On the structure of 1-designs with at most two block intersection numbers

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March 8, 2007

Abstract

We introduce the notion of an unrefinable decomposition of a 1-design with at most two block intersection numbers, which is a certain decomposition of the 1-designs collection of blocks into other 1-designs. We discover an infinite family of 1-designs with at most two block intersection numbers that each have a unique unrefinable decomposition, and we give a polynomial-time algorithm to compute an unrefinable decomposition for each such design from the family. Combinatorial designs from this family include: finite projective planes of order n; SOMAs, and more generally, partial linear spaces of order (s,t) on $(s+1)^2$ points; as well as affine designs, and more generally, strongly resolvable designs with no repeated blocks.

AMS classification. 05B05 (primary), 05B25 (secondary)

Keywords. block intersection numbers, SOMAs, unrefinable decompositions, ud-types, strongly resolvable designs.

1 Introduction

Let X be a finite non-empty set of v elements, called *points*, and let $\mathcal{B}$ be a finite non-empty multiset of k-subsets of X, called *blocks*. Also, we let $t \geq 0$ and $\lambda \geq 1$ be integers. Then the ordered pair $D = (X, \mathcal{B})$ is called a *t-design*, or more specifically a *t-(v, k, λ) design*, if every t-subset of X is contained in exactly λ blocks.

For all $i = 0, 1, \ldots, t-1$, we denote by λ_i the number of blocks of the t-design D that contain a given set of i points. It follows that λ_i is independent of the choice of the i points and $\lambda_i\binom{k-i}{t-i} = \lambda\binom{v-i}{t-i}$. In particular, $b = \lambda_0$ is the number of blocks, and $\lambda_1 = r$ is the number of blocks that contain each point of X. For $0 \leq x < k$, x is called a *block intersection number of* D if there exists distinct blocks $B, B' \in \mathcal{B}$ such that $|B \cap B'| = x$. If D has repeated blocks then we define k to be a block intersection number of D.

Let $x_1 \neq x_2$ be non-negative integers. A $\mathrm{TID}(v, k; r, \{x_1, x_2\})$, or more simply a *two-intersecting design*, is a 1-(v, k, r) design whose block intersection numbers are contained within the set $\{x_1, x_2\}$. If a two-intersecting design has no repeated blocks then it is said to be *simple*.

A *quasi-symmetric design* is a 2-design with two block intersection numbers. So we can regard two-intersecting designs as a generalisation of quasi-symmetric designs. We refer the reader to [9] for further information regarding quasi-symmetric designs.

We can easily see that a regular graph of degree r on v vertices is the same thing as a simple $\mathrm{TID}(v, 2; r, \{0, 1\})$, and more generally, a partial linear space of order (s, t) on v points is essentially the same thing as a simple $\mathrm{TID}(v, s-1; t-1, \{0, 1\})$.

Two two-intersecting designs $(X, \mathcal{B})$ and $(X', \mathcal{B}')$ are said to be *isomorphic* if there is a bijection from X to X' that induces a bijection from $\mathcal{B}$ to $\mathcal{B}'$.

We are interested in the unrefinable decompositions of a two-intersecting design, which we go on to define.

Let $D = (X, \mathcal{B})$ be a TID$(v, k; r, \{x_1, x_2\})$, where $r \geq 1$.

A *decomposition* of D is a partition $\{\mathcal{B}_1, \ldots, \mathcal{B}_m\}$ of the multiset $\mathcal{B}$ of blocks into m parts say, such that each $(X, \mathcal{B}_i)$ is a TID$(v, k; r_i, \{x_1, x_2\})$, for some $r_i \geq 1$. We then call $(r_1, \ldots, r_m)$ a *type* of D.

It is clear that $\{\mathcal{B}\}$ is one decomposition. If this is the only decomposition then the two-intersecting design D is said to be *indecomposable*; otherwise, we say that D is *decomposable.*

An *unrefinable decomposition* of the TID$(v, k; r, \{x_1, x_2\})$ D is a decomposition $\{\mathcal{B}_1, \ldots, \mathcal{B}_m\}$ of D, such that each $(X, \mathcal{B}_i)$ is indecomposable. Where each $(X, \mathcal{B}_i)$ is a TID$(v, k; r_i, \{x_1, x_2\})$, we call $(r_1, \ldots, r_m)$ an *unrefinable decomposition type (or ud-type)* of D.

We now discuss an example of the unrefinable decompositions of a two-intersecting design.

A *strongly resolvable design* is a 2-design whose blocks can be partitioned into c equivalence classes each of m blocks, such that

(i) every point occurs in a constant number μ blocks in each class, and

(ii) there are constants q_1 and q_2, such that any two blocks belonging to the same class intersect each other in q_1 points, whereas any two blocks belonging to different classes intersect each other in q_2 points.

