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Six families of efficient resolvable designs in three replicates

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Abstract A resolvable incomplete-block design in three replicates is abstractly equivalent to a design for three factors whose effects are additive, but the efficiency factors for the two designs are not the same. Six families of known designs of the second type are used to construct designs of the first type.

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Key words efficiency factor; mutual balance; rectangular lattice; resolvable design; square design

1 Introduction

Let $\mathcal{T}$ be a set of v treatments. We consider resolvable incomplete-block designs Δ for $\mathcal{T}$ in r replicates of blocks of size k, where k divides v. Let $q = v/k$.

The i-th replicate defines a partition P_i of $\mathcal{T}$ into q equivalence classes of size k. Each class is the set of treatments allocated to one block in the i-th replicate. Bailey (1999) showed that properties of the resolvable design can be obtained from properties of the family of r partitions.

For $1 \leq i, j \leq r$, define a matrix X_{ij} whose rows are indexed by the classes of P_i and whose columns are indexed by the classes of P_j. The entry of X_{ij} in the row labelled by class G of P_i and the column labelled by class H of P_j is $|G \cap H|$. Thus, for $i \neq j$, X_{ij} is the incidence matrix of a square block design Δ_{ij} with q blocks and treatments and block size k.

In the case that $r = 2$, Williams, Patterson and John (1976) showed that Δ is optimal among resolvable designs if and only if Δ_{12} is optimal.

In particular, if Δ_{12} is a square balanced incomplete-block design (BIBD) then Δ is optimal among resolvable designs. Examples can be seen in each pair of replicates in the designs in Tables 3–4.

When $r = 2$ and $q = k+1$, the design Δ is optimal when the design Δ_{12} consists of all k-subsets of q. In this case Δ is a simple rectangular lattice design (Harshbarger, 1946). Rectangular lattice designs exist for some larger values of r with $q = k + 1$ and $r \leq q$: they all have the following two properties.

Each Δ_{ij}, for $i \neq j$, is a square BIBD. (1)

When i, j and l are all different, the partitions P_i, P_j and P_l are mutually balanced. (2)

The property of mutual balance was introduced by Preece (1966a, 1966b), and further explained by Preece (1976) and Bailey (1999). In the presence of condition (1), it means that, when i, j and l are all different,

$$X_{ij}X_{jl}X_{li} + X_{il}X_{lj}X_{ji} = \psi_{ijl}I + \phi_{ijl}J \tag{3}$$

for some integers ψ_{ijl} and ϕ_{ijl}, where I is the identity matrix and J is the all-1 matrix.

The motivation for mutual balance comes from consideration of designs for responses on $\mathcal{T}$ when P_i, P_j and P_k represent fixed effects. That is, if element t of $\mathcal{T}$ is in class G of P_i, class H of P_j and class K of P_l, then the expected value of the response Y_t on t is given by

$$\mathrm{E}(Y_t) = \lambda_G + \mu_H + \nu_K. \tag{4}$$

Preece (1966a, 1966b) showed that, if Δ_{ij}, Δ_{il} and Δ_{jl} are all square BIBDs, then the information matrix for the λ-parameters in model (4) is completely symmetric if and only if condition (3) is true.

It is not always true that an optimal resolvable design for r replicates remains optimal when one or more replicates are removed. For example, Table 1 shows two designs with $r = 3$, $v = 6$ and $k = 2$. Here A, D and E denote the harmonic mean, geometric mean and minimum, respectively, of the canonical efficiency factors. Design Γ is a rectangular lattice, and is optimal, but cannot be extended to any of the optimal 4-replicate designs. Design Ξ is slightly worse than Γ but can be extended to optimal 4- and 5-replicate designs. Nonetheless, Ξ is not much inferior to Γ, and all the contractions Γ_{ij} and Ξ_{ij} of both designs are optimal square designs.

