

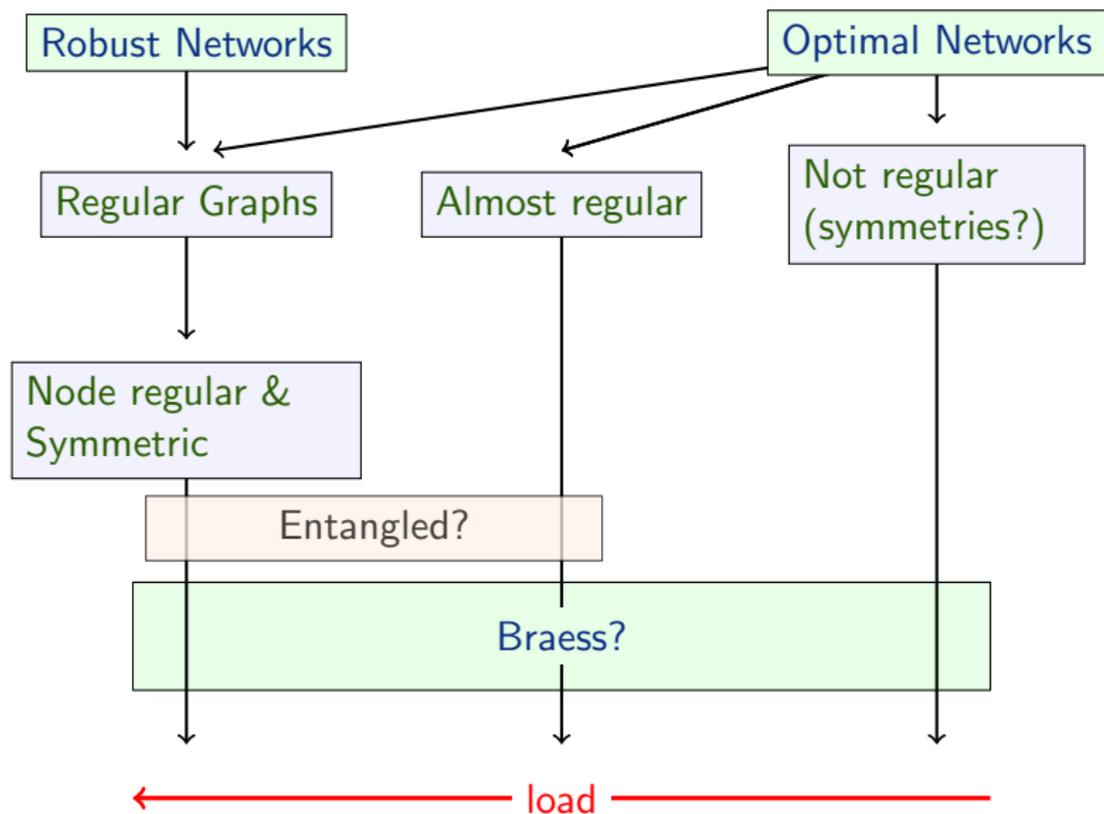
Optimal Networks, Congestion and Braess' Paradox

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Outline



Optimal Network

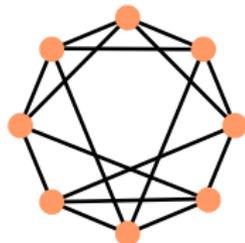
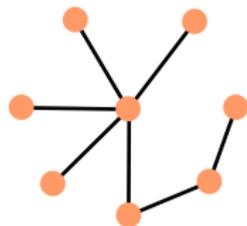
We are interested in how to deliver efficiently information in a communications network (e.g. minimising the transit time of information)

Possible Approaches

- ▶ Given a network 'find' an algorithm to optimise the delivery of packets
- ▶ Given a packet delivering algorithm 'build' a network that is optimal for this algorithm

R. Guimerà *et al.*, Optimal network topologies for local search with congestion, *Physical Review Letters*, **89**, 2002

Robust Networks



Why?

- ▶ Increasing use of networks as infrastructure (e-commerce)
- ▶ Increase threat of disruption of the communications due to failure or attacks (lack of robustness).

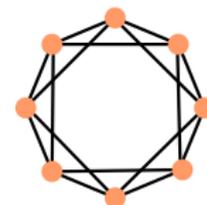
Solution

All the nodes 'look' the same so no node is special.

Removing one node will not disrupt the 'flow of information'.