S. S. Shrikhande and D. Raghavaro [10] have shown that a 2-(v, k, λ) design satisfying (i) also satisfies (ii) if, and only if, $b = v + c - 1$ (where b is the number of blocks and c is the number of equivalence classes of the design). So such a design can easily be shown to be a TID$(v, k; r, \{q_1, q_2\})$ which has a type of $(\mu, \mu, \ldots, \mu)$ (c times), where $r = bk/v$, $q_1 = (\mu - 1)k/(m - 1)$ and $q_2 = \mu k/m = k^2/v$.

An *affine design* is a strongly resolvable design such that every point occurs in exactly one block from each class. An example of an affine design is an *affine plane of* *order* n, which is a 2-$(n^2, n, 1)$ design. It is clear that such a design is a simple $\mathrm{TID}(n^2, n; n+1, \{0, 1\})$.

A *finite projective plane of order* n is a 2-$(n^2+n+1, n+1, 1)$ design, and thus can easily be shown to be a simple $\mathrm{TID}(n^2+n+1, n+1; n+1, \{x, 1\})$, where the non-negative integer $x \neq 1$.

It is known that a finite projective plane of order n exists if, and only if, an affine plane of order n exists. Also, it is known that a finite projective plane of order n exists, when n is a prime power. However, the existence of one when n is not a prime power is a still a major unsolved problem.

In this paper, we show that every simple $\mathrm{TID}(v, k; r, \{x_1, x_2\})$ is indecomposable, when $x_2 < k^2/v$, $x_1 < x_2$ and $r \geq 1$, and has a unique unrefinable decomposition, when $x_2 = k^2/v$, $x_1 < x_2$ and $r \geq 1$. For this latter case, we give a polynomial-time algorithm to compute an unrefinable decomposition of such a two-intersecting design.

Our results show that every finite projective plane of order n is indecomposable, in particular, such designs have a unique unrefinable decomposition. Also, our results show that every strongly resolvable design with no repeated blocks has a unique unrefinable decomposition, and show that every partial linear space of order (s, t) on $(s+1)^2$ points has a unique unrefinable decomposition. Note that these results on the finite projective plane and on strongly resolvable designs were previously known. Uniqueness of unrefinable decomposition of strongly resolvable designs and indecomposability of finite projective planes follows from more general results of Y. J. Ionin and Shrikhande (see [6, Proposition 2.5] and [7, Theorem 5.1.15]).

The results in this paper are shown by studying graphs on two-intersecting designs, and we define these graphs in Section 4. In Section 6, we discuss our results and the methods used to obtain them.

The motivation for studying the unrefinable decompositions of two-intersecting designs comes from SOMAs and from Steiner triple systems. We discuss SOMAs in the following section, and Steiner triple systems in Section 3.

2 SOMAs

Closely related to partial linear spaces of order (s,t) are SOMAs, which we now formally define.

Let $r \geq 0$ and $k \geq 2$ be integers. A SOMA, or more specifically a SOMA(r,k), is an $k \times k$ array A each of whose entries is a r-subset of a rk-set Ω (the *symbol-set*), such that every symbol of Ω occurs exactly once in each row and exactly once in each column of A, and every 2-subset of Ω is contained in at most one entry of A.

A SOMA(r,k) can be constructed by the superposition of r mutually orthogonal Latin squares (MOLS) of order k with pairwise disjoint symbol-sets. Note that not every SOMA(r,k) can be constructed in this way.

We remark that the name SOMA was introduced by N. C. K. Phillips and W. D. Wallis, in [8], as an acronym for simple orthogonal multi-array.

We now show the connection between SOMA(r,k)s and partial linear spaces of order (s,t).

Let A be a SOMA(r,k) with symbol-set Ω, where $r \geq 1$. We denote by $[k]$ and by $A(i,j)$ the set $\{1,2,\ldots,k\}$ and the (i,j)-entry of the SOMA A respectively.

For each symbol $\alpha \in \Omega$, we let B_α be a subset of $[k]\times[k]$ (Cartesian product) given by the rule that $(i,j) \in B_\alpha$ if and only if $\alpha \in A(i,j)$. We then let $\mathcal{B}_A = \{B_\alpha : \alpha \in \Omega\}$.

It is not difficult to show that the ordered pair $D_A = ([k]\times[k], \mathcal{B}_A)$ is a simple TID$(k^2, k; r, \{0,1\})$, or equivalently a partial linear space of order $(k-1, r-1)$ on $(k+1)^2$ points.

We use the terms decomposition, type, indecomposable, decomposable, unrefinable decomposition and ud-type for the SOMA A to mean the corresponding term for the two-intersecting design D_A.

It is easy to see that the SOMA(r, k) A can be constructed by the superposition of r MOLS of order k exactly when the SOMA(r, k) A has a type of $(1, 1, \ldots, 1)$.

Our interest in SOMAs comes from the fact that a SOMA(r, k) can be constructed when the existence of r MOLS of order k is impossible or unknown. BCC Problem 13.21 ([13] and [4]) asks for constructions of SOMA(k, n)s with precisely this property.

