Balanced colourings of strongly regular graphs ⋆

R. A. Bailey

School of Mathematical Sciences,
Queen Mary, University of London,
Mile End Road, London E1 4NS, UK

Abstract

A colouring of a strongly regular graph is an allocation of colours (or treatments) to the vertices of the graph. Such a colouring is balanced if every pair of distinct colours occurs equally often on the ends of an edge. When the graph is the complete regular multipartite graph a balanced colouring is just a balanced incomplete-block design, or 2-design.

Some constructions are given. For example, colourings of the triangular graph give balanced designs for experiments where the experimental units are unordered pairs of people. An analogue of Fisher's inequality is proved.

In this context, strongly regular graphs may be generalized to arbitrary association schemes. When the association scheme is a collection of circular blocks then a colouring is balanced if the design in blocks is a 2-design and there is undirectional neighbour balance at all distances around the circle.

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1 Introduction

To a statistician designing an experiment, the initial ingredients of a block design are two sets: a set Ω of n plots, partitioned into b blocks of size k; and a set Θ of v treatments. The design is a function f from Ω to Θ; it tells the

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experimenter to put treatment $f(\omega)$ on plot ω. The design is called *binary* if, for each θ in Θ, the inverse image $f^{-1}(\theta)$ contains at most one plot from each block. It is *equireplicate* if there is an integer, usually called r, such that $|f^{-1}(\theta)| = r$ for all θ in Θ. Whether or not it is binary, it is called *balanced* if it is equireplicate and there is an integer, usually called λ, such that, whenever θ and ψ are distinct treatments, then there are exactly λ ordered pairs of plots (α, β) such that α and β are in the same block, $f(\alpha) = \theta$ and $f(\beta) = \psi$. A special case of a balanced binary design is the *complete*-block design, for which $v = k$ and $\lambda = r$. For $k < v$, balanced binary block designs are called balanced incomplete-block designs (BIBDs).

The purpose of this paper is to generalize the structure on Ω while retaining the idea of balance. The partition of Ω into b blocks of size k defines a strongly regular graph on Ω: the vertices are the plots, and two plots are joined by an edge if they are in different blocks. To a graph-theorist, it is more natural to think of the elements of Θ as colours and the function f as a colouring of the vertices of the graph. I shall define balanced colourings on arbitrary strongly regular graphs.

However, strongly regular graphs can themselves be generalized to association schemes, where the definitions and proofs do not become harder. This is the setting for the rest of this paper: assume that there is an association scheme on Ω, define what it means for a function on Ω to be balanced, prove some results about balanced colourings (such as a generalization of Fisher's inequality), and present some classes of balanced colouring.

2 Parameters of balanced colourings

Let $\mathcal{Q}$ be an association scheme on set Ω of size n whose associate classes are subsets $\mathcal{C}_i$ of $\Omega \times \Omega$ for i in an index set $\mathcal{K}$, where $|\mathcal{K}| = s + 1$. For i in $\mathcal{K}$, let A_i be the adjacency matrix of $\mathcal{C}_i$, that is, the $\{0, 1\}$-matrix whose (α, β)-entry is equal to 1 if and only if $(\alpha, \beta) \in \mathcal{C}_i$, and let a_i be its valency. There are constants p_{ij}^k such that

$$A_i A_j = \sum_{k \in \mathcal{K}} p_{ij}^k A_k.$$

Let Θ be a set of v objects, called *colours* or *treatments*. A function $f\colon \Omega \to \Theta$ is called a *colouring* or a *design* respectively. Such a function is defined to be *balanced* if there are integers μ_i, λ_i for i in $\mathcal{K}$ such that

(1) if $\theta \in \Theta$, then there are μ_i ordered pairs (α, β) in $\mathcal{C}_i$ with $f(\alpha) = f(\beta) = \theta$;

(2) if θ and ψ are distinct elements of Θ, then there are λ_i ordered pairs (α, β) in $\mathcal{C}_i$ with $f(\alpha) = \theta$ and $f(\beta) = \psi$.

By convention the numbering is chosen so that $\mathcal{C}_0 = \mathrm{Diag}(\Omega) = \{(\omega, \omega) : \omega \in \Omega\}$. Hence $\lambda_0 = 0$. It is normal to write μ_0 as r. Thus $|f^{-1}(\theta)| = r$ for all θ in Θ and so balanced colourings are equireplicate with

$$vr = n. \tag{1}$$

Fixing θ and counting pairs (α, β) in $\mathcal{C}_i$ for which $f(\alpha) = \theta$ gives

$$ra_i = \mu_i + (v-1)\lambda_i \qquad \text{for } i \text{ in } \mathcal{K}. \tag{2}$$

Counting the number of occurrences of each pair of colours gives

$$\sum_{i \in \mathcal{K}} \mu_i = \sum_{i \in \mathcal{K}} \lambda_i = r^2. \tag{3}$$

Summing Equation (2) gives

$$r \sum_i a_i = \sum_i \mu_i + (v-1) \sum_i \lambda_i,$$

so $rn = vr^2$, which is consistent with Equation (1).

