

Constructing t -designs from t -wise balanced designs

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Abstract

We give a construction to obtain a t -design from a t -wise balanced design. More precisely, given a positive integer k and a t - $(v, \{k_1, k_2, \dots, k_s\}, \lambda)$ design \mathcal{D} , with all block-sizes k_i occurring in \mathcal{D} and $1 \leq t \leq k \leq k_1 < k_2 < \dots < k_s$, the construction produces a t - $(v, k, n\lambda)$ design \mathcal{D}^* , with $n = \text{lcm}(\binom{k_1-t}{k-t}, \dots, \binom{k_s-t}{k-t})$. We prove that $\text{Aut}(\mathcal{D})$ is a subgroup of $\text{Aut}(\mathcal{D}^*)$, with equality when both $\lambda = 1$ and $t < k$. We employ our construction in another construction, which, given a t - (v, k, λ) design with $1 \leq t < k < v$, and a point of this design, yields a t - $(v-1, k-1, (k-t)\lambda)$ design. Many of the t -designs coming from our constructions appear to be new.

1 Introduction

For t a positive integer, a t -wise balanced design \mathcal{D} is an ordered pair (X, \mathcal{B}) , where X is a finite non-empty set (of *points*) and \mathcal{B} is a finite non-empty

multiset of subsets of X (called *blocks*), such that every t -subset of X is contained in a constant number $\lambda > 0$ of blocks. If $v = |X|$ and K is the set of sizes of the blocks, then we call \mathcal{D} a t - (v, K, λ) *design*. If all blocks of \mathcal{D} have the same size k (i.e. $K = \{k\}$), then \mathcal{D} is called a t -*design* or a t - (v, k, λ) *design*.

In this note we give a construction (the $*$ -construction) to obtain a t -design from a t -wise balanced design. More precisely, given a positive integer k and a t - $(v, \{k_1, k_2, \dots, k_s\}, \lambda)$ design \mathcal{D} , with all block-sizes k_i occurring in \mathcal{D} and $1 \leq t \leq k \leq k_1 < k_2 < \dots < k_s$, the $*$ -construction produces a t - $(v, k, n\lambda)$ design \mathcal{D}^* , with $n = \text{lcm}\left(\binom{k_1-t}{k-t}, \dots, \binom{k_s-t}{k-t}\right)$. We prove that $\text{Aut}(\mathcal{D})$ is a subgroup of $\text{Aut}(\mathcal{D}^*)$, with equality when both $\lambda = 1$ and $t < k$. We employ the $*$ -construction in another construction (the $\#$ -construction), which, given a t - (v, k, λ) design with $1 \leq t < k < v$, and a point of this design, yields a t - $(v-1, k-1, (k-t)\lambda)$ design. Many of the t -designs coming from our constructions appear to be new, and although they usually have repeated blocks, they often, via their constructions, have quite large automorphism groups.

2 The $*$ -construction

The input to the $*$ -construction consists of positive integers t and k , and a t - $(v, \{k_1, k_2, \dots, k_s\}, \lambda)$ design \mathcal{D} , with all block-sizes k_i occurring in \mathcal{D} and $1 \leq t \leq k \leq k_1 < k_2 < \dots < k_s$. Now for $i = 1, 2, \dots, s$ define

$$n_i = \binom{k_i - t}{k - t}, \quad n = \text{lcm}(n_1, n_2, \dots, n_s), \quad m_i = \frac{n}{n_i}. \quad (1)$$

The output of the $*$ -construction is a block design \mathcal{D}^* , which we prove below to be a t - $(v, k, n\lambda)$ design.

The point-set of \mathcal{D}^* is that of \mathcal{D} , and to construct the block-multiset \mathcal{B}^* of \mathcal{D}^* we proceed as follows:

- start by setting \mathcal{B}^* to be the empty multiset;
- for each $i = 1, 2, \dots, s$ and for each block $B \in \mathcal{B}$ of size k_i (including repeats) do:
 - insert m_i copies of every k -subset of B into \mathcal{B}^* .

Clearly, \mathcal{D}^* depends on the choice of k as well as on \mathcal{D} . Less obviously, since the t -wise balanced design \mathcal{D} may be t' -wise balanced for some $t' \neq t$, \mathcal{D}^* may depend on the choice of t . When we wish to make these dependencies explicit, we shall use the notation $\mathcal{D}^*(t, k)$ instead of \mathcal{D}^* .