It is known that two MOLS of order 6 do not exist. In contrast, Phillips and Wallis [8] have constructed SOMA(3,6)s of ud-types (1,2) and (3), and shown that no SOMA(4,6) exists. Note that no SOMA(5,6) exists as its existence would imply the existence of a finite projective plane of order 6, which of course does not exist.

Soicher [11] has given constructions of indecomposable SOMA(4,14)s, which each satisfy BCC Problem 13.21. It is known that there exists three MOLS of order 14, but not known whether there exists four such MOLS. We briefly discuss some recent developments in BCC Problem 13.21 for SOMA$(r, 10)$s.

Many examples of two MOLS of order 10 are known to exist, but the existence of three such MOLS is an unsolved problem. Soicher [11] has constructed SOMA(3,10)s of ud-types (1,2) and (3). Soicher, in [12], then went on to construct an indecomposable SOMA(4,10). For the benefit of the reader, we show this SOMA(4,10) in Figure 1.

Soicher, in [11], gave partial results on when a SOMA(r, k) has a unique unrefinable decomposition. In Problem 2 of [11], Soicher then asked whether there exists a SOMA(r, k), with $r \geq 1$, which has more than one unrefinable decomposition. We have first answered this question in the negative, in [2], where we have given a polynomial-time algorithm to compute an unrefinable decomposition of a SOMA(r, k). In Section 5 of this paper, not only do we generalise our answer to Soicher's problem, but we also generalise our algorithm to compute an unrefinable decomposition of a SOMA(r, k).

For further general information on SOMAs, we direct the interested reader to web resources on SOMAs in [12], and in [1].

Figure 1: An indecomposable SOMA(4,10)

1 2 19 20	3 21 22 23	4 5 24 37	6 7 8 25	9 26 27 39	10 28 29 40	11 12 13 30	14 15 31 38	16 32 33 34	17 18 35 36
17 30 31 32	14 18 26 33	9 21 25 28	11 16 35 37	1 6 10 24	12 19 22 39	4 23 36 40	2 3 13 34	5 7 29 38	8 15 20 27
10 11 36 38	8 29 30 34	7 15 22 32	12 23 24 28	13 17 20 37	1 4 14 35	2 21 33 39	16 25 27 40	3 9 18 31	5 6 19 26
9 12 15 29	4 13 27 38	6 31 34 36	5 10 21 30	2 23 25 35	3 8 26 37	7 14 16 20	18 28 32 39	17 19 24 40	1 11 22 33
8 33 35 40	2 5 12 36	3 16 19 38	1 27 29 31	4 11 28 34	18 20 24 25	6 9 22 37	7 10 17 26	15 23 30 39	13 14 21 32
5 25 34 39	6 20 32 40	2 11 18 27	9 17 33 38	14 19 29 36	13 16 23 31	15 24 26 35	1 12 21 37	4 8 10 22	3 7 28 30
4 6 16 21	11 24 31 39	1 26 30 40	13 15 18 19	8 12 32 38	7 27 33 36	3 17 25 29	5 20 22 35	2 14 28 37	9 10 23 34
7 18 23 37	1 16 17 28	13 29 35 39	14 22 34 40	3 5 15 33	2 6 30 38	10 19 27 32	8 9 24 36	11 20 21 26	4 12 25 31
13 22 26 28	10 15 25 37	8 14 17 23	3 20 36 39	7 21 31 40	5 9 11 32	1 18 34 38	4 19 30 33	6 12 27 35	2 16 24 29
3 14 24 27	7 9 19 35	10 12 20 33	2 4 26 32	16 18 22 30	15 17 21 34	5 8 28 31	6 11 23 29	1 13 25 36	37 38 39 40

3 Steiner triple systems

A *Steiner triple system of order* v, or a STS(v), is a 2-$(v, 3, 1)$ block design. It is not difficult to show that such a 2-design is a simple TID$(v, 3; r, \{0, 1\})$, where $r = (v-1)/2$.

C. J. Colbourn and A. Rosa, in [5], used the following terms to describe the decomposability of a STS(v): an indecomposable and a decomposable STS(v) are also known as non-separable and separable STS(v) respectively; a decomposition of an STS(v) that gives a type of $(r_1, \ldots, r_m)$ is called an $(r_1, \ldots, r_m)$-separation of an STS(v); and an unrefinable decomposition of an STS(v) that gives a ud-type of $(r_1, \ldots, r_m)$ is called an $(r_1, \ldots, r_m)$-atomic separation of an STS(v). In [5], Colbourn and Rosa discuss the unrefinable decompositions of an STS(v), when $v \leq 15$. We remark here the number of non-isomorphic STS(v)s of order $v = 1, 3, 7, 9, 13, 15, \ldots$ are known to be 1,1,1,1,2,80,… . In [5], Colbourn and Rosa mention the following:

- an STS(v) is trivially indecomposable, for every $v \in \{1, 3, 7\}$;
- the STS(9) has a resolution into parallel classes, and it only has a ud-type of

(1,1,1,1); and

- both STS(13)s only have a ud-type (3,3).