Example 1 Let $\mathcal{Q}$ be the Petersen graph and $v = 5$. Figure 1 shows a balanced colouring with the following parameters:

$$\begin{array}{llll} n = 10 & a_0 = 1 & a_1 = 3 & a_2 = 6 \\ v = \ \ 5 & r \ = 2 & \mu_1 = 2 & \mu_2 = 0 \\ & & \lambda_1 = 1 & \lambda_2 = 3. \end{array}$$

Example 2 Let $\mathcal{Q}$ be the group-divisible scheme with seven blocks of size four; in other words, $\mathcal{Q}$ is the complement of the complete multipartite graph of valency 24 on 28 vertices. A balanced colouring with $v = 7$ is shown in Figure 2. The parameters are as follows.

$$\begin{array}{llll} n = 28 & a_0 = 1 & a_1 = 3 & a_2 = 24 \\ v = \ \ 7 & r \ = 4 & \mu_1 = 0 & \mu_2 = 12 \\ & & \lambda_1 = 2 & \lambda_2 = 14 \end{array}$$

Fig. 1. Balanced colouring of the Petersen graph with five colours

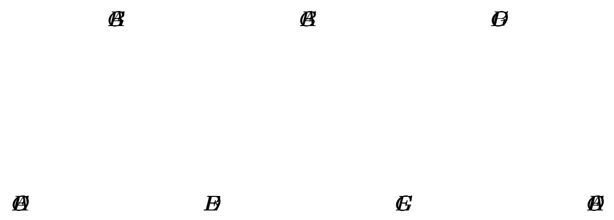

Fig. 2. Balanced colouring of seven blocks of size four

This shows that the balanced incomplete-block design is a special case of a balanced colouring: in the usual notation, $a_1 = k-1$, $\mu_1 = 0$ and $\lambda_1 = \lambda$.

Example 3 Balanced block designs with μ_1 non-zero are also balanced colourings. Here is one with

$$\begin{array}{llll} n = 24 & a_0 = 1 & a_1 = 3 & a_2 = 20 \\ v = \ \ 6 & r \ \ = 4 & \mu_1 = 2 & \mu_2 = 10 \\ & & \lambda_1 = 2 & \lambda_2 = 14. \end{array}$$

All the above examples have $s = 2$, so they are colourings of strongly regular graphs. Here is an example with $s = 3$.

Example 4 Let $\mathcal{Q}$ be the 6-circuit 6, in which $(\alpha, \beta) \in \mathcal{C}_i$ if the distance from α to β is i. Figure 3 shows a balanced design with the following parameters:

$$
\begin{array}{lllll}
n = 6 & a_0 = 1 & a_1 = 2 & a_2 = 2 & a_3 = 1 \\
v = 3 & r = 2 & \mu_1 = 2 & \mu_2 = 0 & \mu_3 = 0 \\
 & & \lambda_1 = 1 & \lambda_2 = 2 & \lambda_3 = 1.
\end{array}
$$

Fig. 3. Balanced colouring of the 6-circuit with three colours

Example 5 In the general Johnson scheme $\mathrm{J}(m,n)$, with $n < m$, Ω consists of the n-subsets of an m-set Γ, and $(\alpha,\beta) \in \mathcal{C}_i$ if $|\alpha \setminus \beta| = i$. Thus the Johnson scheme $\mathrm{J}(6,3)$ also has $s = 3$. Suppose that f allocates five colours to Ω in such a way that $f(\alpha) = f(\Gamma \setminus \alpha)$ for all α in Ω. If $f(\alpha) \neq f(\beta)$, then the distances of β from the four points coloured by $f(\alpha)$ are 1, 1, 2 and 2. Hence we have a balanced colouring with

$$
\begin{array}{lllll}
n = 20 & a_0 = 1 & a_1 = 9 & a_2 = 9 & a_3 = 1 \\
v = 5 & r = 4 & \mu_1 = 4 & \mu_2 = 4 & \mu_3 = 4 \\
 & & \lambda_1 = 8 & \lambda_2 = 8 & \lambda_3 = 0.
\end{array}
$$

One such colouring is:

$$
\begin{array}{ll}
A & \{1,2,3\}\ \{4,5,6\}\ \{1,4,5\}\ \{2,3,6\} \\
B & \{1,2,4\}\ \{3,5,6\}\ \{1,4,6\}\ \{2,3,5\} \\
C & \{2,4,5\}\ \{1,3,6\}\ \{1,3,4\}\ \{2,5,6\} \\
D & \{1,2,5\}\ \{3,4,6\}\ \{1,2,6\}\ \{3,4,5\} \\
E & \{1,3,5\}\ \{2,4,6\}\ \{1,5,6\}\ \{2,3,4\}.
\end{array}
$$

If $s = 1$, then we have the trivial association scheme $\underline{\underline{n}}$ on Ω, in which $\mathcal{C}_1 = (\Omega \times \Omega) \setminus \mathrm{Diag}(\Omega)$. Then every equireplicate function from Ω to Θ is a balanced colouring with $\mu_0 = r$, $\mu_1 = r(r-1)$ and $\lambda_1 = r^2$. In particular, if $v = n$, then every bijection from Ω to Θ is a balanced colouring with $r = 1$.

More generally, if $r = 1$, then neither μ_i nor λ_i can be non-zero for more than one value of i, so balance implies that $s + 1 \leq 2$, that is, that the scheme is trivial. From now on we assume that the scheme $\mathcal{Q}$ is nontrivial. It follows from Equation (1) that there can be no balanced colouring on $\mathcal{Q}$ if n is prime.