Theorem 2.2 of Cheng and Bailey (1991) shows that rectangular lattice designs with $r = q$ are optimal. It is not known whether rectangular lattice designs are optimal for $2 < r < q$. However, their efficiency factors are known (Williams, 1977; Corsten, 1985; Bailey and Speed, 1986) and the designs do appear to be very efficient. Moreover, Heiligers and Sinha (1995) have shown that, when $r = 3$, $q = 7$ and $k = 3$, then one of the two sets of partitions satisfying conditions (1) and (2) gives a design which is optimal for model (4) among designs satisfying condition (1). It therefore seems

Table 1 Designs with $v = 6$ and $k = 2$ (blocks are columns)

Design Γ	Design Ξ
1 3 5 | 1 2 3 | 4 1 2 2 4 6 | 6 4 5 | 6 3 5	1 3 5 | 1 2 3 | 1 2 4 2 4 6 | 6 4 5 | 3 6 5
efficiency factors: $\frac{1}{2}$, $\frac{1}{2}$, $\frac{1}{2}$, $\frac{1}{2}$, 1	efficiency factors: $\frac{1}{3}$, $\frac{1}{2}$, $\frac{1}{2}$, $\frac{5}{6}$, $\frac{5}{6}$
$A = 0.5556,\ D = 0.5743,\ E = 0.5000$	$A = 0.5319,\ D = 0.5656,\ E = 0.3333$

useful to construct other resolvable designs satisfying conditions (1) and (2) for $r = 3$.

Four families of designs are constructed in Section 2. In Section 3, their efficiency factors are calculated. Two further families are presented in Section 4. In Section 5, the designs are compared with each other and with other designs from the literature: all the new designs are highly efficient, and three of the families compare particularly well with other known designs.

2 Four families of designs

Agrawal (1966a, 1966b) and Preece (1966a, 1966b) gave several examples of sets with three partitions satisfying conditions (1) and (2). Agrawal's designs were for rows, columns and treatments, while Preece's were for blocks and two non-interacting sets of treatments. Thus both authors fitted model (4).

When condition (1) is satisfied, then

$$X_{ij}X_{ji} = \alpha I + \beta J,$$

where $\alpha = k(q-k)/(q-1) = k^2(1-\varepsilon)$, $\beta = k(k-1)/(q-1) = k^2\varepsilon/q$ and $\varepsilon = q(k-1)/[(q-1)k]$, which is the efficiency factor for Δ_{ij}. Hence, consideration of the relationship algebra (James, 1957) generated by the three partitions on $\mathcal{T}$ shows that, whenever i, j and l are all different, the values ψ_{ijl} and ϕ_{ijl} in Equation (3) are constants ψ and ϕ satisfying

$$\psi + q\phi = 2k^3 :$$

see Bailey (1999). In what follows we assume that $\psi \neq 0$.

The calculations in Preece (1966a) show that, when conditions (1) and (2) hold, then all contrasts for each of the fixed effects in model (4) have efficiency factor $\mathcal{E}$, where

$$1 - \mathcal{E} = \frac{2(1-\varepsilon) - \psi/k^3}{\varepsilon}. \tag{5}$$

Bailey, Preece and Rowley (1995) showed that there are four infinite families of sets of three partitions satisfying conditions (1) and (2) for values of q which are congruent to 3 modulo 4 and which are prime powers. We

Table 2 Parameters of four families of sets of three partitions, where q is a prime power congruent to 3 modulo 4

Family	k	ψ	$\mathcal{E}$
I	$\frac{q-1}{2}$	$\frac{3q-1}{4}$	$\frac{q-3}{q-1}$
II	$\frac{q-1}{2}$	$-\frac{q+1}{4}$	$\frac{q^2-6q+1}{(q-1)(q-3)}$
III	$\frac{q+1}{2}$	$\frac{1-3q}{4}$	$\frac{(q-3)(q+3)}{(q+1)^2}$
IV	$\frac{q+1}{2}$	$\frac{q+1}{4}$	$\frac{q-1}{q+1}$

omit $q = 3$, which gives a rectangular lattice. The four families have the parameters shown in Table 2.