Colbourn and Rosa, in [5], then used computational results to show that a STS(15) may have more than one ud-type.

4 Graphs on two-intersecting designs

This section is a preliminary section to the following section, where we show the main results of this paper.

Let $x_1 \neq x_2$ be non-negative integers. We let $D = (X, \mathcal{B})$ be an ordered pair consisting of a finite non-empty set X of points and a finite non-empty multiset $\mathcal{B}$ of blocks, where each block is a subset of X, such that all the blocks of D have constant size, and the block intersection numbers of D are contained within the set $\{x_1, x_2\}$.

For all $i = 1, 2$, we define Γ_i^D to be the graph formed by taking as vertices the blocks of D, and joining two vertices of Γ_i^D by an edge whenever the corresponding blocks intersect in exactly x_i points. Similarly, we define $\Gamma_{1,2}^D$ to be the graph formed by taking as vertices the blocks of D, and joining two vertices of $\Gamma_{1,2}^D$ by an edge whenever the corresponding blocks intersect in x_1 or x_2 points.

If D is a quasi-symmetric design with block block intersection numbers x_1 and x_2, where $x_1 < x_2$, then the graph Γ_1^D is also known as the *block graph* of D. Assuming that the block graph of a quasi-symmetric design is connected, it is known that this graph is strongly regular (see [9, Theorem 3.8]).

Let $B \in \mathcal{B}$ be a block. For all $i = 1, 2$, we let $\Gamma_i^D(B)$ be the set of blocks adjacent to the block B in the graph Γ_i^D, and we let $\gamma_i^D(B)$ be the cardinality of the set $\Gamma_i^D(B)$.

5 The structure of simple two-intersecting designs

Let $D = (X, \mathcal{B})$ be a simple TID$(v, k; r, \{x_1, x_2\})$ with $r \geq 1$. Again, we let b be the number of blocks of D.

Lemma 5.1. *The graph* Γ_1^D *is regular of degree*

$$\frac{r(k^2 - vx_2)}{k(x_1 - x_2)} + \frac{(x_2 - k)}{(x_1 - x_2)}.$$

Proof. We let $B \in \mathcal{B}$ be a block. Since the two-intersecting design D is simple, we have that

$$\gamma_1^D(B) + \gamma_2^D(B) = b - 1. \tag{1}$$

We count all blocks other than the block B (with multiplicity) that intersect this block B in a point, and so it follows that

$$\gamma_1^D(B)x_1 + \gamma_2^D(B)x_2 = (r - 1)k.$$

Consequently by Equation 1, we have that

$$\gamma_1^D(B)(x_1 - x_2) + (b - 1)x_2 = (r - 1)k. \tag{2}$$

Counting the point-block incidence pairs of D gives that $bk = vr$, and so $b = vr/k$. By Equation 2, it follows that

$$\gamma_1^D(B)(x_1 - x_2)k + (vr - k)x_2 = (r - 1)k^2.$$

Hence

$$\gamma_1^D(B) = \frac{r(k^2 - vx_2)}{k(x_1 - x_2)} + \frac{(x_2 - k)}{(x_1 - x_2)}.$$

The result now follows. □

In this lemma, if $x_2 = k^2/v$ then $x_1 < x_2$, since $x_2 \leq k$, $x_1 \neq x_2$ and as the graph Γ_1^D is regular of non-negative degree. Also in this lemma, if $x_2 > k^2/v$ then $x_1 < x_2$, since the inequalities $x_2 > k^2/v$ and $x_1 > x_2$ would imply that the graph Γ_1^D is regular of negative degree.

This lemma directly gives the following corollary.

Corollary 5.2. *Let $D' = (X, \mathcal{B}')$ be a simple* TID$(v, k; r', \{x_1, x_2\})$*, for some $r' \geq 1$, where $\mathcal{B}' \subseteq \mathcal{B}$. In addition, suppose that $x_2 = k^2/v$. Then, the sets $\Gamma_1^D(B)$ and $\Gamma_1^{D'}(B)$ are equal, for every block $B \in \mathcal{B}'$.*

We now come to the following theorem.

Theorem 5.3. *Let $x_1 \neq x_2$ be non-negative integers. We let $D = (X, \mathcal{B})$ be an ordered pair that consists of a v-set X of points and a non-empty set $\mathcal{B}$ of blocks, where each block is a k-subset of X, such that any two distinct blocks intersect in precisely x_1 or x_2 points, and $x_2 = k^2/v$. Then D is a* TID$(v, k; r, \{x_1, x_2\})$ *(for some $r \geq 1$) if, and only if, the graph Γ_1^D is regular of degree $\frac{x_2-k}{x_1-x_2}$.*

Proof. For convenience, we let

$$c = \frac{x_2 - k}{x_1 - x_2}. \tag{3}$$

Suppose that D is a simple TID$(v, k; r, \{x_1, x_2\})$, for some $r \geq 1$. Then by Lemma 5.1, we have that the graph Γ_1^D is regular of degree c.