A. H. Dekker & B. D. Colbert, *Network Robustness and Graph Topology*, 27th Australasian Computer Science Conference, 2004

Robust Networks



- ▶ **Node connectivity** $= \kappa$: Minimum number of nodes needed to remove to obtain a disconnected network
- ▶ **Link connectivity** $= \eta$: Minimum number of links needed to remove to obtain a disconnected network.
- ▶ d_{min} : minimum degree in the graph
- ▶ Any graph: $\kappa \leq \eta \leq d_{min}$,

Robust Networks (Dekker & Colbert): They are regular graphs with $\kappa = \eta = d_{min} = d$

Regular and Symmetric Graphs

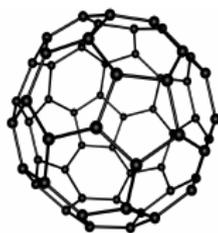
Theorem

(Dekker & Colbert) For any connected node-similar graph of degree d :

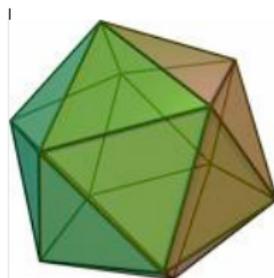
1. $\eta = d$ (link connectivity = degree)
2. if $d \leq 4$, then $\kappa = d$ (node connectivity = degree)
3. if the graph is symmetric, then $\kappa = d$
4. $\kappa \geq 2/3(d + 1)$



regular



node-similar

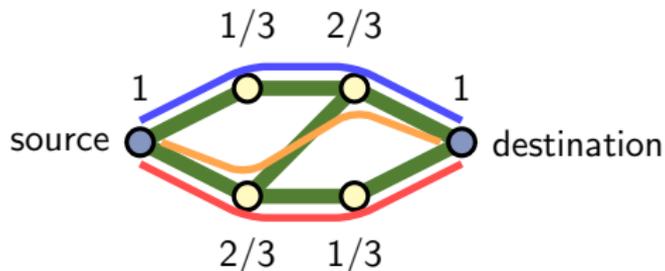


symmetric

The Network

- ▶ Fixed number of nodes and links.
- ▶ Each node is a source of traffic and has a queue (M/M/1).
- ▶ Each node produces the same amount of traffic.
- ▶ Estimate the traffic load at node i using the Betweenness Centrality (assumption: routing using shortest-path)

$$\text{Betweenness} = C_B = \frac{\text{number of shortest-paths that visit a node}}{\text{number of different shortest paths}}$$



Delay and Congestion

Total number of packets on the network, $n(t)$: (from Little's law)

$$\frac{dn(t)}{dt} = \Lambda N - \frac{n(t)}{\bar{\tau}}.$$

N = number of nodes, Λ = average traffic per node, $\bar{\tau}$ = average delay.

$\Lambda N \rightarrow$ traffic going in

$n(t)/\bar{\tau} \rightarrow$ traffic going out

For low load $\Lambda \ll 1$, $\bar{\tau} \approx$ average shortest-path.

Delay and Congestion

Average time

$$\bar{\tau} = \frac{1}{N} \sum_{i=1}^N \bar{W}_i$$

Average time that a packet spends in queue i .

$$\bar{W}_i = \rho_i / (1 - \rho_i)(1/\mu), \quad \text{where } \rho_i = \lambda_i / \mu$$

μ = service rate

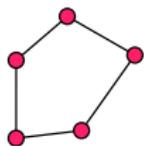
Steady state solution $dn(t)/dt = 0$ gives:

$$\bar{n} = \sum_{i=1}^N \frac{\Lambda(N-1)}{\mu_i(N-1) - \Lambda C_B(i)}$$

Congestion (queue node m)

$$\Lambda_c = (\mu(N-1)) / C_B(m)$$

olraRing.pdf!minipage
entagonNet.pdf!

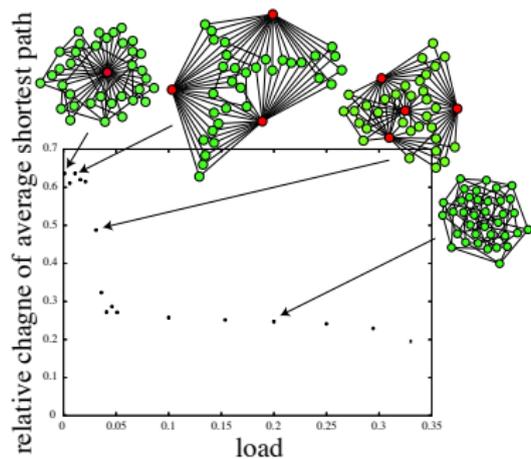
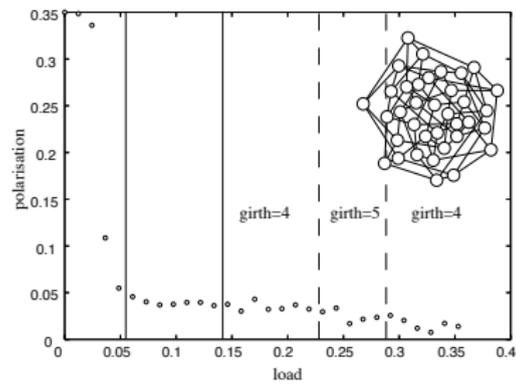


b skeleton

we can evaluate the optimal networks as a function of the load

Transition: 1-star \Rightarrow 3-star \Rightarrow 5-star \Rightarrow 7-star ...

