The CRC Handbook of Combinatorial Designs

Edited by

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1 Infinite designs

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1.1 Cardinal arithmetic

1.1 **Remark** In order to motivate the "right" generalisation of t-designs to infinite sets, we have to say a bit about the arithmetic of infinite cardinal numbers. We work in Zermelo–Fraenkel set theory with the Axiom of Choice (ZFC). We assume that the definition and basic properties of ordinal numbers are known.

1.2 **Definition** A *cardinal number* is an ordinal number which is not bijective with any smaller ordinal number. The *cardinality* of a set X is the unique cardinal number which is bijective with X.

1.3 **Example** The natural numbers $0, 1, 2, \ldots$ are cardinal numbers; the smallest infinite cardinal number is $\aleph_0$, the cardinality of the set of all natural numbers.

1.4 **Theorem** Let α, β be cardinals with α infinite. Then $\alpha + \beta = \alpha \cdot \beta = \max\{\alpha, \beta\}$. If also $2 \leq \beta \leq 2^\alpha$. Then $\beta^\alpha = 2^\alpha > \alpha$.

1.2 The definition of a t-design

1.5 **Remark** The usual definition of a t-design requires some strengthening when extending to infinite sets, if we wish to retain the property that (for example) the complement of a t-design is a t-design. The authors [?] proposed the following definition, and gave examples to show that relaxing these conditions admits structures which should probably not be called t-designs.

1.6 **Definition** Let t be a positive integer, v an infinite cardinal, k and $\overline{k}$ cardinals with $k+\overline{k} = v$, and Λ a $(t+1) \times (t+1)$ matrix with rows and columns indexed by $\{0, \ldots, t\}$ with (i, j) entry a cardinal number if $i + j \leq t$ and blank otherwise. Then a *simple infinite* t-$(v, (k, \overline{k}), \Lambda)$ *design* consists of a set V of points, and a set $\mathcal{B}$ of subsets of V, having the properties

- $|B| = k$ and $|V \setminus B| = \overline{k}$ for all $B \in \mathcal{B}$;
- for $0 \leq i + j \leq t$, let $x_1, \ldots, x_i, y_1, \ldots, x_j$ be distinct points of V. Then the number of elements of B containing all of $x_1, \ldots, x_i$ and none of $y_1, \ldots, y_j$ is precisely Λ_{ij}.
- No block contains another block.

1.7 **Remark** Sometimes, where there is no confusion, we refer more briefly to a t-(v, k, λ) design, where $\lambda = \Lambda_{t,0}$, as in the finite case.

1.8 **Remark** In a non-simple infinite design, repeated blocks are allowed, but the last condition should be replaced by

- No block strictly contains another block.

1.3 Examples and results

1.9 **Example** A 2-$(\aleph_0, \aleph_0, \Lambda)$ design: the points are the vertices of the random graph (Rado's graph), and the blocks are the maximal cliques and co-cliques. Here, $\lambda_{i,j} = 2^{\aleph_0}$ for all $i + j \leq 2$, so $b = r = \lambda > v = k$, where $b = \lambda_{0,0} = |\mathcal{B}|$, and $r = \lambda_{1,0}$.

1.10 **Definition** For t finite, v an infinite cardinal, and k an arbitrary cardinal, an $S(t, k, v)$ *infinite Steiner system* consists of a set V of v points and a family $\mathcal{B}$ of k-subsets of V called blocks, such that any t points lie in exactly one block, and (if $v = k$) no block contains every point.

1.11 **Theorem** An infinite Steiner system is an infinite t-design according to the earlier definition. Moreover, Steiner systems exist for all t, k, v with $t < k < v$ and t finite and v infinite. If k is also finite, then a large set of Steiner systems exists; this is a partition of the set of all k-subsets into Steiner systems.

1.12 **Theorem** *(Infinite analogue of Fisher's inequality)* Let $\mathcal{D}$ be an infinite t-(v, k, Λ) design with t finite. Then $b = v$, unless λ is infinite and either

- $\lambda < v$ and $k > r$ (infinite), in which case, $b \leq v$; or
- $\lambda > v$, in which case k is infinite and $b > v$.

1.13 **Remark** There is no "characterization of equality" as in the finite case. Indeed we have the following:

1.14 **Theorem** Let s, t, λ, μ be positive integers satisfying

$$t \leq \mu \text{ if and only if } s \leq \lambda.$$

Then there exists a countable design with the properties

(a) every t points lie in exactly λ blocks;
(b) every s blocks intersect in exactly μ points.

1.15 **Example** In the case $s = t = 2$, $\lambda = \mu = 1$, we have the *free projective planes*.

1.16 **Remark** Some remarkable highly symmetric projective planes are constructed using stability theory [?, ?]. Space does not permit the discussion of these interesting infinite designs, nor their connections with logic, except for the following consequence.

1.17 **Remark** There is no infinite analogue of Block's lemma. In fact, this result does not even hold for infinite Steiner systems:

1.18 **Theorem** (Evans) [?] Let v be an infinite cardinal and s a positive integer. Suppose that $t \geq 2$, $s \leq k/t$ and $k > t$, then there exists an infinite $S(t, k, v)$ with a block-transitive automorphism group that acts with s point-orbits.

1.4 See Also

[?]	A comprehensive introduction to infinite designs including examples of designs and structures that are not designs; contains much of the information given in this section.
[?]	A survey discussing Hrushovski's amalgamation method and its use in constructing an ω-categorical pseudoplane.