On the other hand, suppose that the ordered pair D is such that the graph Γ_1^D is regular of degree c.

For each point $p \in X$, we let r_p be the number of blocks that contain the point p. Note that r_p may be zero, for some point $p \in X$.

By counting the point-block incidence pairs of D, it follows that

$$\sum_{p \in X} r_p = bk, \tag{4}$$

where we recall that b is the cardinality of the set $\mathcal{B}$ of blocks.

We have that any two distinct blocks of the ordered pair D intersect in exactly x_1 or x_2 points. So each point $p \in X$ corresponds to exactly $\binom{r_p}{2}$ edges of the graph $\Gamma^D_{1,2}$. Our supposition is that the graph Γ^D_1 is regular of degree c on b vertices, and hence the graph Γ^D_2 is regular of degree $b-1-c$. Consequently by counting the number of points given by the graph $\Gamma^D_{1,2}$ (with multiplicity), we have that

$$\sum_{p \in X} \binom{r_p}{2} = \frac{bcx_1}{2} + \frac{b(b-1-c)x_2}{2}.$$

Into this equation, we substitute Equations 3 and 4, and the equation that $x_2 = k^2/v$. We expand and simplify the resulting equation, and so we can show that

$$\sum_{p \in X} {r_p}^2 = \frac{b^2k^2}{v}. \tag{5}$$

By Equations 4 and 5, we can easily observe that

$$\sum_{p \in X} \left(r_p - \frac{bk}{v} \right)^2 = 0.$$

Thus $r_p = bk/v$, for every point $p \in X$.

Hence, the ordered pair D is a $\mathrm{TID}(v, k; r, \{x_1, x_2\})$, where $r = bk/v \geq 1$.

The result now follows. □

This theorem leads to the following main result.

Theorem 5.4. *Let $D = (X, \mathcal{B})$ be a simple $\mathrm{TID}(v, k; r, \{x_1, x_2\})$ with $r \geq 1$, such that $x_2 = k^2/v$ (and so $x_1 < x_2$). We let $\{\mathcal{B}_1, \ldots, \mathcal{B}_m\}$ be a partition of the set $\mathcal{B}$ of blocks into m parts say, such that each part $\mathcal{B}_i$ is the set of vertices for a connected component of the graph Γ^D_1. Then $\{\mathcal{B}_1, \ldots, \mathcal{B}_m\}$ is a unique unrefinable decomposition of D.*

Proof. Lemma 5.1 shows that the graph Γ^D_1 is regular of degree $\frac{x_2-k}{x_1-x_2}$. Thus by Theorem 5.3, we have that each $D_i = (X, \mathcal{B}_i)$ is a $\mathrm{TID}(v, k; r_i, \{x_1, x_2\})$, for some $r_i \geq 1$. Hence $\{\mathcal{B}_1, \ldots, \mathcal{B}_m\}$ is a decomposition of D.

Let $D' = (X, \mathcal{B}')$ be a simple $\mathrm{TID}(v, k; r', \{x_1, x_2\})$, for some $r' \geq 1$, such that $\mathcal{B}' \subseteq \mathcal{B}$. Since $\{\mathcal{B}_1, \ldots, \mathcal{B}_m\}$ is a partition of the set $\mathcal{B}$ of blocks, there exists a part $\mathcal{B}_j$ say, which is not disjoint from the set $\mathcal{B}'$.

Consider the graphs Γ_1^D and $\Gamma_1^{D'}$. Corollary 5.2 shows that the sets $\Gamma_1^{D'}(B)$ and $\Gamma_1^D(B)$ are equal, for every block $B \in \mathcal{B}'$. Thus, if a block $B \in \mathcal{B}' \cap \mathcal{B}_j$ then $\Gamma_1^D(B) \subseteq \mathcal{B}' \cap \mathcal{B}_j$. Therefore by induction on the length of paths in the graph Γ_1^D, we can show that $\mathcal{B}_j$ is a subset of $\mathcal{B}'$. Hence, if the sets $\mathcal{B}'$ and $\mathcal{B}_j$ of blocks are not disjoint, for some $j = 1, \ldots, m$, then $\mathcal{B}_j \subseteq \mathcal{B}'$.

By the arguments above, it follows that each $\mathrm{TID}(v, k; r_i, \{x_1, x_2\})$ $D_i = (X, \mathcal{B}_i)$ is indecomposable. Furthermore, it follows that if the $\mathrm{TID}(v, k; r', \{x_1, x_2\})$ D' is indecomposable then $\mathcal{B}' = \mathcal{B}_j$, for some $j = 1, \ldots, m$.