3 General results

If f is a colouring from Ω to Θ, let X_f be its incidence matrix; that is, X_f is the $\Omega \times \Theta$ matrix with $X_f(\omega, \theta) = 1$ if $f(\omega) = \theta$ and $X_f(\omega, \theta) = 0$ otherwise. Let J_Θ be the $\Theta \times \Theta$ matrix all of whose entries are 1, and let I_Θ be the identity matrix on Θ. Then the condition for f to be balanced can be written as

$$X_f' A_i X_f = \mu_i I_\Theta + \lambda_i (J_\Theta - I_\Theta) \qquad \text{for } i \text{ in } \mathcal{K}. \tag{4}$$

Lemma 1 *Suppose that $\mathcal{Q}$ is an association scheme on the set Ω and that $f\colon \Omega \to \Theta$ is a balanced colouring with $|\Theta| = v$. Let Ψ be a set of size m, where m divides v, and let g be a function from Θ to Ψ such that $|g^{-1}(\psi)| = v/m$ for all ψ in Ψ. Then the composite function $gf\colon \Omega \to \Psi$ is a balanced colouring of $\mathcal{Q}$.*

PROOF. Put $t = v/m$. Since g is equireplicate, $X_g' I_\Theta X_g = t I_\Psi$ and $X_g' J_\Theta X_g = t^2 J_\Psi$; in other words, g is a balanced colouring of the trivial association scheme $\underline{\underline{v}}$ on Θ. Therefore, for i in $\mathcal{K}$,

$$\begin{aligned}
X_{gf}' A_i X_{gf} &= X_g' X_f' A_i X_f X_g \\
&= X_g'((\mu_i - \lambda_i) I_\Theta + \lambda_i J_\Theta) X_g \\
&= t(\mu_i - \lambda_i) I_\Psi + t^2 \lambda_i J_\Psi \\
&= t(\mu_i + \lambda_i(t-1)) I_\Psi + t^2 \lambda_i (J_\Psi - I_\Psi),
\end{aligned}$$

and so gf satisfies Condition (4). $\square$

Given a colouring $f\colon \Omega \to \Theta$ and two distinct colours θ and ϕ, define the vector $u_{\theta\phi}$ in $\mathbb{R}^\Omega$ by

$$u_{\theta\phi}(\omega) = \begin{cases} +1 & \text{if } f(\omega) = \theta; \\ -1 & \text{if } f(\omega) = \phi; \\ 0 & \text{otherwise.} \end{cases}$$

Let U be the subspace of $\mathbb{R}^\Omega$ spanned by the vectors $u_{\theta\phi}$ for θ, ϕ in Θ. Then U has dimension $v-1$; it is called the subspace of *colour contrasts*. The matrix P of orthogonal projection onto U is given by

$$P = r^{-1}X_f X_f' - n^{-1}J_\Omega. \tag{5}$$

The set $\mathcal{A}$ of all real linear combinations of the adjacency matrices of $\mathcal{Q}$ is a commutative algebra called the Bose–Mesner algebra of $\mathcal{Q}$. It has dimension $s+1$. Its elements are symmetric matrices, so they have $s+1$ mutual eigenspaces, whose direct sum is $\mathbb{R}^\Omega$ and whose orthogonal projectors form a basis for $\mathcal{A}$: see [8, Chapter 17]. Statisticians call these eigenspaces *strata*.

Let W be a stratum with eigenprojector S. The colours are said to be *orthogonal* to W if $Su_{\theta\phi} = 0$ for all pairs of distinct colours θ and ϕ; in other words, if $SP = 0$.

In a block design, the strata W_0, W_1 and W_2 have dimensions 1, $b-1$ and $b(k-1)$ respectively. If the design is equireplicate, the colours are orthogonal to W_0; in a complete-block design they are also orthogonal to W_1; in a balanced incomplete-block design they are not orthogonal to W_1 but Fisher's Inequality [10] tells us that $v-1 \leq b-1$. This can be generalized to balanced colourings on arbitrary association schemes.

Theorem 2 *Let $\mathcal{Q}$ be an association scheme on a set Ω. Let $f\colon\Omega \to \Theta$ be a balanced colouring of $\mathcal{Q}$ with v colours. Let W be a stratum of $\mathcal{Q}$. Then either the colours are orthogonal to W or $v-1 \leq \dim W$.*

PROOF. Let S be the projector onto W. Since S is a linear combination of the adjacency matrices of $\mathcal{Q}$ and f is balanced, $X_f'SX_f$ is a linear combination of I_Θ and J_Θ: say $X_f'SX_f = \rho(I_\Theta - v^{-1}J_\Theta) + \sigma v^{-1}J_\Theta$.

If W is the 1-dimensional stratum spanned by the all-1 vector, then colours are orthogonal to W. Otherwise, $SJ_\Omega = 0$. But $J_\Omega X_f = rX_fJ_\Theta$, so then $X_f'SX_fJ_\Theta = 0$ and so $\sigma = 0$. Hence Equation (5) shows that $PSP = r^{-2}X_fX_f'SX_fX_f' = r^{-2}\rho X_f(I_\Theta - v^{-1}J_\Theta)X_f'$. If $\rho = 0$, then $(SP)'SP = PSP = 0$, and so $SP = 0$: hence colours are orthogonal to W. If $\rho \neq 0$, then $v-1 = \operatorname{rank}(X_f'SX_f) \leq \operatorname{rank} S = \dim W$. $\square$

An alternative way of describing a balanced block design is as a balanced incidence of a set Θ of v treatments in a set Δ of b blocks. The incidence is given by a $\Delta\times\Theta$ matrix N with non-negative integer entries: the entry $N(\delta,\theta)$ is referred to as the number of times that treatment θ occurs in block δ. The incidence is *proper* if all row-sums of N are the same; equireplicate if all column-sums are the same; balanced if it is equireplicate and $N'N$ is a linear combination of I_Θ and J_Θ. Therefore N is a balanced block design if and only if $N'I_\Delta N$ and $N'J_\Delta N$ are linear combinations of I_Θ and J_Θ.

