeta(s)$ is non-zero in the region $\mathrm{Re}(s) \geq 1$.

These analytic properties of $\zeta(s)$ lead to deep number-theoretic consequences, for example:

The prime number theorem

The fact that $\zeta(s)$ is non vanishing on $\mathrm{Re}(s) = 1$ is equivalent to the famous prime number theorem, which states

$$\pi(x) \sim \frac{x}{\log x}$$

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The Riemann-zeta function is an example of a L-function.

All L-functions are given by a Dirichlet series of the form $\sum_{n>0} \frac{a_n}{n^s}$ and are expected to have certain nice properties. However in many cases, several of these properties are not known, and it is an important and challenging problem to prove them for any new class of L-functions.

Special value results

Special value results

We have $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$, and in general, for any positive integer m, we have

$$\frac{\zeta(2m)}{\pi^{2m}} \quad \in \quad \mathbb{Q}.$$

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Similarly we have $1 - \frac{1}{3} + \frac{1}{5} - \ldots = \frac{\pi}{4}$ which is a special case of the fact that for any positive integer m,

$$\frac{1 - \frac{1}{3^{2m-1}} + \frac{1}{5^{2m-1}} - \ldots}{\pi^{2m-1}} = \frac{L(2m-1, \chi_4)}{\pi^{2m-1}} \in \mathbb{Q},$$

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The above results are special cases of a broad conjecture, due to Deligne, that asserts that critical special values of motivic L-functions are algebraic numbers up to a certain period.

An example with modular forms

The Ramanujan Δ-function is defined as follows:

$$\Delta(z) = e^{2\pi iz} \prod_{n=1}^{\infty} (1 - e^{2\pi inz})^{24} = \sum_{n=1}^{\infty} \tau(n) e^{2\pi inz}.$$

$\Delta(z)$ is a holomorphic cusp form of weight 12.

We put $L(s, \Delta) = \sum_{n=1}^{\infty} \frac{\tau(n)}{n^s}$.

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Theorem (Manin, Shimura)

There exist real numbers r^+, r^- such that

- $$\frac{L(m, \Delta)}{r^- \cdot \pi^m} \in \mathbb{Q} \quad \text{for } m = 1, 3, 5, 7, 9, 11$$
- $$\frac{L(m, \Delta)}{r^+ \cdot \pi^m} \in \mathbb{Q} \quad \text{for } m = 2, 4, 6, 8, 10$$

All these special value results fall under a broad conjecture due to Deligne :

Deligne's conjecture

Let $L(s, M)$ be the L-function associated to a "motive". Then there exists a set S of "critical points" such that for all $m \in S$,

$$\frac{L(m, M)}{\text{(certain periods)}} \in \overline{\mathbb{Q}}$$

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Results in the spirit of Deligne are known for several automorphic L-functions that are cohomological in a certain sense.

- Classical holomorphic modular forms of arbitrary weight and character, and their Rankin-Selberg convolutions (Shimura).
- More generally, L-functions of regular algebraic representations on $\mathrm{GL}_n \times \mathrm{GL}_m$ with $m + n$ odd (Raghuram – Harder).
- Triple product L-functions (Garrett, Harris, Gross, Kudla,..)

Upper half plane and congruence subgroups

Let

$$\mathbb{H} = \{x + iy : x \in \mathbb{R}, y > 0\}$$

be the upper half plane. It is a homogeneous space for $SL_2(\mathbb{R})$ under the action

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \mapsto \frac{az+b}{cz+d}.$$

In fact, this extends to an action on $\mathbb{H}^* = \mathbb{H} \cup \mathbb{R} \cup \{\infty\}$. The $SL_2(\mathbb{R})$-invariant measure on $\mathbb{H}$ is $\frac{dx\ dy}{y^2}$.
The subset $\mathbb{Q} \cup \{\infty\}$ of $\mathbb{H}^*$ is known as the set of cusps.