Hence $\{\mathcal{B}_1, \ldots, \mathcal{B}_m\}$ is a unique unrefinable decomposition of D as required. □

This theorem clearly answers a problem of Soicher's on the unrefinable decompositions of SOMAs ([11, Problem 2]) in the negative. Also, this theorem shows directly that every strongly resolvable design with no repeated blocks has a unique unrefinable decomposition, and shows that every partial linear space of order (s, t) on $(s + 1)^2$ points has a unique unrefinable decomposition.

Note that this theorem gives a polynomial-time algorithm for determining a unrefinable decomposition of the $\mathrm{TID}(v, k; r, \{x_1, x_2\})$ D, since we are computing all the connected components of the graph Γ_1^D.

We come to another main result of this paper.

Theorem 5.5. *Every simple* $\mathrm{TID}(v, k; r, \{x_1, x_2\})$ *is indecomposable, when* $x_2 < k^2/v$, $x_1 < x_2$ *and* $r \geq 1$.

Proof. Let $D = (X, \mathcal{B})$ be a simple $\mathrm{TID}(v, k; r, \{x_1, x_2\})$ with $r \geq 1$, where $x_2 < k^2/v$ and $x_1 < x_2$.

For a proof by contradiction, suppose that $\{\mathcal{B}_1, \mathcal{B}_2\}$ is a decomposition of D. Then, we have that $D_1 = (X, \mathcal{B}_1)$ is a TID$(v, k; r_1, \{x_1, x_2\})$, for some r_1 such that $r > r_1 \geq 1$. Consider the graphs Γ_1^D and $\Gamma_1^{D_1}$. Since $\mathcal{B}_1 \subseteq \mathcal{B}$, it follows that

$$0 \leq \gamma_1^{D_1}(B) \leq \gamma_1^D(B), \tag{6}$$

for each block $B \in \mathcal{B}_1$.

By Lemma 5.1, we have that

$$\gamma_1^{D_1}(B) = \frac{r_1(k^2 - vx_2)}{k(x_1 - x_2)} + \frac{x_2 - k}{x_1 - x_2} \tag{7}$$

and

$$\gamma_1^D(B) = \frac{r(k^2 - vx_2)}{k(x_1 - x_2)} + \frac{x_2 - k}{x_1 - x_2}, \tag{8}$$

for each block $B \in \mathcal{B}_1$.

It is clear that our supposition gives that

$$\frac{k^2 - vx_2}{x_1 - x_2} < 0 \quad \text{and} \quad \frac{x_2 - k}{x_1 - x_2} \geq 0,$$

as $k \geq x_2$. We recall that $r > r_1 \geq 1$. Consequently by Equations 7 and 8, we have that

$$\gamma_1^{D_1}(B) > \gamma_1^D(B),$$

for every block $B \in \mathcal{B}_1$, which clearly contradicts Equation 6.

Thus $\{\mathcal{B}\}$ is a unique decomposition of D. Hence D is indecomposable.

The result now follows. □

Note here that this theorem shows that every finite projective plane of order n is indecomposable, in particular, such designs have a unique unrefinable decomposition.

Again, we let D be a simple TID$(v, k; r, \{x_1, x_2\})$ with $r \geq 1$. A trivial sufficient condition for D to have a unique unrefinable decomposition is that $r = 1$. In contrast, the following corollary shows a non-trivial sufficient condition for D to have a unique unrefinable decomposition, and this corollary follows directly from Theorems 5.3 and 5.4.

Corollary 5.6. *Every simple* $\mathrm{TID}(v,k;r,\{x_1,x_2\})$ *has a unique unrefinable decomposition, when* $x_2 \le k^2/v$, $x_1 < x_2$ *and* $r \ge 1$.

We conclude this section with the following theorem.

Theorem 5.7. *Let* $D = (X,\mathcal{B})$ *be a simple* $\mathrm{TID}(v,k;r,\{x_1,x_2\})$ *with* $r \ge 1$, *such that* $x_2 > k^2/v$ *(and so* $x_1 < x_2$*). Then the graph* Γ_1^D *is connected.*

Proof. First, we consider the degree of each vertex of the graph Γ_1^D.

Lemma 5.1 shows that the graph Γ_1^D is regular of degree c, where

$$c = \frac{r(k^2 - vx_2)}{k(x_1 - x_2)} + \frac{x_2 - k}{x_1 - x_2}.$$

Into this equation, we substitute the equation $r = bk/v$, where b is the cardinality of the set $\mathcal{B}$ of blocks. Therefore

$$c = \frac{b(k^2 - vx_2)}{v(x_1 - x_2)} + \frac{x_2 - k}{x_1 - x_2} \tag{9}$$

We let $D' = (X,\mathcal{B}')$ be an ordered pair, where $\mathcal{B}'$ is the set of vertices for a connected component of the graph Γ_1^D. Let b' be the cardinality of the set $\mathcal{B}'$. So each block contained within $\mathcal{B}'$ is a k-subset of X, and any two distinct blocks of $\mathcal{B}'$ intersect in precisely x_1 or x_2 points.