Let F be the finite field of order q, and let S be the set of non-zero squares in F, and N the set of non-squares in F. Then $|S| = |N| = (q-1)/2$ and $F = S \cup N \cup \{0\}$. Let L and M be the $F \times F$ matrices with $L(f, g) = 1$ if $g - f \in S$ and $M(f, g) = 1$ if $g - f \in N$ and other entries zero. Thus $L + M + I = J$, because -1 is not a square in F.

For Families I and II, put $\mathcal{T} = S \times F$. For h in F, define the partition Q_h of $\mathcal{T}$ whose classes are the sets

$$\{(s, f - hs) : s \in S\}$$

indexed by f in F. Thus $k = (q-1)/2$. Put $P_1 = Q_0$ and $P_2 = Q_1$. Then $X_{12} = L$. Put $P_3 = Q_h$, where h is a fixed element of F different from 0 and 1. If $h \in N$ and $h - 1 \in S$ then $X_{23} = X_{31} = L$ and the design comes from Family I. For all other values of h, the matrices X_{12}, X_{23} and X_{31} consist of two Ls and one M or two Ms and one L, giving designs in Family II. Examples of designs from the two families are shown in Tables 3 (for $q = 7$) and 4 (for $q = 11$). The good design identified by Heiligers and Sinha (1995) belongs to Family I.

For Families III and IV, replace S by $S \cup \{0\}$, so that $k = (q+1)/2$, and proceed as above. In Family III we have $X_{12} = X_{23} = X_{31} = I + L$ while in Family IV the three incidence matrices X_{12}, X_{23} and X_{31} consist of two copies of $I + L$ and one of $I + M$ or vice versa. Examples are shown in Tables 3 and 4.

For a design Δ in any of the four families, the permutation π_g of $\mathcal{T}$ is an automorphism of the design, where g is a fixed element of F and

$$\pi_g \colon (s, f) \mapsto (s, f + g) \qquad \text{for } s \text{ in } S \text{ and } f \text{ in } F.$$

Table 3 Designs for $q = 7$ and $k = 3$ or $k = 4$

Treatment array (columns are classes of Q_0)

	0	1	2	3	4	5	6
1	a	b	c	d	e	f	g
2	h	i	j	k	l	m	n
4	o	p	q	r	s	t	u
0	v	w	x	y	z	A	Z

Q_1 (classes are columns)

g	a	b	c	d	e	f
m	n	h	i	j	k	l
r	s	t	u	o	p	q
v	w	x	y	z	A	Z

Q_2 (classes are columns)

f	g	a	b	c	d	e
k	l	m	n	h	i	j
u	o	p	q	r	s	t
v	w	x	y	z	A	Z

Q_3 (classes are columns)

e	f	g	a	b	c	d
i	j	k	l	m	n	h
q	r	s	t	u	o	p
v	w	x	y	z	A	Z

Family	I	II	III	IV
Treatments	First three rows of treatment array		All four rows of treatment array	
k	3	3	4	4
P_1	Q_0	Q_0	Q_0	Q_0
P_2	Q_1	Q_1	Q_1	Q_1
P_3	Q_3	Q_2	Q_3	Q_2
multiplicity of efficiency factor 1	2	2	9	9
other efficiency factors with multiplicity 6	0.7778	0.4154	0.5833	0.8552
	0.8657	0.6289	0.8993	0.6950
	0.3565	0.9557	0.5174	0.4499

Thus $\{\pi_g : g \in F\}$ is a semi-regular group of automorphisms of Δ, abstractly isomorphic to the additive group of F. If q is prime then this group is cyclic and so Δ belongs to the class of α-designs introduced by Patterson and Williams (1976b).

3 Efficiency factors

The design Δ has $3q$ blocks. Since $3q \leq kq = v$, there are at least $q(k-3)$ efficiency factors equal to 1, and the remaining efficiency factors are those of the dual design Δ' for $3q$ treatments in v blocks of size 3.