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Definition of $\Gamma_0(N)$

For a positive integer N we define the Hecke congruence subgroup

$$\Gamma_0(N) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : c \equiv 0 \pmod{N}\}$$

Modular forms

Definition of modular forms and cusp forms

Fix positive integers N, k. A modular form of level N and weight k is a holomorphic function f on $\mathbb{H}$ satisfying the following properties:

- For any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$,

$$f(\gamma z) = (cz + d)^k f(z).$$

- f is holomorphic at the cusps.

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Remarks.

1. Examples of modular forms that are not cusp forms: Eisenstein series.
2. All modular forms have a Fourier expansion $f(z) = \sum_{n\geq 0} a_n e^{2\pi i n z}$.

Petersson inner products and Hecke operators

The set of cusp forms of level N and weight k forms a finite dimensional vector-space. There exists a natural inner product on this space called the Petersson inner product.

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Petersson inner product

Let f,g be modular forms of level N, weight k. Then we define their inner product

$$< f, g >:= \frac{1}{[SL_2(\mathbb{Z}) : \Gamma_0(N)]} \int_{\Gamma_0(N)\backslash\mathbb{H}} f(z)\overline{g(z)} y^k \frac{dxdy}{y^2}.$$

There exist certain Hermitian operators on the space of cusp forms of level N called Hecke operators. One can canonically attach L-functions to cusp forms that are eigenforms for (almost) all the Hecke operators.

Two cusp forms

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Assume that $k > l$ and all the Fourier coefficients a_n, b_n are algebraic numbers. Define

$$L(s, f \otimes g) = \zeta(2s) \sum_{n=1}^{\infty} \frac{a_n b_n}{n^{\frac{k+l-2}{2}+s}}.$$

This is called the Rankin–Selberg L-function.

Shimura's result on the Rankin–Selberg L-function

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Theorem 1 (Shimura, 1976)

For $m \in \frac{1}{2}\mathbb{Z}$, define

$$C(m, f, g) = \frac{L(m, f \otimes g)}{\pi^{2m+k} \langle f, f \rangle}.$$

Then, for $m \in \left[\frac{1}{2}, \frac{k-l}{2}\right] \cap \left(\mathbb{Z} + \frac{k+l}{2}\right)$, we have $C(m, f, g) \in \overline{\mathbb{Q}}$

Shimura's proof (when $N = 1$)

1. **Integral representation**: Define a (non-holomorphic) Eisenstein series

$$E_\lambda(z, s) = \sum_{(m,n)\in\mathbb{Z}^2} (mz + n)^{-\lambda} |mz + n|^{-2s}$$

Then, up to simple factors,

$$L(s + \frac{k + l - 2}{2}, f \otimes g) = \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} \overline{f(z)} g(z) E_{k-l}(z, s+1-k) y^{s-1} dxdy.$$

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4. **Arithmeticity of Petersson inner products**: For any modular form h of weight k with algebraic Fourier coefficients, one has
$$\frac{\langle f,h\rangle}{\langle f,f\rangle} \in \overline{\mathbb{Q}}.$$

Nearly holomorphic modular forms

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Definition 2

A nearly holomorphic modular form on $\mathbb{H}$ of weight k, polynomial degree p and level N is a continuous function on $\mathbb{H}$ that

- Transforms like a modular form for $\Gamma_0(N)$.
- Is a sum of finitely many terms of the form $y^{-k}h(z)$ where $p \geq k \geq 0$ is an integer and $h(z)$ is holomorphic.

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- For each p, k, N, we get a finite dimensional vector space.
- Shimura: All nearly holomorphic modular forms have the rationality of projection property. (explicit differential operators)
- If f is a nearly holomorphic modular form of degree p, then $(\frac{d}{dz} + \frac{k}{2iy})f$ is a nearly holomorphic modular form of degree $p+1$.

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3. Nearly holomorphic Siegel modular forms have been indispensable for studying special L-values via the method of integral representations. However, their structural properties and place in the Langlands framework have not yet been fully understood. This greatly reduces their applicability.
4. A better understanding of nearly holomorphic modular forms on general spaces in the adelic language is key to solving many open problems on special values of L-functions.

Why study Siegel modular forms?

