1 Real numbers (review)

In *Convergence and Continuity*, the real numbers $\mathbb{R}$ were introduced axiomatically, though most of the axioms were glossed over as intuitive or familiar. The one that was stressed, because it characterises the real numbers, and distinguishes them from the rationals is the axiom of *completeness*. Informally, this expresses the property of $\mathbb{R}$ that it has "no gaps". More formally, this is done by insisting that every subset $S \subset \mathbb{R}$ that is bounded above has a least upper bound in $\mathbb{R}$.

That approach, however, uses the fact that $\mathbb{R}$ is ordered, whereas we want to build the theory of metric spaces on a notion of *distance*. Recall

Definition 1.1. A sequence (s_n) in $\mathbb{R}$ is a *Cauchy sequence* if, given any $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that $|s_n - s_m| \leq \varepsilon$ for all $n, m \geq N_\varepsilon$.

Completeness of the reals can be expressed as follows: for every Cauchy sequence (s_n) there is a number $\alpha \in \mathbb{R}$ such that $s_n \to \alpha$. It turns out that these two ways of expressing completeness of $\mathbb{R}$

- every bounded subset has a least upper bound in $\mathbb{R}$;
- every Cauchy sequence has a limit in $\mathbb{R}$,

are equivalent. The second is appropriate for us here.

If we look now at the two central concepts of the *Convergence and Continuuity* module, we see that they too are expressed in terms of distance. A sequence (s_n) converges to $\alpha \in \mathbb{R}$ if, roughly, the *distance* between s_n and α can be made arbitrarily small by taking n sufficiently large. And a function $f : \mathbb{R} \to \mathbb{R}$ is continuous at α if the *distance* between $f(x)$ and $f(\alpha)$ can be made arbitrarily small by making the *distance* between x and α sufficiently small. (NB. The above are *not* rigorous definition of convergence of sequences and continuity of functions. So go to your notes or a textbook for rigorous definition, and verify that the only property of real numbers that is needed is that a natural notion of distance can be defined on them. A suitable choice for the distance between x and y is $|x - y|$.)

One of the aims of this course will be to set up a framework within which we can state and prove generalisations of familiar results about $\mathbb{R}$. Example of results that hold in much greater generality are (a) every continuous function $f : [a, b] \to \mathbb{R}$ on a bounded interval $[a, b]$ is bounded, and (b) every sequence (s_n) in $[a, b]$ has a convergent subsequence. That framework is described in the following section.

MTH6126: Metric Spaces

Mark Jerrum*

October 4, 2010

*Based on notes provided by Yu Safarov.

2 Metrics and norms

Definition 2.1. Let X be a non-empty set. A function $\varrho : X \times X \to \mathbb{R}$ is called a *metric* on X if it satisfies

M1. $\varrho(x, y) > 0$ if $x \neq y$, and $\varrho(x, x) = 0$,

M2. $\varrho(x, y) = \varrho(y, x)$,

M3. $\varrho(x, z) \leq \varrho(x, y) + \varrho(y, z)$,

where x, y and z are arbitrary elements of X.

Example 2.2. $\mathbb{R}$ with the usual (Euclidean) metric $\varrho(x, y) = |x - y|$.

Example 2.3 (Euclidean or ℓ^2-distance)**.** $\mathbb{R}^n$ with the Euclidean metric

$$\varrho(x, y) = \left(\sum_{i=1}^{n} |x_i - y_i|^2 \right)^{1/2},$$

where x_i and y_i are coordinates of the points x and y respectively.

Example 2.4 (Manhattan or ℓ^1-distance)**.** $\mathbb{R}^n$ with the metric

$$\varrho(x, y) = |x_1 - y_1| + |x_2 - y_2| + \cdots + |x_n - y_n| .$$

Example 2.5 (ℓ^∞-distance)**.** $\mathbb{R}^n$ with the metric

$$\varrho(x, y) = \max \big\{ |x_1 - y_1|, |x_2 - y_2|, \ldots, |x_n - y_n| \big\} .$$

Example 2.6. $C[a, b]$, the set of all continuous real functions on $[a, b]$, with the metric

$$\varrho(f, g) = \sup_{x \in [a,b]} |f(x) - g(x)| . \tag{1}$$

Example 2.7. $B(S)$, the set of all bounded real functions on a set S, with the metric

$$\varrho(f, g) = \sup_{x \in S} |f(x) - g(x)| . \tag{2}$$

Note that when $S = [a, b]$, equations (1) and (2) define the same metric. However the sets on which those metrics are defined are different in the two examples. (We return to this point later.)