Order the $3q$ blocks by replicate and, within replicate, by the same ordering on the index set F. Then the concurrence matrix C for Δ' is equal

Table 4 Designs for $q = 11$ and $k = 5$ or $k = 6$

Treatment array (columns are classes of Q_0)

	0	1	2	3	4	5	6	7	8	9	10
1	a	b	c	d	e	f	g	h	i	j	k
3	l	m	n	o	p	q	r	s	t	u	v
4	A	B	C	D	E	F	G	H	I	J	K
5	L	M	N	O	P	Q	R	S	T	U	V
9	α	β	γ	δ	ϵ	ζ	η	θ	ι	κ	λ
0	μ	ν	ξ	π	ρ	σ	τ	υ	ϕ	χ	ψ

Q_1 (classes are columns)

k	a	b	c	d	e	f	g	h	i	j
t	u	v	l	m	n	o	p	q	r	s
H	I	J	K	A	B	C	D	E	F	G
R	S	T	U	V	L	M	N	O	P	Q
γ	δ	ϵ	ζ	η	θ	ι	κ	λ	α	β
μ	ν	ξ	π	ρ	σ	τ	υ	ϕ	χ	ψ

Q_2 (classes are columns)

j	k	a	b	c	d	e	f	g	h	i
q	r	s	t	u	v	l	m	n	o	p
D	E	F	G	H	I	J	K	A	B	C
M	N	O	P	Q	R	S	T	U	V	L
ϵ	ζ	η	θ	ι	κ	λ	α	β	γ	δ
μ	ν	ξ	π	ρ	σ	τ	υ	ϕ	χ	ψ

Q_3 (classes are columns)

i	j	k	a	b	c	d	e	f	g	h
n	o	p	q	r	s	t	u	v	l	m
K	A	B	C	D	E	F	G	H	I	J
S	T	U	V	L	M	N	O	P	Q	R
η	θ	ι	κ	λ	α	β	γ	δ	ϵ	ζ
μ	ν	ξ	π	ρ	σ	τ	υ	ϕ	χ	ψ

Family	I	II	III	IV
Treatments	First three rows of treatment array		All four rows of treatment array	
k	5	5	6	6
P_1	Q_0	Q_0	Q_0	Q_0
P_2	Q_1	Q_1	Q_1	Q_1
P_3	Q_2	Q_3	Q_2	Q_3
multiplicity of efficiency factor 1	24	24	35	35
other efficiency factors with multiplicity 10	0.7333	0.4789	0.6111	0.8232
	0.8248	0.6642	0.8540	0.6854
	0.4418	0.8770	0.5349	0.4914

to

$$\begin{bmatrix} kI & X_{12} & X_{13} \\ X_{21} & kI & X_{23} \\ X_{31} & X_{32} & kI \end{bmatrix}.$$

It is clear that the efficiency factors for the two degrees of freedom between replicates are both 1, so we need to find the eigenvalues of C on the space W consisting of vectors of the form $[x^\top \mid y^\top \mid z^\top]^\top$ where x, y and z are contrast vectors in $\mathbb{R}^F$; here $x^\top$ denotes the transpose of x. For calculating on W, we may treat J as zero in any of the nine $q \times q$ submatrices, so

$$(C-kI)^2 = \begin{bmatrix} 2\alpha I + 2\beta J & X_{13}X_{32} & X_{12}X_{23} \\ X_{23}X_{31} & 2\alpha I + 2\beta J & X_{21}X_{13} \\ X_{32}X_{21} & X_{31}X_{12} & 2\alpha I + 2\beta J \end{bmatrix}$$
$$= \begin{bmatrix} 2\alpha I & X_{13}X_{32} & X_{12}X_{23} \\ X_{23}X_{31} & 2\alpha I & X_{21}X_{13} \\ X_{32}X_{21} & X_{31}X_{12} & 2\alpha I \end{bmatrix}$$

and, therefore,

$$(C-kI)^3 = \begin{bmatrix} \psi I + \phi J & 3\alpha X_{12} + \beta k J & 3\alpha X_{13} + \beta k J \\ 3\alpha X_{21} + \beta k J & \psi I + \phi J & 3\alpha X_{23} + \beta k J \\ 3\alpha X_{31} + \beta k J & 3\alpha X_{32} + \beta k J & \psi I + \phi J \end{bmatrix}$$
$$= 3\alpha(C-kI) + \psi I.$$

Hence the eigenvalues of $C - kI$ on W are the zeros x_1, x_2, x_3 of the polynomial

$$x^3 - 3\alpha x - \psi. \tag{6}$$

Thus $x_1 + x_2 + x_3 = 0$.