It is useful to extend the idea of balanced colourings on association schemes to balanced incidences. Let $\mathcal{Q}$ be an association scheme on Ω with adjacency matrices A_i for i in $\mathcal{K}$. Let N be an $\Omega \times \Theta$ matrix with non-negative integer entries; that is, an incidence of colours (elements of Θ) in plots (elements of Ω). We shall say that N is *proper* if all row-sums of N are the same; *equireplicate* if all column-sums are the same; and *balanced* for $\mathcal{Q}$ if $N'A_iN$ is a linear combination of I_Θ and J_Θ for all i in $\mathcal{K}$. Thus a balanced colouring is a proper balanced incidence in which all row-sums are equal to 1.

It is also useful to generalize balance to partial balance, as in [6,11]. Suppose that, in addition to the association scheme $\mathcal{Q}$ on Ω, there is an association scheme $\mathcal{P}$ on Θ. The $\Omega\times\Theta$ incidence N is *partially balanced* with respect to $\mathcal{P}$ and $\mathcal{Q}$ if $N'A_iN$ is in the Bose–Mesner algebra of $\mathcal{P}$ for all i in $\mathcal{K}$. The proof of the following proposition is straightforward, and so is omitted.

Proposition 3 *Let Z be a $\Delta\times\Omega$ incidence and let X be an $\Omega\times\Theta$ incidence. Let $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$ be association schemes on Θ, Ω and Δ respectively. Then*

(1) if Z and X are both proper, then so is ZX;
(2) if Z and X are both equireplicate, then so is ZX;
(3) if Z is partially balanced with respect to $\mathcal{Q}$ and $\mathcal{R}$, and X is partially balanced with respect to $\mathcal{P}$ and $\mathcal{Q}$, then ZX is partially balanced with respect to $\mathcal{P}$ and $\mathcal{R}$.

Two special cases of part (3) of Proposition 3 are interesting. If X is balanced for $\mathcal{Q}$ then $\mathcal{P}$ is trivial and so ZX is balanced for $\mathcal{R}$. If Z is balanced for $\mathcal{R}$ then $\mathcal{Q}$ is trivial and we obtain Lemma 1.

4 Orthogonal block structures

If F is a partition of Ω into blocks of equal size, let R_F be the $\Omega\times\Omega$ matrix with $R(\alpha,\beta)=1$ if α and β are in the same block and $R(\alpha,\beta)=0$ otherwise. Thus a colouring $f\colon\Omega\to\Theta$ turns (Ω,F) into a balanced block design if there are integers r, μ and λ such that $X_f'I_\Omega X_f = rI_\Theta$ and $X_f'R_FX_f = \mu I_\Theta + \lambda(J_\Theta - I_\Theta)$.

A set $\mathcal{F}$ of partitions of Ω is called an *orthogonal block structure* if it contains the two trivial partitions and satisfies some other conditions which ensure that the set of all real linear combinations of the matrices R_F, for F in $\mathcal{F}$, is the Bose–Mesner algebra of an association scheme on Ω: see [4]. Condition (4) shows that a colouring of an orthogonal block structure is balanced if and only if it turns (Ω, F) into a balanced block design for every F in $\mathcal{F}$.

If $\mathcal{F}$ contains two non-trivial partitions, one finer than the other, then any colouring of Ω is called a *nested block design*. Balanced nested block designs were introduced by Preece [19]; a recent catalogue is in [16]. Suppose that there are b_1 large blocks and b_2 small blocks. The stratum dimensions are 1, $b_1 - 1$, $b_2 - b_1$ and $n - b_2$. Since $b_2 - b_1 > b_1 - 1$, Theorem 2 shows that in a nested balanced block design one of the following happens:

(1) $v \leq b_1$;
(2) every treatment occurs in each large block equally often, and $v \leq b_2 - b_1 + 1$;
(3) every treatment occurs in each small block equally often.

A special case of (2) is known as Bose's Inequality [5]; see [12, p. 227].

The other orthogonal block structure with two non-trivial partitions is the $m_1 \times m_2$ rectangle. Theorem 2 shows that, for a balanced colouring,

either colours are orthogonal to rows, which means that every colour occurs equally often in every row,
or $v \leq m_1$;

and similarly for columns. Orthogonality in both directions gives a design like a Latin square; orthogonality in one direction only gives a design like a Youden square. Balanced row-column designs include all generalized Youden designs, which were introduced by Kiefer [13].

Another orthogonal block structure for which balanced colourings have been investigated is the nested row-column structure, which consists of two or more rectangular blocks. Balanced nested row-column designs were introduced in [19]. Further constructions are given in [1,17,23].

In a designed experiment, the structure on the plots is usually an orthogonal block structure: see [18]. The remainder of this paper discusses colourings on other association schemes identified in [3] as potential structures on the plots in a designed experiment.

5 Triangular schemes

The triangular scheme $\mathrm{T}(m)$ is the Johnson scheme $\mathrm{J}(m, 2)$. It is appropriate for experiments where the experimental units are pairs of people undertaking a cooperative task, as is done in some experiments on computer vision, and also for experiments on hybrid plants where the gender of the parent is immaterial.

Here are four constructions of balanced colourings on $\mathrm{T}(m)$.

Construction 1 If m is even then a balanced colouring of $\mathrm{T}(m)$ with $m-1$ colours is given by a 1-factorization of the complete graph K_m on m vertices, which is equivalent to a symmetric idempotent Latin square of order m. These exist for all even m: see [7,9]. All the edges in each 1-factor receive the same colour.