For each point $p \in X$, we let r_p be the number of blocks of D' that each contain the point p. Note that each integer r_p may be zero.

By counting the point-block incidence pairs of D', we have that

$$\sum_{p \in X} r_p = b'k. \tag{10}$$

We count the number of points given by the graph $\Gamma_{1,2}^{D'}$ (with multiplicity). Thus

$$\sum_{p \in X} \binom{r_p}{2} = \frac{b'cx_1}{2} + \frac{b'(b'-1-c)x_2}{2}.$$

Therefore by Equation 9, we can show that

$$\sum_{p\in X} {r_p}^2 - \sum_{p\in X} r_p = b' \left(\frac{b(k^2 - vx_2)}{v} + (b'x_2 - k) \right).$$

Consequently by Equation 10, it follows that

$$\sum_{p\in X} {r_p}^2 = \frac{b'(k^2 - vx_2)b}{v} + b'^2 x_2. \tag{11}$$

Finally, we use this equation together with the following easily observed inequality that

$$\sum_{p\in X} \left(r_p - \frac{b'k}{v} \right)^2 \geq 0 \tag{12}$$

to deduce that the graph $\Gamma_1^{D'}$ is connected.

We expand and simplify Equation 12. Consequently as X is a v-set of points, and by Equation 10, we can show that

$$\sum_{p\in X} {r_p}^2 - \frac{b'^2 k^2}{v} \geq 0.$$

Into this inequality, we substitute Equation 11. We factorise the resulting equation, and thus

$$b'(b' - b) \left(x_2 - \frac{k^2}{v} \right) \geq 0. \tag{13}$$

It is clear that $b' > 0$, $b' - b \leq 0$ and $\left(x_2 - \frac{k^2}{v} \right) > 0$, since $b \geq b' \geq 1$ and as $x_2 > k^2/v$. Therefore by equation 13, we have that

$$b'(b' - b) \left(x_2 - \frac{k^2}{v} \right) = 0,$$

which gives that $b' - b = 0$. So the sets $\mathcal{B}'$ and $\mathcal{B}$ of blocks are equal.

Hence, the graph Γ_1^D is connected as required. □

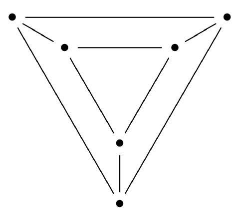
6 Outcomes and discussion

In the previous section, we have shown that every simple TID$(v, k; r, \{x_1, x_2\})$ is indecomposable, when $x_2 < k^2/v$, $x_1 < x_2$ and $r \geq 1$, and has a unique decomposition, when $x_2 = k^2/v$, $x_1 < x_2$ and $r \geq 1$. So every simple TID$(v, k; r, \{x_1, x_2\})$ has a unique unrefinable decomposition, when $x_2 \leq k^2/v$, $x_1 < x_2$ and $r \geq 1$.

We now discuss the unrefinable decompositions of simple TID$(v, k; r, \{x_1, x_2\})$s, with $r \geq 1$, where $x_2 > k^2/v$ (and so $x_1 < x_2$).

We can easily see that a regular graph of degree r on v vertices is essentially the same thing as a simple TID$(v, 2; r, \{0, 1\})$. So the complete graph $\mathbf{K}_5$ on 5 vertices and the 3-prism, which is shown in Figure 2, are a TID(5,2;4,{0,1}) and a TID(6,2;3,{0,1}) respectively.

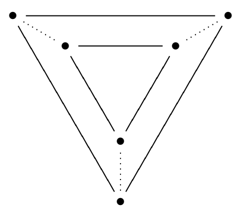
Figure 2: The 3-prism



It is easy to see that each unrefinable decomposition of $\mathbf{K}_5$ consists of two edge-disjoint 5-cycles. Note that although the graph $\mathbf{K}_5$ has more than one unrefinable decomposition it has a unique ud-type of (2,2).

Two unrefinable decompositions of the 3-prism are shown in Figure 3. A ud-type of (1,2) and of (1,1,1) of the 3-prism arises from the unrefinable decompositions shown in Figure 3(a) and Figure 3(b) respectively.

