Self-dual, not self-polar

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Abstract

The smallest number of points of an incidence structure which is self-dual but not self-polar is 7. For non-binary structures (where a "point" may occur more than once in a "block") the number is 6.

Key words: incidence structure, self-dual, self-polar

At the IPM workshop in Tehran in August 2003, the third author asked for a small (preferably the smallest) example of an incidence structure whose incidence matrix N is self-dual but not self-polar. That is, N is a zero-one matrix such that there exist permutation matrices P_1, P_2 with $P_1 N = N^\top P_2$, but there does not exist a permutation matrix P_3 with $P_3 N = N^\top P_3^\top$.

Such matrices are known to exist, but the proofs depend on rather subtle properties of groups of Lie type and the matrices themselves are rather large. In this paper, we show:

Theorem 1 *The smallest order of a self-dual but not self-polar incidence structure is 7. Up to isomorphism there are exactly eight incidence structures on 7 points with this property.*

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Preprint submitted to Elsevier Science *8 December 2003*

PROOF. Let G be the bipartite incidence graph of an incidence structure. A duality of the structure is an isomorphism from the structure to its dual, which is thus an automorphism of G interchanging the two bipartite blocks. Such an automorphism would have order 2 if and only if the duality is a polarity. So in our case, G is a bipartite graph admitting an automorphism σ which interchanges the two bipartite blocks, but no such automorphism of order 2. Raising σ to an odd power if necessary, we may assume that its order is 2^d, with $d > 1$.

Suppose first that $d = 2$. We claim that if G has at most 12 vertices, then G also admits an automorphism τ of order 2 interchanging the bipartite blocks. Let σ have 4-cycles (a_i, b_i, c_i, d_i) for $i \in I$ and 2-cycles (e_j, f_j) for $j \in J$. We may assume that the points a_i, c_i and e_j form one bipartite block and b_i, d_i, and f_j the other.

Our first candidate for τ will be a permutation whose structure on the set $\{a_i, b_i, c_i, d_i\}$ is either $(a_i, b_i)(c_i, d_i)$ or $(a_i, d_i)(b_i, c_i)$, and which has cycles (e_j, f_j) for $j \in J$. Since τ agrees with σ on the union of the cycles of length 2, it preserves all edges and non-edges here. Moreover, if, say, f_j is joined to a_i, then it is also joined to c_i, while e_i is joined to b_i and d_i. So edges between 2-cycles and 4-cycles are preserved by τ. Also, edges within a 4-cycle are obviously preserved by τ. This shows that, if σ has only one 4-cycle, then τ is an automorphism.

Consider two 4-cycles of σ, say $(a_1, \ldots, d_1)$ and $(a_2, \ldots, d_2)$. Then a_1 is joined to both, one or neither of b_2 and d_2, and the other edges between the cycles follow from this. If a_1 is joined to both or neither, then either choice of τ on each cycle preserves these edges. If a_1 is joined to one of b_2 and d_2, then we can (and must) take $\tau = (a_1, b_1)(c_1, d_1)(a_2, d_2)(b_2, c_2)\ldots$. So if σ has two 4-cycles then an automorphism τ exists. Hence we may assume that σ has three 4-cycles and there are 12 vertices altogether.

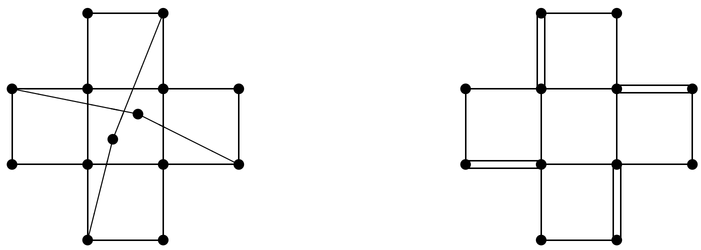
In this case, we can still choose τ unless a point of each cycle is joined to one point in each of the others; then the three requirements for τ conflict. The graph formed by the edges between the 4-cycles is a 12-gon, and the three 4-cycles are the "squares" formed by the diagonals of length 3. Whichever set of squares are chosen to be edges, there is always a reflection of the 12-gon interchanging the two bipartite blocks.

Now suppose that there are 14 vertices. The argument shows that the induced subgraph on the 4-cycles must be of the above form, and there is one 2-cycle (e, f). Now, in order to destroy the reflection symmetry, we must take a set S_1 of one or two squares to be edges of the graph, and a set S_2 of one or two squares whose vertices are joined alternately to e and f (where S_2 is not equal or complementary to S_1); moreover, we can choose whether or not to join e

and f. This gives eight graphs forming four complementary pairs within $K_{7,7}$; the numbers of edges are 20, 21, 24 (twice), 25 (twice), 28 and 29. It is simple to check that the eight graphs all have the required property and are pairwise non isomorphic.

A similar but easier argument shows that no such graph on 14 or fewer vertices can have a duality of order 8 but none of order 2 or 4. $\square$

If we allow multiple edges, then 12 vertices suffice: we can take the 12-gon whose vertices are the integers mod 12 and edges $\{i, i+1\}$, duplicate the edges $\{3i, 3i+1\}$, and add the diagonals $\{3i, 3i+3\}$. Similar arguments show that no smaller number of vertices is possible.



Graphs with fewest vertices, without and with multiple edges, are shown in the above figure; the 12-gon is the outer boundary.

More generally, if we take a regular 2^d-gon, erect a square on each side, and join one new vertex of each square alternately to one of two further vertices preserving the cyclic symmetry, we obtain a bipartite graph having $2(3 \cdot 2^{d-1} + 1)$ vertices, whose automorphism group is cyclic of order 2^d, such that an automorphism interchanges the two bipartite blocks if and only if it has order 2^d. That is, there is an incidence structure with $3 \cdot 2^{d-1} + 1$ points having a duality of order 2^d but none of smaller order.