Construction 2 If m is odd then a balanced colouring of $\mathrm{T}(m)$ with m colours is given by a near-1-factorization of K_m, which is equivalent to a symmetric unipotent Latin square of order m. These exist for all odd m: see [7,9]. This gives the colouring in Example 1, because the edges of the Petersen graph form one of the associate classes of $\mathrm{T}(5)$.

Construction 3 If m is even then there is a neighbour-balanced design for m treatments in $m/2$ linear blocks of size m: see [2,14]; this is equivalent to a decomposition of K_m into Hamiltonian paths. Giving all edges in each path the same colour creates a balanced colouring of $\mathrm{T}(m)$ with $m/2$ colours.

Construction 4 If m is odd there is a 2-factorization of K_m. This gives a balanced colouring of $\mathrm{T}(m)$ with $(m-1)/2$ colours.

Every colouring f of $\mathrm{T}(m)$ induces an incidence N of colours in the parent set Γ. For example, Construction 1 gives a colouring in which every colour is incident once with every member of the parent set, while Construction 3 gives a colouring in which each colour is incident once with two members of the parent set and twice with all the others.

Theorem 4 *Let $f\colon \Omega \to \Theta$ be a colouring, where Ω consists of the 2-subsets of an m-set Γ. Then f is balanced if and only if its induced incidence N of colours in Γ is a balanced block design.*

PROOF. Let Z be the $\Gamma \times \Omega$ incidence matrix; that is, $Z(\gamma, \omega) = 1$ if $\gamma \in \omega$ and $Z(\gamma, \omega) = 0$ otherwise. Then $N = ZX_f$. The incidence Z is balanced for the trivial scheme $\underline{\underline{m}}$ on Γ, so if f is a balanced colouring then Proposition 3 shows that N is also a balanced block design.

The adjacency matrices of $\mathrm{T}(m)$ are

$$A_0 = I_\Omega, \qquad A_1 = Z'Z - 2A_0 \qquad \text{and} \qquad A_2 = J_\Omega - A_1 - A_0.$$

Therefore $N'N = X_f'Z'ZX_f = X_f'A_1X_f + 2X_f'A_0X_f$. The replication of each colour in N is twice its replication in f.

Suppose that N is a balanced block design. Then $N'N$ is a linear combination of I_Θ and J_Θ. Moreover, N is equireplicate, so f is also equireplicate, which implies that $X_f'A_0X_f$ is a scalar multiple of I_Θ and $X_f'J_\Omega X_f$ is a scalar multiple of J_Θ. Therefore $X_f'A_iX_f$ is a linear combination of I_Θ and J_Θ for $i = 0$, 1 and 2 and so f is a balanced colouring. $\square$

Example 6 Figure 4 shows a balanced colouring of T(7) whose induced balanced block design is not binary.

	1	2	3	4	5	6	7
1		A	G	A	E	E	G
2			B	A	B	F	F
3				C	B	C	G
4					D	C	D
5						E	D
6							F
7							

Fig. 4. Balanced colouring of T(7) with seven colours

The balanced block design of colours induced in Γ has m blocks of size $m-1$ and replication $2r$. However, not every balanced block design with even replication and m blocks of size $m-1$ arises from a balanced colouring of $\mathrm{T}(m)$.

Example 7 Let $m = 9$ and $v = 4$. The following is a balanced block design for 4 treatments in 9 blocks of size 8 with replication 18.

Block 1 = Block 2 = Block 3 = $AABBCCDD$

Block 4 = $AAAABBBB$ Block 5 = $AAAACCCC$

Block 6 = $AAAADDDD$ Block 7 = $BBBBCCCC$

Block 8 = $BBBBDDDD$ Block 9 = $CCCCDDDD$

This cannot be realised as a colouring of T(9), because there is no colour (treatment) that can go on the plot $\{4, 9\}$.

Theorem 4, Fisher's Inequality and Equation (1) show that if there is a balanced colouring of $\mathrm{T}(m)$ with v colours then $v \leq m$ and v divides $m(m-1)/2$. For $m \leq 15$, Constructions 1–4 and Lemma 1 provide balanced colourings for all v satisfying these conditions except for $m = 9$ and $v = 6$. There is no balanced block design for 6 treatments in 9 blocks of size 8 so there is no balanced colouring of T(9) with 6 colours.

Figure 5 shows a balanced colouring of T(16) with 6 colours. As 6 does not divide either 15 or 16, this is not obtainable from Constructions 1–4 and Lemma 1.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1		B	F	F	A	A	B	C	D	E	E	F	B	D	A	C
2			C	F	F	E	A	B	C	D	D	B	F	A	C	E
3				D	F	D	E	A	B	C	C	D	A	F	E	B
4					E	C	D	E	A	B	B	A	C	E	F	D
5						B	C	D	E	A	A	C	E	B	D	F
6							D	F	F	C	A	C	B	A	E	D
7								E	F	F	B	C	B	A	E	D
8									A	F	C	B	A	E	D	C
9										B	D	D	C	B	A	E
10											E	A	E	D	C	B
11												B	A	E	D	C
12													F	D	A	F
13														F	C	E
14															F	B
15																F
16																

Fig. 5. Balanced colouring of T(16) with six colours

Half of Theorem 4 generalizes easily to Johnson schemes, using Proposition 3. Let Ω and Δ be the set of p-subsets and q-subsets respectively of an m-set Γ, where $m > p > q$. If $f\colon \Omega \to \Theta$ is a balanced colouring of $\mathrm{J}(m,p)$ and Z is the incidence of Δ in Ω then the incidence ZX_f is balanced for $\mathrm{J}(m,q)$.