Theorem 2.1 *Let k be a positive integer and let $\mathcal{D} = (X, \mathcal{B})$ be a t - $(v, \{k_1, k_2, \dots, k_s\}, \lambda)$ design, with all block-sizes k_i occurring in \mathcal{D} and $1 \leq t \leq k \leq k_1 < k_2 < \dots < k_s$. Then $\mathcal{D}^* = \mathcal{D}^*(t, k) = (X, \mathcal{B}^*)$ is a t - $(v, k, n\lambda)$ design, where $n = \text{lcm}(n_1, n_2, \dots, n_s)$ and $n_i = \binom{k_i-t}{k-t}$.*

Proof. Let T be any t -subset of X . Suppose that B is a block of \mathcal{B} of size k_i containing T . Then the number of k -subsets of B which contain T is $n_i = \binom{k_i-t}{k-t}$. Each of these k -subsets is added to \mathcal{B}^* exactly $m_i = n/n_i$ times. Hence B contributes exactly $n_i m_i = n$ blocks containing T to \mathcal{B}^* . Now T is contained in exactly λ blocks in \mathcal{B} , and so in exactly $n\lambda$ blocks in \mathcal{B}^* . ■

We have defined n to be $\text{lcm}(n_1, n_2, \dots, n_s)$. We could have chosen n to be any common multiple of $\{n_1, n_2, \dots, n_s\}$, but, in order to keep $n\lambda$ as small as possible, we choose the least common multiple. We also remark that the $*$ -construction works perfectly well when $s = 1$, that is, when \mathcal{D} is a t -design.

Example 1 Let \mathcal{D} be the 2- $(11, \{3, 5\}, 1)$ design with point-set $X = \{1, 2, \dots, 9, T, E\}$ (here $T = 10$ and $E = 11$), and block-multiset $\mathcal{B} =$

$\{167, 18E, 19T, 268, 279, 2TE, 369, 37E, 38T, 46T, 478, 49E, 56E, 57T, 589, 12345\}$

(see [1, p.187]).

(a) Suppose $t = k = 2$. Here $k_1 = 3, k_2 = 5$, and each $n_i = n = m_i = 1$. So $\mathcal{D}^*(2, 2)$ is the 2- $(11, 2, 1)$ design consisting of all the 2-subsets of X .

(b) The case $t = 2, k = 3$ is more interesting. Here $k = k_1 = 3, k_2 = 5, n_1 = 1, n_2 = 3, n = 3, m_1 = 3, \text{ and } m_2 = 1$. So $\mathcal{D}^* = \mathcal{D}^*(2, 3)$ is a 2- $(11, 3, 3)$ design, an $(11, 55, 15, 3, 3)$ -BIBD. The block-multiset of \mathcal{D}^* consists of three copies of each block of \mathcal{D} of size 3, together with all the 3-subsets of $\{1, 2, 3, 4, 5\}$.

The $*$ -construction was found as a result of looking for 2-designs with repeated blocks to help fill up Preece's catalogue [4]. Many new examples coming from this construction have since gone into the catalogue.

3 The #-construction

Let $\mathcal{T} = (X, \mathcal{B})$ be a t -(v, k, λ) design with $1 \leq t < k < v$, and let $x \in X$. We employ the *-construction in a new construction (the #-construction) which produces a t -($v - 1, k - 1, (k - t)\lambda$) design when given input \mathcal{T} and x . The #-construction proceeds as follows:

Let $X' = X \setminus \{x\}$, and let \mathcal{B}' be the multiset consisting of all $B \setminus \{x\}$ with $B \in \mathcal{B}$ (counting repeats). Denote the resulting block design (X', \mathcal{B}') by $\mathcal{T} \setminus x$, which is a t -($v - 1, \{k - 1, k\}, \lambda$) design (whose isomorphism class may depend on the choice of x). Next, apply the *-construction with input $t, k - 1$ and $\mathcal{T} \setminus x$ to obtain $(\mathcal{T} \setminus x)^*(t, k - 1)$, a t -($v - 1, k - 1, (k - t)\lambda$) design. We denote this output of the #-construction by $\mathcal{T}^\#(t, x)$.

Example 2 Start with the large Witt design \mathcal{W} , the unique (up to isomorphism) 5-(24, 8, 1) design; see [3, Chapter 8], where \mathcal{W} is called the Mathieu design \mathcal{M}_{24} . Now \mathcal{W} is also a 4-(24, 8, 5) design, a 3-(24, 8, 21) design, and a 2-(24, 8, 77) design. Let x be a point of \mathcal{W} (it matters not which one, since the automorphism group M_{24} of \mathcal{W} acts transitively (in fact 5-transitively) on the point-set of \mathcal{W}). Then $\mathcal{W}^\#(5, x)$ is a 5-(23, 7, 3) design, $\mathcal{W}^\#(4, x)$ is a 4-(23, 7, 20) design, $\mathcal{W}^\#(3, x)$ is a 3-(23, 7, 105) design, and $\mathcal{W}^\#(2, x)$ is a 2-(23, 7, 462) design.