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- They can be related to the number of ways of representing a quadratic form by another.
- Paramodular conjecture: L-functions of rational abelian surface are L-functions of Siegel cusp forms.

Siegel modular forms of degree 2

Definition of Sp_4

For a commutative ring R, we denote by $\mathrm{Sp}_4(R)$ the set of 4×4 matrices $A \in \mathrm{GL}_4(R)$ satisfying the equation $A^t JA = J$ where $J = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}$.

Definition of $\mathbb{H}_2$

Let $\mathbb{H}_2$ denote the set of complex 2×2 matrices Z such that $Z = Z^t$ and $\mathrm{Im}(Z)$ is positive definite.

$\mathbb{H}_2$ is a homogeneous space for $\mathrm{Sp}_4(\mathbb{R})$ under the action

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} : Z \mapsto (AZ + B)(CZ + D)^{-1}$$

The space $S_\rho(\Gamma)$

Let Γ be a congruence subgroup of $\mathrm{Sp}_4(\mathbb{Z})$ and (ρ, V) a representation of $\mathrm{GL}_2(\mathbb{C})$.

Siegel modular forms

A holomorphic vector valued Siegel modular form of degree 2, level Γ and weight ρ is a holomorphic V-valued function F on $\mathbb{H}_2$ satisfying

$$F(\gamma Z) = \rho(CZ + D)F(Z),$$

for any $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma$,
If in addition, F vanishes at the cusps, then F is called a cusp form.

We define $S_\rho(\Gamma)$ to be the space of cusp forms as above.

For any weight k, and any integer $p \geq 0$, we can define nearly holomorphic Siegel modular forms of degree 2, polynomial degree p, weight k for Γ.

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- This has been done in many special cases, but not in full generality.
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  2. Shimura, Mizumoto, Boecherer, Kozima: some other Γ. For $\rho \neq k$, only $\Gamma = \mathrm{Sp}_4(\mathbb{Z})$ has been done.
- Main difficulties: 1) Vector valued functions are hard! 2) Nearly holomorphic forms are not very well understood for arbitrary Γ.

Some recent work with Pitale and Schmidt

We prove the following result which resolves these difficulties.

Theorem 3 (Pitale – S –Schmidt, 2014)

There exists a linear map that takes elements of $S_\rho(\Gamma)$ to nearly holomorphic cusp forms of weight k, such that

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1. *Given by explicit Lie algebra operators at the group level, easy to restate in classical language.*
2. *All nearly holomorphic cusp forms are obtained in this way, and this is a direct sum decomposition.*

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2. *All nearly holomorphic cusp forms are obtained in this way, and this is a direct sum decomposition.*
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Some recent work with Pitale and Schmidt

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Moreover, the adelization of nearly holomorphic cusp forms give cuspidal automorphic representations whose parameters can be easily written down.

- The above theorem can be used to understand nearly holomorphic Siegel cusp forms of scalar weight in terms of holomorphic Siegel cusp forms of vector weight, OR vice versa!

- Puts nearly holomorphic forms of degree 2 squarely in the Langlands setup.

- Key idea: understand the Lie algebra operators that move around vectors in terms of differential operators and prove that any nearly holomorphic form is annihilated by a suitable composition of these (lots of combinatorics!)

- Some key features of our result:
  1. Everything is done adelically, so no level restrictions.
  2. Very explicit.

Application to special values

Ongoing work with Pitale and Schmidt should lead to the following result:

Theorem 4 (Pitale-S-Schmidt, in progress)

Let $\rho = \det^l sym^m$, m even. $F \in S_\rho(\Gamma)$ be an eigenfunction of Hecke operators, with algebraic Fourier coefficients. Then for all even integers r, $2 \leq r \leq l - 2$,

$$\frac{L_{st}(r, F)}{\pi^{2l+m+3r-3}\langle F, F\rangle} \in \overline{\mathbb{Q}}$$

and moreover this ratio behaves nicely under actions of $Aut(\mathbb{C})$.

- Generalizes all previous known special value results for this L-function.
- Has applications to special values of sym^4 L-function.

Key steps of the proof

The following are the two key steps of our special value result.

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Thank you for listening!