Example 8 Take $m = 6$, $p = 3$ and $q = 2$. Let the adjacency matrices of $\mathrm{J}(6,3)$ be $A_0, \ldots, A_3$ as in Example 5, and let the adjacency matrices of $\mathrm{J}(6,2)$ be B_0, B_1 and B_2. Then

$$\begin{aligned} Z'B_0Z &= 3A_0 + A_1 \\ Z'B_1Z &= 6A_0 + 6A_1 + 4A_2 \\ Z'B_2Z &= \phantom{6A_0 + {}} 2A_1 + 5A_2 + 9A_3. \end{aligned}$$

For the colouring f given in Example 5, $X_f'A_0X_f = X_f'A_3X_f = 4I_\Theta$ and $X_f'A_1X_f = X_f'A_2X_f = 4I_\Theta + 8(J_\Theta - I_\Theta)$. Therefore $(ZX_f)'B_0(ZX_f) = 16I_\Theta + 8(J_\Theta - I_\Theta)$, $(ZX_f)'B_1(ZX_f) = 64I_\Theta + 80(J_\Theta - I_\Theta)$, and $(ZX_f)'B_2(ZX_f) = 64I_\Theta + 56(J_\Theta - I_\Theta)$.

6 Schemes of Latin-square type

Plots in an agricultural field trial are often laid out in a two-dimensional array. Typically the plots are not square and one direction is the direction of ploughing, so the $m_1 \times m_2$ rectangle is the appropriate association scheme. However, in some trials with hand-cultivation of plots there may be no intrinsic difference between the directions if $m_1 = m_2 = m$, and so the association scheme $\mathrm{L}(2, m)$ can be appropriate.

More generally, an association scheme of Latin-square type $\mathrm{L}(t, m)$ is constructed from $t - 2$ mutually orthogonal Latin squares of order m, where $2 \leq t \leq m - 1$. The set Ω consists of the m^2 cells of a square array. If $\alpha \neq \beta$ then $(\alpha, \beta) \in \mathcal{C}_1$ if α and β are in the same row or α and β are in the same column or α and β have the same letter in any of the Latin squares; otherwise $(\alpha, \beta) \in \mathcal{C}_2$. In particular, $\mathrm{L}(2, m)$ is formed from the $m \times m$ rectangle by merging the associate classes 'same row' and 'same column'.

Figure 6 shows a balanced colouring of $\mathrm{L}(2, 6)$ with 9 colours which is not balanced for the 6×6 rectangle considered as an orthogonal block structure: see also [15,20,21].

A	B	C	D	E	F
D	E	F	H	I	G
G	H	I	C	A	B
B	I	D	E	G	C
E	C	G	A	F	H
H	F	A	I	B	D

Fig. 6. Balanced colouring of $\mathrm{L}(2, 6)$

A	A	C
A	B	B
C	B	C

Fig. 7. Balanced colouring of $\mathrm{L}(2, 3)$

The scheme $\mathrm{L}(t, m)$ defines an incomplete-block design known as a square-lattice design. It has tm blocks of size m, and its treatments are the cells of the square array. Each row is a block, and so is each column. For each of the $t - 2$ Latin squares, all the cells with each letter form a block: see [12, p. 260] or [24].

Theorem 5 *A colouring of $L(t, m)$ is balanced if and only if its induced incidence in the blocks of the square-lattice design is a balanced block design.*

PROOF. This is similar to the proof of Theorem 4. $\square$

Any Latin square orthogonal to the original $t - 2$ squares, or among those squares, is a balanced colouring of $\mathrm{L}(t, m)$ with m colours, but there are many other possibilities, such as the colouring in Figure 7 for $t = 2$ and $m = 3$.

A related association scheme consists of several blocks, each of type $\mathrm{L}(t, m)$. Some balanced colourings for the case $t = 2$ are given by [22].

7 Circuits

It is tempting to conclude from Example 4 that the cyclic scheme $\textcircled{n}$ has a balanced colouring with v colours whenever v divides n. However, Theorem 2 imposes a severe restriction on colourings for such schemes and, indeed, for all inverse Abelian-group schemes; that is, those where Ω is an Abelian group and (α, β) is in the same class as (γ, δ) if and only if $\alpha^{-1}\beta \in \{\gamma^{-1}\delta, \delta^{-1}\gamma\}$.

Theorem 6 *If there is a balanced colouring of an inverse Abelian-group scheme with v colours, then $v \leq 3$.*

PROOF. In an inverse Abelian-group scheme every stratum either has dimension 1, being spanned by an irreducible character of the group taking only real values, or has dimension 2, being spanned by a pair of complex conjugate irreducible characters of the group. The space of colour contrasts cannot be orthogonal to every stratum, so Theorem 2 shows that $v - 1 \leq 2$. $\square$

Let G be an Abelian group of order n and let $\textcircled{G}$ be its corresponding association scheme. If v divides n and $v \in \{2, 3\}$, then there is a balanced colouring of $\textcircled{G}$ with v colours. The following constructions give two possibilities for $v = 3$; analogous constructions work for $v = 2$. Here $r = n/v$ and $l(\theta, \phi, g) = |\{(\alpha, \beta) : f(\alpha) = \theta,\ f(\beta) = \phi,\ \alpha^{-1}\beta = g\}| = |\{\alpha : f(\alpha) = \theta,\ f(\alpha g) = \phi\}|$. For $\theta \neq \phi$ we have $l(\theta, \phi, g) = l(\phi, \theta, g^{-1})$ and so $l(\theta, \phi, g) + l(\theta, \phi, g^{-1})$ is always equal to $l(\phi, \theta, g) + l(\phi, \theta, g^{-1})$. Hence it suffices to show that $l(A, A, g) = l(B, B, g)$ for $v = 2$; for $v = 3$ it suffices to check that $l(A, A, g) = l(B, B, g) = l(C, C, g)$ and that $l(A, B, g) = l(B, C, g) = l(C, A, g)$.