Example 2.8 (discrete metric)**.** For any set X, define the metric ϱ by

$$\varrho(x,y) = \begin{cases} 1 & \text{if } x \neq y; \\ 0 & \text{otherwise.} \end{cases}$$

Of course, it is necessary to check that all the above examples are indeed metrics, i.e., that they satisfy Definition 2.1. Usually it is easy to check conditions M1 and M2 (symmetry), but M3 may require more work. We'll consider just the more important or trickier examples here, leaving the others as exercises.

Consider for instance the metric ϱ on $\mathbb{R}$ defined in Example 2.2. It is clear that ϱ satisfies M1 and M2. For M3, we need to show that $\varrho(x,z) \leq \varrho(x,y) + \varrho(y,z)$, for all $x,y,z \in \mathbb{R}$. But this is just the assertion that $|x-z| \leq |x-y| + |y-z|$ for all $x,y,z \in \mathbb{R}$, which is the familiar "triangle inequality" for real numbers.

Example 2.3 (continued). Conditions M1 and M2 are easy to check. (Do this!) For the triangle inequality M3 we need to check that

$$\sqrt{\sum\nolimits_i (x_i - z_i)^2} \leq \sqrt{\sum\nolimits_i (x_i - y_i)^2} + \sqrt{\sum\nolimits_i (y_i - z_i)^2},$$

for all points $x = (x_i)$, $y = (y_i)$, $z = (z_i)$ in $\mathbb{R}^n$. Here, and in what follows, all summations are over $i = 1, \ldots, n$.

For brevity, write u_i for $x_i - y_i$ and v_i for $y_i - z_i$ and note that $x_i - z_i = u_i + v_i$. The inequality we need to prove becomes

$$\sqrt{\sum\nolimits_i (u_i + v_i)^2} \leq \sqrt{\sum\nolimits_i u_i^2} + \sqrt{\sum\nolimits_i v_i^2}\,.$$

Since both sides of this inequality are positive, we can square both of them to obtain an equivalent inequality

$$\sum\nolimits_i (u_i + v_i)^2 \leq \sum\nolimits_i u_i^2 + \sum\nolimits_i v_i^2 + 2\sqrt{\sum\nolimits_i u_i^2}\sqrt{\sum\nolimits_i v_i^2},$$

which simplifies to

$$\sum\nolimits_i u_i v_i \leq \sqrt{\sum\nolimits_i u_i^2}\sqrt{\sum\nolimits_i v_i^2}.$$

But this is the well-known Cauchy-Schwarz inequality.

Example 2.7 (continued). Suppose $f, g, h : S \to \mathbb{R}$ are bounded real-valued functions on S. Since f and g are bounded, so also is $f - g$. So $\varrho(f, g) = \sup_{x \in S} |f(x) - g(x)| = 0$ is the supremum of a bounded set and hence well defined.

It is clear that $\varrho(f, g) \geq 0$, Also,

$$\begin{aligned} \varrho(f, g) = 0 \;&\Leftrightarrow\; \sup_{x \in S} |f(x) - g(x)| = 0 \\ &\Leftrightarrow\; |f(x) - g(x)| = 0, \text{ for all } x \in S \\ &\Leftrightarrow\; f(x) = g(x), \text{ for all } x \in S. \end{aligned}$$

So M1 holds.

For M2, note that $\varrho(f, g) = \sup_{x \in S} |f(x) - g(x)| = \sup_{x \in S} |g(x) - f(x)| = \varrho(g, f)$.

M3 requires only a little more work. Let $\alpha \in S$ be arbitrary. By definition, $\varrho(f, g) = \sup_{x \in S} |f(x) - g(x)|$ and so $|f(\alpha) - g(\alpha)| \leq \varrho(f, g)$. Similarly, $|g(\alpha) - h(\alpha)| \leq \varrho(g, h)$. By the triangle inequality for real numbers, $|f(\alpha) - h(\alpha)| \leq |f(\alpha) - g(\alpha)| + |g(\alpha) - h(\alpha)| \leq \varrho(f, g) + \varrho(g, h)$. But $\alpha \in S$ was chosen arbitrarily, so $\varrho(f, h) = \sup_{x \in S} |f(x) - h(x)| \leq \varrho(f, g) + \varrho(g, h)$.

