Amalgamating idempotent latin squares

Ph. D. Project

A latin square of order $n$ is an $n \times n$ matrix on $n$ symbols such that each symbol occurs exactly once in each row and each column. If the set of symbols is $\{1, 2, \cdots, n\}$, then the latin square is idempotent if symbol $i$ occurs in cell $(i, i)$. An idempotent latin square is also characterized by it being the multiplication table of an idempotent quasigroup.

An amalgamation of a latin square $L$ of order $n$ on symbols $\{1, 2, \cdots, n\}$ may be described in the following way. Let $R = \{\rho_1, \rho_2, \cdots, \rho_r\}$, $S = \{\sigma_1, \sigma_2, \cdots, \sigma_s\}$ and $T = \{\tau_1, \tau_2, \cdots, \tau_t\}$ be three compositions of $n$ (A composition of $n$ is a vector of positive integers adding to $n$). Let $A$ be the following $r \times s$ matrix with $t$ symbols, say $\nu_1, \cdots, \nu_t$. If the sum of the numbers of occurrences of symbols $\tau_1 + \cdots + \tau_{k-1} + 1, \cdots, \tau_k$ in the set of cells

$\{(\lambda, \mu) : \rho_1 + \rho_2 + \cdots + \rho_{i-1} + 1 \leq \lambda \leq \rho_1 + \rho_2 + \cdots + \rho_i,$
$\sigma_1 + \sigma_2 + \cdots + \sigma_{j-1} + 1 \leq \mu \leq \sigma_1 + \sigma_2 + \cdots + \sigma_j\}$

of $L$ is $x$, then cell $(i, j)$ of $A$ contains $\nu_k x$ times. The matrix $A$ has the following properties:

(i) Row $i$ of $A$ contains symbol $\nu_k \rho_i \tau_k$ times, counting repetitions,
(ii) Column $j$ of $A$ contains symbol $\nu_k \sigma_j \tau_k$ times, counting repetitions,
(iii) Cell $(i, j)$ of $A$ contains $\rho_i \sigma_j$ symbols, counting repetitions.

The matrix $A$ is called the $(R, S, T)$-amalgamation of $L$.

If we did not know that $A$ was derived from a latin square, but simply that $A$ was an $r \times s$ matrix, with multiple entries in its cells, with symbols $\nu_1, \cdots, \nu_t$, satisfying the numerical conditions (i), (ii) and (iii) for some compositions $R, S$ and $T$ of $n$, then we call $A$ an $(R, S, T)$-outline latin square.

Theorem 1. An $(R, S, T)$-outline latin square is the $(R, S, T)$-amalgamation of some latin square $L$.

Various slight generalizations of this result have been proved, in particular in [11]. A number of analogues have been proved, in particular for amalgamated
Hamiltonian decomposition of complete graphs of odd order in [9], for triple systems of even index in [5], and for complete hypergraphs in [2]. In these analogues usually some additional structural conditions are needed as well as the numerical conditions.

A number of embedding theorems for partial latin squares are easy consequences of Theorem 1. One well-known consequence is Ryser’s theorem, giving necessary and sufficient numerical conditions for an $r \times s$ rectangular partial latin square to be completable to a latin square of order $n$. There are embedding theorems for partial latin squares which are much more involved than Ryser’s theorem which are also consequences of Theorem 1 (see for example [10]). Courtiel and Vaughan [13] have an application of Theorem 1 to rectangular gerechte designs, and Cameron, Hilton and Vaughan [3] used some analogous ideas to prove some very similar results about generalized Sudoku designs.

An unexpected application of a slight generalization of Theorem 1 was found by Glebsky, Gordon and Rubio to a rather different area of mathematics. It was to approximations of topological groups by unimodular groups [6] and of unimodular groups by finite quasigroups [7]. (A locally compact group $G$ is called unimodular if the left Haar measure on $G$ is equal to the right one. It is shown that $G$ is unimodular if and only if it is approximable by finite quasigroups.)

The beginning of the study of embeddings of partial latin squares seems to go back to Universal Algebra with Trevor Evans [4]. In 1960 he showed that any partial latin square on $n$ symbols of order $n$ can be embedded in a latin square of order $t$ for any $t \geq 2n$. The question arose whether a finite partial idempotent quasigroup could be finitely embeddable in an idempotent quasigroup. Lindner, at that time a student of Evans, solved the problem in his Ph.D thesis, and the result was published in 1971 [12]. He suggested that a partial idempotent latin square of order $n$ could be embedded in an idempotent latin square of order $t$ for all $t \geq 2n + 1$. This difficult problem was solved in 1982 [1] by Andersen, Hilton and Rodger. The arguments used by Evans for embedding partial latin squares do not adapt at all easily for partial idempotent latin squares.

The project is to find and prove an amalgamation theorem for idempotent latin squares. Sufficient is now known about them for the project to be feasible. Various interesting corollaries concerning the embedding of partial idempotent latin squares, unproved at present, can be expected to follow in the same way that Theorem 1 has numerous corollaries. Moreover applications to locally compact groups, like the results of Glebsky, Gordon and Rubio, may well be possible.

References