The eigenvalues of C are $3k$, 0 (with multiplicity 2) and $x_i + k$ with multiplicity n_i say, where $n_1 + n_2 + n_3 = 3(q-1)$. Now $3qk = \operatorname{trace} C = 3k + n_1x_1 + n_2x_2 + n_3x_3 + 3(q-1)k$, so $n_1x_1 + n_2x_2 + n_3x_3 = 0$. If $n_1 = n_2$ then $(n_1 - n_3)x_3 = 0$; since x_3 cannot be zero if $\psi \neq 0$, we have $n_1 = n_2 = n_3$. If n_1, n_2 and n_3 are all different then all the zeros of the polynomial (6) must be integers; as this does not happen for any of the designs discussed in this paper, we can now assume that $n_1 = n_2 = n_3 = q - 1$.

Each eigenvalue $x + k$ of C, other than $3k$, gives a canonical efficiency factor $1 - (x+k)/(3k) = z/(3k)$ of Δ'. Hence the canonical efficiency factors of Δ are 1, with multiplicity $q(k-3)+2$, and $z_1/(3k)$, $z_2/(3k)$, $z_3/(3k)$, where the z_i are the zeros of

$$z^3 - 6kz^2 + 3(4k^2 - \alpha)z + (\psi + 6\alpha k - 8k^3): \tag{7}$$

these last three each have multiplicity $q - 1$.

Now, $z_1z_2z_3 = 8k^3 - 6\alpha k - \psi$, so the following relation holds for the geometric mean D of the canonical efficiency factors:

$$D^{qk-1} = \left(\frac{z_1z_2z_3}{27k^3}\right)^{q-1} = \left(\frac{8k^3 - 6\alpha k - \psi}{27k^3}\right)^{q-1}. \tag{8}$$

Similarly, the following relation holds for the harmonic mean A of the canonical efficiency factors:

$$(qk-1)A^{-1} = q(k-3) + 2 + 3k(q-1)\left(\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right)$$
$$= q(k-3) + 2 + \frac{9k(q-1)(4k^2 - \alpha)}{8k^3 - 6\alpha k - \psi}. \tag{9}$$

In Family I, polynomial (6) becomes

$$x^3 - \frac{3(q+1)}{4}x - \frac{3q-1}{4} = (x+1)\left(x^2 - x - \frac{(3q-1)}{4}\right),$$

whose zeros are -1 and $(1 \pm \sqrt{3q})/2$. Hence the canonical efficiency factors are

$$\frac{2q}{3(q-1)} \quad \text{and} \quad \frac{2q-3 \pm \sqrt{3q}}{3(q-1)},$$

each with multiplicity $q-1$, and 1, with multiplicity $q(q-7)/2+2$.

In Family II, polynomial (6) becomes

$$x^3 - \frac{3(q+1)}{4}x + \frac{q+1}{4}.$$

Put $y = x/\sqrt{q+1}$. If x is a zero of the polynomial then

$$4y^3 - 3y = -\frac{1}{\sqrt{q+1}}$$

and so $y = \cos\theta$ where $\cos 3\theta = -1/\sqrt{q+1}$. Hence the canonical efficiency factors are

$$\frac{2(q-1) - 2\sqrt{q+1}\cos\theta}{3(q-1)},$$

where $\cos 3\theta = -1/\sqrt{q+1}$, each with multiplicity $q-1$, and 1, with multiplicity $q(q-7)/2+2$.