Construction 5 Let H be a subgroup of G of index 3. Assign the colours A, B, C to the cosets of H in G. If $h \in H$ then $l(A, A, h) = l(B, B, h) = l(C, C, h) = r$ and $l(\theta, \phi, h) = 0$ if $\theta \neq \phi$. If $g \notin H$ then g and g^2 are in different cosets of H. Suppose that $f(1) = A$ and $f(g) = B$. Then $(f(\alpha), f(\alpha g)) = (A, B)$ if and only if $\alpha \in H$. Similarly, $(f(\alpha), f(\alpha g)) = (B, C)$ if and only if $\alpha \in Hg$, while $(f(\alpha), f(\alpha g)) = (C, A)$ if and only if $\alpha \in g^2 H$. Hence $l(A, B, g) = l(B, C, g) = l(C, A, g) = r$ while $l(A, A, g) = l(B, B, g) = l(C, C, g) = 0$.

When $\textcircled{G} = \textcircled{9}$ we may take $H = \{0, 3, 6\}$ and obtain the colouring in Figure 8.

Fig. 8. Balanced colouring of 9 using Construction 5

Fig. 9. Balanced colouring of 9 using Construction 6

Construction 6 Let h be an element of G of order 3, let $H = \langle h \rangle$ and let L be a transversal for H in G. For x in L, put $f(x) = A$, $f(hx) = B$ and $f(h^2x) = C$. Then f is a colouring because L is a transversal. Now,

$$\begin{aligned} l(A,A,g) &= \left|\left\{(x,y) \in L \times L : x^{-1}y = g\right\}\right| \\ &= \left|\left\{(hx,hy) \in hL \times hL : x^{-1}y = g\right\}\right| = l(B,B,g) \\ &= \left|\left\{(h^2x,h^2y) \in h^2L \times h^2L : x^{-1}y = g\right\}\right| = l(C,C,g). \end{aligned}$$

Similarly,

$$\begin{aligned} l(A,B,g) &= \left|\left\{(x,hy) \in L \times hL : x^{-1}hy = g\right\}\right| \\ &= \left|\left\{(hx,h^2y) \in hL \times h^2L : x^{-1}hy = g\right\}\right| = l(B,C,g) \\ &= \left|\left\{(h^2x,y) \in h^2L \times L : x^{-1}hy = g\right\}\right| = l(C,A,g). \end{aligned}$$

When G = 9 we may take $h = 3$ and $L = \{0, 4, 5\}$ to obtain the colouring in Figure 9.

A designed experiment may have circular structure in time (days of the week) or space (plots equi-distant from a fixed feature). It is more common for the structure to consist of several blocks, each with circular structure. Then a balanced colouring needs a balanced block design in the blocks as well as requiring balance at the distances around the circle. If the colouring is binary in blocks of size k then it is equivalent to a perfect decomposition of K_v into k-cycles in which the cycles also form a balanced block design. An example with $k = 5$ and $v = 6$ is in Figure 10.

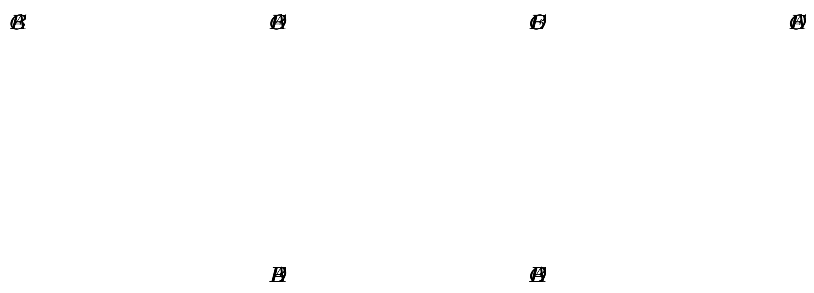

Fig. 10. Balanced colouring of six copies of ⑤ with six colours

8 Ordered pairs

Sometimes the experimental units are ordered pairs of distinct elements from an m-set. This happens in experiments on hybrid plants if the gender of the parent is important, and also in experiments where two people have to co-operate but play different roles, such as sending and receiving a message.

The appropriate association scheme is Pair(m). Put $\bar{\alpha} = (y, x)$ if $\alpha = (x, y)$; call $\bar{\alpha}$ the *mirror-image* of α. If $\beta \neq \alpha$ then

$(\alpha, \beta) \in \mathcal{C}_1$ if $\beta = \bar{\alpha}$;
$(\alpha, \beta) \in \mathcal{C}_2$ if α and β are in the same row or column;
$(\alpha, \beta) \in \mathcal{C}_3$ if $\bar{\alpha}$ and β are in the same row or column but $\beta \neq \bar{\alpha}$;
$(\alpha, \beta) \in \mathcal{C}_4$ otherwise.

Denote the corresponding adjacency matrices $B_1, \ldots, B_4$. Then an equireplicate colouring f of Pair(m) is balanced if and only if all of $X_f' B_1 X_f$, $X_f' B_2 X_f$ and $X_f' B_3 X_f$ are linear combinations of I_Θ and J_Θ.