Example 3 Start with a projective plane $\mathcal{P} = (X, \mathcal{B})$ of order $m \geq 2$, a 2-($m^2 + m + 1, m + 1, 1$) design. Now, given any $x \in X$, construct $\mathcal{P}^\#(2, x)$, which is a 2-($m^2 + m, m, m - 1$) design.

4 Automorphism groups

The automorphism group of a t -wise balanced design $\mathcal{D} = (X, \mathcal{B})$, denoted $\text{Aut}(\mathcal{D})$, is the group consisting of all the permutations of X which leave the block-multiset \mathcal{B} invariant. We now investigate the relationship of the automorphism groups of \mathcal{D} and $\mathcal{D}^*(t, k)$. For a block $B \in \mathcal{B}$, we let $\text{mult}(B)$ denote its multiplicity in \mathcal{B} .

Theorem 4.1 *Let k be a positive integer, let $\mathcal{D} = (X, \mathcal{B})$ be a t -($v, \{k_1, k_2, \dots, k_s\}, \lambda$) design, with all block-sizes k_i occurring in \mathcal{D} and $1 \leq t \leq k \leq k_1 < k_2 < \dots < k_s$, and let $\mathcal{D}^* = \mathcal{D}^*(t, k) = (X, \mathcal{B}^*)$ be the t -design obtained from the *-construction. Then*

- (i) $\text{Aut}(\mathcal{D}) \subseteq \text{Aut}(\mathcal{D}^*)$;
(ii) if $\lambda = 1$ and $t < k$, then $\text{Aut}(\mathcal{D}) = \text{Aut}(\mathcal{D}^*)$.

Proof. (i) Let $\alpha \in \text{Aut}(\mathcal{D})$. Let B^* be an arbitrary block in \mathcal{B}^* , hence there is a block $B \in \mathcal{B}$ which contains B^* as a k -subset. Suppose that $\alpha(B) = C$ for some block $C \in \mathcal{B}$, and that $\alpha(B^*) = C^*$. Then clearly C^* is a k -subset of C , a block of \mathcal{B} , hence $C^* \in \mathcal{B}^*$. Now we must show that $\text{mult}(C^*) = \text{mult}(B^*)$ (in \mathcal{B}^*) to conclude that $\alpha \in \text{Aut}(\mathcal{D}^*)$.

Fix i . Let B_1, B_2, \dots, B_d be the *distinct* blocks of \mathcal{B} of size k_i which contain B^* , and let C_1, C_2, \dots, C_e be the distinct blocks of \mathcal{B} of size k_i which contain C^* . Now, because $\alpha \in \text{Aut}(\mathcal{D})$, we must have $d = e$ and for every j with $1 \leq j \leq d$ there must exist a unique j' with $1 \leq j' \leq d$ for which $\alpha(B_j) = C_{j'}$. Hence $\text{mult}(B_j) = \text{mult}(C_{j'})$ since α preserves block multiplicities.

Now let f_i be the number of blocks (counting multiplicities) of \mathcal{B} of size k_i which contain B^* , and let g_i be the number of blocks (counting multiplicities) of \mathcal{B} of size k_i which contain C^* . Then $g_i = \sum_{j'=1}^d \text{mult}(C_{j'}) = \sum_{j=1}^d \text{mult}(B_j) = f_i$, and so, in \mathcal{B}^* , we have $\text{mult}(C^*) = \sum_{i=1}^s g_i m_i = \sum_{i=1}^s f_i m_i = \text{mult}(B^*)$ (m_i defined in (1)), as required. Hence $\alpha \in \text{Aut}(\mathcal{D}^*)$.

(ii) We first note that, because $\lambda = 1$, then $\text{mult}(B) = 1$ for every block $B \in \mathcal{B}$. Secondly, if R^* is an arbitrary block in \mathcal{B}^* then, again because $\lambda = 1$, there is a unique block $R \in \mathcal{B}$, with $R^* \subseteq R$.

Now let $\gamma \in \text{Aut}(\mathcal{D}^*)$. We must show that, for every block $B \in \mathcal{B}$, we have $\gamma(B) \in \mathcal{B}$. Then, from above, $\text{mult}(\gamma(B)) = 1 = \text{mult}(B)$, so $\gamma \in \text{Aut}(\mathcal{D})$. This will show that $\text{Aut}(\mathcal{D}^*) \subseteq \text{Aut}(\mathcal{D})$; part (i) then gives the result.