In Family III and Family IV, each matrix X_{ij} is obtained by subtracting the corresponding matrix for Family I or II, respectively, from J. Therefore, the values x in polynomial (6) are the negatives of those for Families I and II. Therefore, the canonical efficiency factors for Family III are

$$\frac{2q}{3(q+1)} \quad \text{and} \quad \frac{2q+3 \pm \sqrt{3q}}{3(q+1)},$$

each with multiplicity $q-1$, and 1, with multiplicity $q(q-5)/2+2$. Similarly, those for Family IV are

$$\frac{2(q+1) + 2\sqrt{q+1}\cos\theta}{3(q+1)},$$

where $\cos 3\theta = -1/\sqrt{q+1}$, each with multiplicity $q-1$, and 1, with multiplicity $q(q-5)/2+2$.

4 Two further families of designs

Cameron (1972) and Cameron and Seidel (1973) studied systems of linked square designs for q treatments in blocks of size k. Although these systems are not all realisable as partitions of a set of size qk, their incidence matrices satisfy condition (1) and the following strong form of condition (3):

$$X_{ij}X_{jl} = \gamma X_{il} + \delta J. \tag{10}$$

Bailey (1999) showed that $\gamma k + \delta q = k^2$, $\gamma^2 = \alpha$ and $\delta = k(k-\gamma)/q$. Condition (10) implies that

$$X_{ij}X_{jl}X_{li} = \gamma X_{il}X_{li} + \delta J X_{li} = \gamma(\alpha I + \beta J) + \delta k J,$$

so $\psi = 2\alpha\gamma$ and $\phi = 2\gamma\beta + 2\delta k$.

Rectangular lattice designs satisfy condition (10) with $k = q-1$, $\gamma = -1$ and $\delta = q-1$. Cameron (2003) gave two series of sets of partitions satisfying conditions (1) and (10). They have $q = 4^m$ and $k = 2^{2m-1} \pm 2^{m-1}$, for positive integers m. For both series, $\alpha = 2^{2m-2}$ and so $\gamma = \pm 2^{m-1}$. But q divides $k(k-\gamma)$, so γ must be negative when $k = 2^{2m-1} - 2^{m-1}$ and positive when $k = 2^{2m-1} + 2^{m-1}$.

These families are summarized in Table 5. The two smallest designs in Families V and VI can be derived from the tables in Preece and Cameron (1975).

Table 5 Parameters of three families of sets of three partitions satisfying condition (10): here RL denotes rectangular lattice

Family	q	k	γ	ψ	$\mathcal{E}$
RL	any	$q-1$	-1	-2	$\dfrac{q(q-3)}{(q-1)(q-2)}$
V	4^m	$2^{2m-1} - 2^{m-1}$	-2^{m-1}	-2^{3m-2}	$\dfrac{2^{m-1}(2^m-3)}{(2^m-1)(2^{m-1}-1)}$
VI	4^m	$2^{2m-1} + 2^{m-1}$	$+2^{m-1}$	$+2^{3m-2}$	$\dfrac{2^{m-1}(2^m+3)}{(2^m+1)(2^{m-1}+1)}$

When condition (10) is satsified,

$$(C - kI)^2 = 2\alpha I + \gamma(C - kI)$$

on the space W, so the eigenvalues of $C - kI$ on W are the zeros of the polynomial $x^2 - \gamma x - 2\alpha$, which is $(x+\gamma)(x-2\gamma)$. Let n_1 and n_2 be the multiplicities of $-\gamma$ and 2γ as eigenvalues of $C - kI$. Then $0 = -n_1\gamma + 2n_2\gamma$

and $n_1 + n_2 = 3(q-1)$ and so $n_1 = 2(q-1)$ and $n_2 = q-1$. The canonical efficiency factors are equal to

$$\frac{2k+\gamma}{3k} \quad \text{and} \quad \frac{2(k-\gamma)}{3k},$$

with multiplicities $2(q-1)$ and $q-1$ respectively, and 1, with multiplicity $q(k-3)+2$. Hence

$$D^{qk-1} = \left(\frac{2(2k+\gamma)^2(k-\gamma)}{27k^3}\right)^{q-1}$$

and

$$(qk-1)A^{-1} = q(k-3) + 2 + 3k(q-1)\left(\frac{2}{2k+\gamma} + \frac{1}{2(k-\gamma)}\right).$$

Also, $E = 2(k-\gamma)/(3k)$ if γ is positive but $E = (2k+\gamma)/(3k)$ if γ is negative.