Example 2.5 is a discrete analogue of Example 2.7, so you should be able to adapt the argument above to that example too.

Definition 2.9. We call the pair (X, ϱ) a *metric space* if X is a non-empty set and ϱ is a metric on X.

Note that M3 expresses a general triangle inequality for general metric spaces: if x, y, z are points in X then the distance from x to z cannot exceed the sum of the distances from x to y and from y to z.

Definition 2.10. If (X, ϱ) is a metric space and $A \subset X$ then ϱ is also a metric on A. The metric space (A, ϱ) is called a *subspace* of (X, ϱ).

Example 2.11. A continuous function on a bounded closed interval is always bounded. Therefore $C[a, b]$ is a subspace of $B[a, b]$ whenever $-\infty < a < b < +\infty$. (Refer to Examples 2.6 and 2.7. We already noted that the metrics agree.) In fact $C[a, b]$ is a strict subspace of $B[a, b]$ since there are bounded functions that are not continuous.

If X is a linear space, it is often possible to express the metric ϱ in terms of a function of one variable that can be thought of as the length of each element (i.e., its distance from 0).

Definition 2.12. Let X be a vector space over $\mathbb{R}$ (or over $\mathbb{C}$). A function $\|\cdot\| : X \to \mathbb{R}$ is called a *norm* on X if it satisfies

N1. $\|x\| > 0$ if $x \neq 0$, and $\|0\| = 0$,

N2. $\|\lambda x\| = |\lambda| \, \|x\|$ for all $x \in X$ and $\lambda \in \mathbb{R}$ (or $\lambda \in \mathbb{C}$),

N3. $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$.

A linear space with a norm is called a *normed* space.

A normed space can be viewed quite naturally as a metric space.

Theorem 2.13. *Suppose X is a linear space with norm $\|\cdot\|$. The function $\varrho(x, y) = \|x - y\|$ is a metric on X.*

Proof. M1 follows from N1, since $\varrho(x, x) = \|x - x\| = \|0\| = 0$, and $\varrho(x, y) = \|x - y\| > 0$, when $x \neq y$. M2 follows from N2, through the chain of equalities

$$\varrho(x, y) = \|x - y\| = \|-1 \cdot (y - x)\| = |-1| \cdot \|y - x\| = \varrho(y, x).$$

For M3, observe that

$$\varrho(x, y) + \varrho(y, z) = \|x - y\| + \|y - z\| \geq \|x - y + y - z\| = \|x - z\| = \varrho(x, z),$$

where the inequality is N3. □

However, not every metric arises in this way; one can have a metric ϱ on a vector space X such that $\varrho(x, 0)$ does not have the properties of a norm.

Example 2.14. In Examples 2.2, 2.3 and 2.4, the metrics are generated by the norms $\|x\| = |x|$, $\|x\| = (\sum_{i=1}^{n} |x_i|^2)^{1/2}$ and $\|x\| = \sum_{i=1}^{n} |x_i|$ respectively.

Example 2.15. The metrics (1) and (2) are generated by the norms

$$\|f\| = \sup_{x \in [a,b]} |f(x)| \qquad \text{and} \qquad \|f\| = \sup_{x \in S} |f(x)| \, .$$

A further way to obtain metric spaces is by taking a product of other metric spaces. For example, if (X_1, ϱ_1) and (X_2, ϱ_2) are both metric spaces, then so is $(X_1 \times X_2, \varrho)$, where $\varrho : (X_1 \times X_2)^2 \to \mathbb{R}$ is defined by any of:

- $\varrho\big((x_1, x_2), (y_1, y_2)\big) = \varrho_1(x_1, y_1) + \varrho_2(x_2, y_2)$,
- $\varrho\big((x_1, x_2), (y_1, y_2)\big) = \sqrt{\varrho_1(x_1, y_1)^2 + \varrho_2(x_2, y_2)^2}$, or
- $\varrho\big((x_1, x_2), (y_1, y_2)\big) = \max\big\{\varrho_1(x_1, y_1), \varrho_2(x_2, y_2)\big\}$.

It's not particularly difficult to prove the above claim, but we won't do so here.