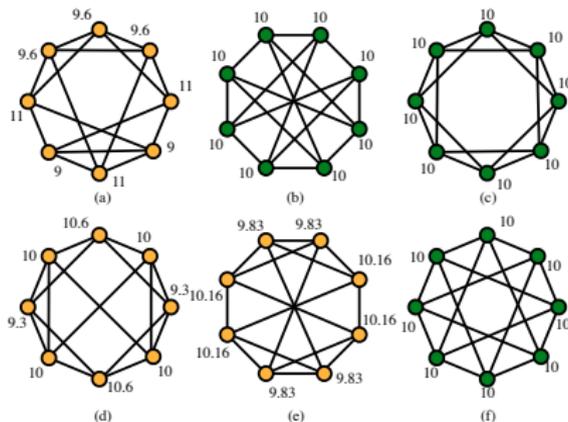
From Stars to Regular Graphs



$$2(\text{number of nodes}) = \text{number of links}$$

Which Regular Graphs?

- ▶ These graphs are all regular



- ▶ The sum of the betweenness is the same $\sum_i C_B(i) = 80$
- ▶ The average shortest path is the same.
- ▶ but the nodes congest at different loads
- ▶ the girth is different

Entangled Networks. Synchronisation

Very briefly (from review by Donetti *et al.*)

$$\frac{dx_i}{dt} = F(x_i) - \sigma \sum_j L_{ij} H(x_j)$$

$F(x_i)$ describes the evolution, $H(x_j)$ the coupling between neighbours and σ is a constant.

L_{ij} is the Laplacian matrix

$$L_{ij} = \begin{cases} -1 & \text{if there is a link between } i \text{ and } j \\ k_i & \text{if } j=i, \text{ and } k_i \text{ is the degree of node } i \\ 0 & \text{if there is no link between } i \text{ and } j \end{cases}$$

L. Donetti *et al.* Optimal network topologies: Expanders, Cages, Ramanujan graphs, Entangled networks and all that. arXiv:cond:mat/0605565

Entangled Networks. Synchronisation

'Robust' synchronised state if the ratio $Q = \lambda_N/\lambda_2$ is as small as possible; $\lambda_i =$ eigenvalue of L . (Barahona and Pecora, Wang and Chen)

Properties:

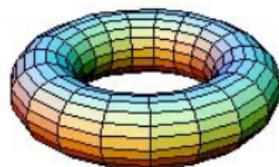
- ▶ Homogeneous regular networks (entangled networks Donetti *et al*)
- ▶ long loops (large girth)

But synchronisation is not necessary a property wanted in communication networks (route flapping).

Large girth means that if a link fails, there is a long detour when delivering the information

Adding Links

- ▶ In a rectangular–toroidal network the addition of a small amount of random links reduces the value of the critical load in–spite of the increased connectivity between the nodes (Fukś and Lawniczak)

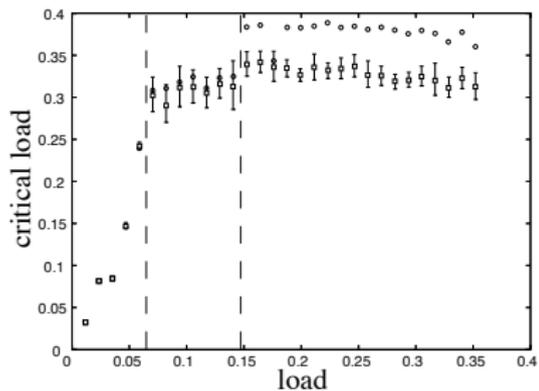


Braess' paradox: Each user chooses to minimise its expected delay by choosing an 'optimal' route. The addition of an extra link and hence route choice could reduce the delay. The paradox is that this is usually true for uncongested networks but it is less likely to be true for a congested network.

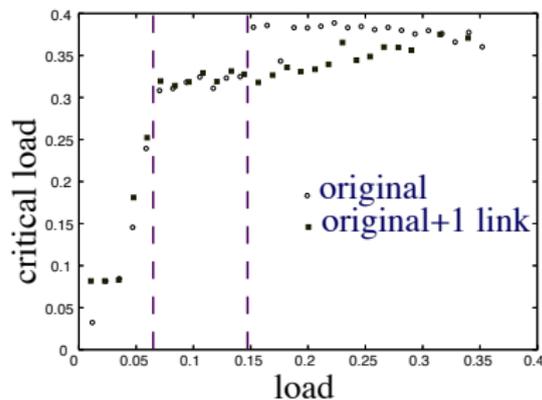
D. Braess, *Unternehmensforschung*, 12:258-268, 1968

Networks with Qs, Kelly *et al*, Calvert *et al*

Adding Links. Braess' Paradox



adding one link



adding one link and then
optimising

More Questions and Some Conclusions

- ▶ Low loads: For simple networks ($L = N$) it can be solved, can we use symmetry (group Th.) to obtain (an approximation of) the optimal solution?
- ▶ 'middle' of the range loads: Look like entangled networks, are they?
 - ▶ desirable qualities: adding new links has a small effect on the performance of the network (Braess).
- ▶ High loads: They seem to be node-regular networks (or even Symmetric).
 - ▶ Girth changes with the load and is relatively large (perhaps not a good characteristic in communication networks).

Possible future work

- ▶ Nodes that are not equivalent (fast queues).