Fix i . Let B be an arbitrary block of \mathcal{B} of size k_i , and let B^* be an arbitrary k -subset of B , and let $\gamma(B^*) = C^*$. Now, because $\gamma \in \text{Aut}(\mathcal{D}^*)$, then $C^* \in \mathcal{B}^*$. So, from above, there is a unique block $C \in \mathcal{B}$, with $C^* \subseteq C$. We will show that $\gamma(B) = C$.

First we show that $\gamma(B) \subseteq C$. Suppose that $\gamma(B) \not\subseteq C$, then there is an element $x \in B \setminus B^*$ with $\gamma(x) \notin C$. Let D be a $(k-1)$ -subset of $B^* \subseteq B$, then $D^* = \{x\} \cup D$ is a k -subset of $B \in \mathcal{B}$, so $D^* \in \mathcal{B}^*$. Hence $E^* = \gamma(D^*) \in \mathcal{B}^*$, and there is a block $E \in \mathcal{B}$ with $E^* \subseteq E$. Now $E \neq C$ because $\gamma(x) \in E$ but $\gamma(x) \notin C$. Hence E and C are distinct blocks of \mathcal{B} . However, $\gamma(D) \subseteq E$, and $D \subseteq B^*$ so $\gamma(D) \subseteq \gamma(B^*) = C^* \subseteq C$. Now $t < k$ so $t \leq k-1 = |\gamma(D)|$. Now let T be any t -subset of $\gamma(D)$, then the distinct blocks E and C both contain T , a contradiction since $\lambda = 1$. Hence $\gamma(B) \subseteq C$.

To show that $C \subseteq \gamma(B)$ we show that $\gamma^{-1}(C) \subseteq B$ by noting that $\gamma^{-1}(C^*) = B^*$, and so the proof follows as above. Hence $\gamma(B) = C$ and, since i was arbitrary, the result is proved. ■

Example 4 We take \mathcal{D} to be the 2-(11, {3, 5}, 1) design of Example 1. Then $|\text{Aut}(\mathcal{D})| = 120$; indeed $\text{Aut}(\mathcal{D})$ is isomorphic to $\text{Sym}(5)$, and acts naturally as this group on the subset $\{1, 2, 3, 4, 5\}$ of the point-set (checked using GAP [2] and its DESIGN package [5]).

(a) $\mathcal{D}^*(2, 2)$ is the complete 2-(11, 2, 1) design. Hence $\text{Aut}(\mathcal{D}) \subseteq \text{Aut}(\mathcal{D}^*(2, 2)) = \text{Sym}(11)$, illustrating Theorem 4.1(i), and also showing that if $\lambda = 1$ and $t = k$ then $\text{Aut}(\mathcal{D}) \neq \text{Aut}(\mathcal{D}^*(t, k))$ is possible (see Theorem 4.1(ii)).

(b) $\mathcal{D}^* = \mathcal{D}^*(2, 3)$ is a 2-(11, 3, 3) design with $|\text{Aut}(\mathcal{D}^*)| = 120$ (double checked with the DESIGN package). This illustrates Theorem 4.1(ii).

Example 5 This example shows that if $\lambda > 1$ then $\text{Aut}(\mathcal{D}) \neq \text{Aut}(\mathcal{D}^*(t, k))$ is possible, even when $t < k$. We apply the #-construction to the projective plane \mathcal{P} of order 4, to obtain a 2-(20, 4, 3) design $\mathcal{P}^\# = \mathcal{P}^\#(2, x) = (X, \mathcal{B})$, which has a point-transitive automorphism group of order 5760. Then, we take $x \in X$ and obtain a 2-(19, {3, 4}, 3) design $\mathcal{D} = \mathcal{P}^\# \setminus x$ (using the notation of Section 3). (The choice of x does not affect the isomorphism class of \mathcal{D} since $\mathcal{P}^\#$ is point-transitive). Finally, construct a 2-(19, 3, 6) design $\mathcal{D}^* = \mathcal{D}^*(2, 3)$. It turns out that $|\text{Aut}(\mathcal{D})| = 288$, but $|\text{Aut}(\mathcal{D}^*)| = 576$. The construction of these designs and the determination of their automorphism groups was done using the DESIGN package.

Example 6 The DESIGN package shows that, up to isomorphism, there are exactly four 2-(11, {4, 5}, 2) designs (not counting the unique 2-(11, 5, 2) design). These designs \mathcal{D} have automorphism groups of orders 6, 8, 12, and 120, as do the corresponding $\mathcal{D}^*(2, 4)$, which are (believed to be new) 2-(11, 4, 6) designs. Note that these examples show that the converse of Theorem 4.1(ii) does not hold.

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