Thus, for Family V, the non-unit canonical efficiency factors are

$$\frac{2^{m+1}-3}{3(2^m-1)} \quad \text{and} \quad \frac{2^{m+1}}{3(2^m-1)}$$

with multiplicities $2(2^{2m}-1)$ and $2^{2m}-1$, while, for Family VI, they are

$$\frac{2^{m+1}+3}{3(2^m+1)} \quad \text{and} \quad \frac{2^{m+1}}{3(2^m+1)}$$

with multiplicities $2(2^{2m}-1)$ and $2^{2m}-1$.

The two smallest designs are summarized in Table 6.

Table 6 Summary of designs for $q = 16$ and $k = 6$ or $k = 10$

Family	V	VI
k	6	10
multiplicity of efficiency factor 1	50	114
other efficiency factor with multiplicity 30	0.5555	0.7333
other efficiency factor with multiplicity 15	0.8888	0.5333

5 Discussion

Table 7 gives the parameters of the nine smallest designs in the families defined in this paper, together with their A-, D- and E-measures of overall efficiency.

Table 7 Comparison of designs

v	k	design	$\mathcal{E}$	ψ	A	D	E	triangles
21	3	I	0.6667	5	0.5975	0.6518	0.3565	252
		II	0.3333	−2	0.6199	0.6595	0.4154	126
		C32			0.5975	0.6518	0.3565	252
		PW			0 5967	0.6524	0.2814	210
28	4	III	0.6250	−5	0.7190	0.7484	0.5174	630
		IV	0.7500	2	0.7108	0.7459	0.4499	756
		PW			0.7152	0.7473	0.4240	672
		PWH			0.7190	0.7484	0.5174	630
55	5	I	0.8000	8	0.7459	0.7832	0.4418	2640
		II	0.7000	−3	0.7520	0.7850	0.4789	2310
		PW			0.7440	0.7830	0.3176	2508
		PWH			0.7530	0.7852	0.4550	2244
66	6	III	0.7778	−8	0.7949	0.8218	0.5349	4620
		IV	0.8333	3	0.7918	0.8209	0.4914	4950
		PW			0.7924	0.8211	0.4047	4752
		PWH			0.7942	0.8216	0.4922	4686
96	6	V	0.6667	−16	0.7859	0.8153	0.5555	5760
		PW			0.7712	0.8141	0.2467	6912

C32 design given by John, Wolock and David (1972)
PW designs given by Patterson and Williams (1976b)
PWH designs given by Patterson, Williams and Hunter (1978)

5.1 Pairs of designs

For each design in each of the six families, the value $\mathcal{E}$ is the value of A, D and E for estimating P_1, P_2 or P_3 in model (4). Table 7 shows two designs for the first four values of v and k. In every case, the design with the lower value of $\mathcal{E}$ has higher values of A, D and E for the three-replicate resolvable design. In particular, the optimal design found by Heiligers and Sinha (1995) is not optimal as a resolvable block design. At first sight, this seems surprising, because it is the opposite of what happens when $r = 2$. However, Equation (5) shows that $\mathcal{E}$ increases with ψ, while Equations (9), (8) and (7) show that A, D and E decrease as ψ increases.