In the previous section, we have shown that every simple TID$(v, k; r, \{x_1, x_2\})$ has a unique unrefinable decomposition, when $x_2 \leq k^2/v$, $x_1 < x_2$ and $r \geq 1$. We


(a) ud-type (1,2)

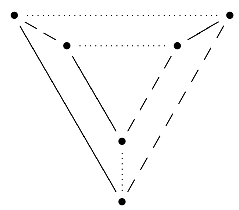

(b) ud-type (1,1,1)

Figure 3: Two unrefinable decompositions of the 3-prism

now let D be a simple $\mathrm{TID}(v, k; r, \{x_1, x_2\})$ with $r \geq 1$, where $x_2 > k^2/v$ (and so $x_1 < x_2$). The examples discussed above give that D may have more than one unrefinable decomposition, in which case, these unrefinable decomposition may give different ud-types. Consequently, we come to the following problem:

Problem 1. *Again, we let D be a simple $\mathrm{TID}(v, k; r, \{x_1, x_2\})$ with $r \geq 1$, where $x_2 > k^2/v$ (and so $x_1 < x_2$). What are the sufficient conditions for D to have a unique unrefinable decomposition? More generally, what are the sufficient conditions for D to have a unique ud-type?*

Many solutions to this problem are known for regular graphs: For example, a regular graph on v vertices that is a cycle has a unique unrefinable decomposition, and a regular bipartite graph has a unique ud-type. On the subject of two-intersecting designs which have a unique ud-type, Colbourn and Rosa [5] mention that both STS(13)s only have a ud-type of (3,3).

We end this section by discussing the techniques used in the previous section to compute an unrefinable decomposition of a two-intersecting design.

Let $D = (X, \mathcal{B})$ be a simple $\mathrm{TID}(v, k; r, \{x_1, x_2\})$ with $r \geq 1$. Theorem 5.4 gives a method for determining an unrefinable decomposition of D, when $x_2 = k^2/v$ and $x_1 < x_2$. In contrast, Theorem 5.7 shows that this same method always gives the decomposition $\{\mathcal{B}\}$ of D, when $x_2 > k^2/v$ and $x_1 < x_2$. We recall that the proof of Theorem 5.5 uses the degree of each vertex of the graph Γ_1^D to deduce that D is indecomposable, when $x_2 < k^2/v$ and $x_1 < x_2$. It is an easy exercise to show that this method does not give an unrefinable decomposition of D, when $x_2 > k^2/v$ and $x_1 < x_2$. In conclusion, we are unable to use the techniques developed in this paper to compute an unrefinable decomposition of D, when $x_2 > k^2/v$ and $x_1 < x_2$. This leads naturally to the following problem:

Problem 2. *Is there an efficient method to compute an unrefinable decomposition of a simple* $\mathrm{TID}(v, k; r, \{x_1, x_2\})$ *with* $r \geq 1$*, when* $x_2 > k^2/v$ *(and so* $x_1 < x_2$*)?*

Acknowledgements. The results in this paper generalise some results from my PhD thesis. I would like to thank my thesis supervisor Leonard H. Soicher for the many discussions on SOMAs and on related topics. I am grateful to the EPSRC for their financial support during my PhD. A thanks also to Peter J. Cameron for suggesting the name of two-intersecting designs, and Bill Jackson for providing an example of a two-intersecting design which has more than one ud-type (i.e. the 3-prism). In addition, I would like to thank the referees for providing constructive comments and help in improving the paper.

References

[1] John Arhin,
Further SOMA Update,
`http://www.maths.qmul.ac.uk/~arhin/soma/somas.html`.

[2] John Arhin, On the construction of SOMAs and related partial linear spaces, *PhD thesis*, University of London, 2006.

[3] E. F. Bricknell, A few results in message authentication,*Congressus Numerantium* **43** (1984) 141–154.

[4] P.J. Cameron,
British Combinatorial Conference Problem List,
`http://www.maths.qmul.ac.uk/~pjc/bcc/allprobs.pdf`.

[5] C. J. Colbourn A. Rosa, Triple Systems, *Oxford Sci. Pub.*, 1999.

[6] Y. J. Ionin M. S. Shrikhande, Resolvable pairwise balanced designs, *Journal of Statistical Planning and Inference* **72** (1998), 393–405.

[7] Y. J. Ionin M. S. Shrikhande, Combinatorics of symmetric designs, *New Mathematical Monographs* **5**, Cambridge University Press, 2006.

[8] N. C. K. Phillips W. D. Wallis, All solutions to a tournament problem, *Congressus Numerantium*, **114** (1996) 193–196.

[9] M. S. Shrikhande S. S. Sane, Quasi-Symmetric Designs, *London Mathematical Society Lecture Notes* Series **164**, Cambridge University Press, 1991.

[10] S. S Shrikhande D. Raghavaro, Affine α-resolvable incomplete block designs, *Contributions to Statistics* (1964) 471–480, Pergamon Press, Oxford.

[11] L. H. Soicher, On the structure and classification of SOMAs: generalizations of mutually orthogonal Latin squares, *Electronic Journal of Combinatorics* **6** (1999), #R32, 15 pp.

[12] Leonard H. Soicher,
SOMA Update, `http://www.maths.qmul.ac.uk/~leonard/soma/`.

[13] Research Problems section, *Discrete Mathematics* vol. **197**.