The mirror-image pairs form $m(m-1)/2$ blocks of size two. Equations (1) and (2) give

$$m(m-1) = vr = \mu_1 v + \lambda_1 v(v-1). \tag{6}$$

If $\lambda_1 > 0$ then $v \leq m$. If $\lambda_1 = 0$ then f is *symmetric* in the sense that $f(\bar{\omega}) = f(\omega)$ for all ω, and so the induced block design in the m rows is the same as the induced block design in the m columns. The union of these two block designs is balanced, by Proposition 3, so each of them individually is balanced. Thus Fisher's Inequality gives $v \leq m$ in this case too.

Construction 7 Every symmetric colouring f of Pair(m) is derived from a colouring g of T(m) by putting $f((x,y)) = f((y,x)) = g(\{x,y\})$. Then $X_f' B_1 X_f = rI_\Theta$ and $X_f' B_2 X_f = X_f' B_3 X_f = 2X_g' A_1 X_g$, so f is balanced if and only if g is.

Construction 8 If $v = m$ and $\lambda_1 > 0$, then Equation (6) shows that $\mu_1 = 0$ and $\lambda_1 = 1$, so the mirror-image pairs consist of each pair of distinct colours once. Suppose that every row and every column contains $m-1$ colours, so that the colouring is a unipotent Latin square with the diagonal missing. Then rows and columns both form balanced incomplete-block designs, so

$$X_f' B_2 X_f = 2(m-2)(J_\Theta - I_\Theta).$$

Given a colour θ, there is a diagonal cell (x,x) which contains θ in the full Latin square. Cells related by $\mathcal{C}_3$ to cells ω coloured by θ consist of all rows except row x and all columns except column x, omitting the mirror-images $\bar{\omega}$ in both cases. Hence

$$X_f'(2B_1 + B_3)X_f = 2(m-1)I_\Theta + 2(m-2)(J_\Theta - I_\Theta).$$

Therefore the colouring is balanced.

If m is coprime to 6, then a Latin square with these properties can be constructed by labelling the rows, columns and letters by the integers modulo m and putting letter $x + 2y$ in cell (x,y). For $v = m = 5$ this gives the colouring in Figure 11. If m is a prime power other than 2 or 3, then there is a similar construction using the field of m elements and a field element z not in $\{0, 1, -1\}$: put letter $x + zy$ in cell (x,y).

	1	2	3	4	5
1		B	C	D	E
2	D		A	B	C
3	B	C		E	A
4	E	A	B		D
5	C	D	E	A	

Fig. 11. Balanced colouring of Pair(5) with five colours

	1	2	3	4	5
1		A	B	C	D
2	C		A	D	B
3	A	D		B	C
4	B	C	D		A
5	D	B	C	A	

Fig. 12. Balanced colouring of Pair(5) with four colours

Construction 9 Another possibility when $v = m$ is to put $f((x,y)) = x$ for all x and y. Then $X_f' B_1 X_f = J_\Theta - I_\Theta$, $X_f' B_2 X_f = (m-1)(m-2)I_\Theta + (m-2)(J_\Theta - I_\Theta)$, and $X_f' B_3 X_f = 2(m-2)(J_\Theta - I_\Theta)$. Therefore the colouring is balanced.

Construction 10 If $v = m - 1$ and $\lambda_1 > 0$, then Equation (6) shows that we must have $\lambda_1 = 1$ and $\mu_1 = 2$. If there is an idempotent Latin square of order m, then it gives a colouring f with

$$X_f'(2B_0 + B_2)X_f = X_f'(2B_1 + B_3)X_f = 2mJ_\Theta.$$

If the square has the further property that mirror-image pairs consist of each unordered pair of colours (including repeated colours) just once, then $X_f'B_1X_f = (m+2)I_\Theta + (J_\Theta - I_\Theta)$ and so f is balanced. The prolongation of any Latin square used in Construction 8 has these properties. Figure 12 shows an example with $m = 5$ and $v = 4$ constructed using the field with four elements.

	1	2	3	4	5	6	7
1		A	G	G	E	A	E
2	F		B	A	A	F	B
3	C	G		C	B	B	G
4	A	D	A		D	C	C
5	D	B	E	B		E	D
6	E	E	C	F	C		F
7	G	F	F	D	G	D	

Fig. 13. Balanced colouring for Pair(7) with seven colours

We have seen that, if the mirror-image pairs, the rows and the columns all form balanced block designs, then $X_f'B_1X_f$ and $X_f'B_2X_f$ are linear combinations of I_Θ and J_Θ. Figure 13 shows such a colouring with $v = m$; Figures 14 and 15 with $v = m - 1$. However, it does not follow that $X_f'B_3X_f$ is also a linear combination of I_Θ and J_Θ. Figures 13 and 14 are interesting because in each case the balanced block design in the rows has different values of μ and λ from the balanced block design in the columns. Nonetheless, both colourings are balanced for the pair association scheme. In Figure 15 the colouring gives a balanced block design in mirror-pairs, in rows and in columns, but the colouring is not balanced because

$$X_f'B_3X_f = \begin{bmatrix} 2 & 5 & 9 \\ 5 & 6 & 5 \\ 9 & 5 & 2 \end{bmatrix}.$$

	1	2	3	4
1		A	A	B
2	A		C	C
3	C	B		A
4	B	C	B	

Fig. 14. Balanced colouring for Pair(4) with three colours

	1	2	3	4
1		A	A	B
2	B		C	B
3	C	C		A
4	C	B	A	

Fig. 15. Unbalanced colouring for Pair(4) with three colours

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