This behaviour is also linked to Paterson's (1983) conjecture about triangles in the variety-concurrence graph $\mathcal{G}$ of Δ (Patterson and Williams, 1976a). Triangles in $\mathcal{G}$ are of two types. Some arise from sets of three treatments in the same block. There are $3qk(k-1)(k-2)$ such triangles. The remaining triangles have one edge contributed by each replicate. The number of such triangles is equal to $3\,\mathrm{trace}(X_{12}X_{23}X_{31} + X_{13}X_{32}X_{21}) - 6qk$. However, $\mathrm{trace}(X_{12}X_{23}X_{31} + X_{13}X_{32}X_{21}) = q(\psi + \phi) = (q-1)\psi + 2k^3$. Hence $\mathcal{E}$ and the number of triangles increase together.

Paterson's conjecture, which seems most likely to be true for designs where all concurrences are 0 or 1, is that optimal designs are to be found among those whose graphs have fewest triangles. Although none of A, D, E is monotonically related to the number of triangles across the whole range of designs, the relation does seem to be monotonic at the top end. Thus the apparent paradox is consistent with Paterson's conjecture.

5.2 Comparison with other designs

How do these designs compare with other designs in the literature? Clatworthy (1973) gives no designs with these parameters, resolvable or not. John, Wolock and David (1972) give a cyclic design C32 for 21 treatments in 21 blocks of size 3. It has $A = 0.5975$ and can be shown to be isomorphic to the poorer of the two new designs for these parameters in Table 7; certainly the other of those two designs is an improvement on C32.

As shown in Section 2, when q is prime the designs in this paper are α-designs, so it is useful to compare them with good α-designs given elsewhere. In the notation of Patterson, Williams and Hunter (1978), an α-design is defined by an $r \times k$ array α whose entries are integers modulo q. Let B be the permutation matrix of the q-cycle $(1\,2\,\ldots\,q)$. If Δ is the design defined by α then X_{ij} is a sum of powers of B, the indices being $\alpha_{ip} - \alpha_{jp}$ for $p = 1$, $\ldots$, k. Hence the eigenvalues of C are the eigenvalues of the $r \times r$ matrices obtained by replacing B in C by the complex q-th roots of unity.

For $q = 7$ and $k = 3$, Patterson and Williams (1976b) give

$$\alpha = \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 6 & 5 \end{array}.$$

Hence $X_{12} = I + B^{-1} + B^{-2}$, $X_{13} = I + B + B^2$ and $X_{23} = I + B^2 + B^4$. Table 7 shows the values of A, D and E. This design is worse than both the Family I design and the Family II design.

Patterson and Williams (1976b) also give designs for the other four pairs of values of v and k in Table 7. Their efficiency factors are summarized in Table 7. For $q = 7$ and $k = 4$, and for $q = 11$ and $k = 6$, their design is E-worse than both the Family III design and the Family IV design, but intermediate with regard to both A and D. Their design with $q = 11$ and $k = 5$ is worse in all respects than the designs from Families I and II.

Patterson, Williams and Hunter (1978) give a list of recommended α-designs for $k \geq 4$. Their design for $q = 7$ and $k = 4$ has α-array

$$\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 4 \\ 0 & 3 & 6 & 5 \end{array},$$

which has the property that

$$\begin{aligned}\{\alpha_{2p} - \alpha_{1p} : p = 1,\ \ldots,\ 4\} &= \{\alpha_{3p} - \alpha_{2p} : p = 1,\ \ldots,\ 4\}\\ &= \{\alpha_{1p} - \alpha_{3p} : p = 1,\ \ldots,\ 4\}\\ &= \{0, 1, 2, 4\},\end{aligned}$$

which is the set of squares modulo 7. Hence this design is isomorphic to the Family III design.

Their designs for $q = 11$, $k = 5$ and $q = 11$, $k = 6$ are summarized in Table 7. The design with $k = 5$ is slightly better than the Family II design with respect to both A and D, but E-worse. The design with $k = 6$ is slightly worse than the Family III design with respect to both A and D, and considerably E-worse.

Table 7 also compares two designs with $q = 16$, $k = 6$. The one from Family V has no triangles of the second type. It is superior to the design given by Patterson and Williams (1976b).

Overall, it appears that the designs from Families II, III and V are highly efficient, even though the examples with $q = 11$, $k = 5$ show that they may not always be optimal.

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