Giant improvement of time–delayed feedback control by spatio–temporal filtering

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Control of spatio–temporal chaos by the time–delay autosynchronization method is improved by several orders of magnitude. Unstable time periodic patterns are efficiently stabilized if one employs filters and couplings which originate from the Floquet eigenvalue problem of the unstable orbit. We illustrate our scheme by an application to a globally coupled reaction–diffusion model which describes charge transport in semiconductor devices.

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Introduction. Control has become one of the most rapidly developing areas in nonlinear science within the last decade [1]. Whereas early investigations had focused on chaos control in low–dimensional dynamical systems the focus now shifts towards the application in spatially extended dynamics [2–6].

Control is of course a well developed discipline in engineering and applied mathematical sciences. However, a new aspect of recent research in physics is the emphasis on non–invasive methods, i.e. control schemes where the control force finally vanishes if the desired state is reached. Such an idea is potentially fruitful in chaotic systems where the huge number of different unstable periodic orbits allows for stabilization of various states by applying tiny control power [7]. One scheme where the control force is constructed from time–delayed measured signals (*Pyragas control* or *time–delay autosynchronization*) [8] has turned out to be very robust and quite simple and universal to apply. Deeper theoretical understanding of the corresponding differential–difference equations has been gained just recently [9–11].

In the present work we focus on the control and selection of spatio–temporal patterns in spatially extended systems. Attempts in that direction have been made recently in the context of optical systems [12,13,5] by selecting certain spatial Fourier modes for generating appropriate control forces. Such an approach essentially corresponds to a suitable spatial filtering of the control signal. The strictly noninvasive version with asymptotically vanishing control force is limited to the case of spatially periodic patterns. It is the purpose of this letter to propose a control scheme which can be more widely applied, also to non–periodic patterns, and which leads to a dramatic improvement of the control domain. It thus represents a highly efficient scheme to select and stabilize various spatio–temporal patterns in spatially extended systems.

We apply our method to a nonlinear reaction–diffusion system of activator–inhibitor type with a global coupling. The relevant dynamical variables are given by the spatially extended activator $a(x,t)$ and the global inhibitor $u(t)$. In dimensionless units the equations of motion take the form

$$\begin{aligned}\partial_t a(x,t) &= g(u-a) - Ta + \partial_x^2 a - K f_a(x,t) \\ \partial_t u(t) &= \alpha[j_0 - (u - \frac{1}{L}\int_0^L a dx)] - K f_u(t)\end{aligned} \qquad (1)$$

where the nonlinearity is given by $g(j) = j/(j^2+1)$, α is an inverse timescale, T is an internal parameter determining the range of bistability, and j_0 is the external control parameter. Such a reaction–diffusion system is representative for a wide class of globally coupled bistable models which may arise, e.g., in chemical reaction systems, electrochemistry, or semiconductor physics. It was originally derived for charge transport in a semiconductor heterostructure [14], where $u(t)$ is the voltage across the device, and $j(x,t) \equiv u(t) - a(x,t)$ is the local current density in the sample. We shall consider Neumann boundary conditions on a one–dimensional spatial domain of size L. For $K = 0$ and suitable choices of the parameters the model exhibits spatio–temporal chaos and unstable time periodic spatio–temporal spiking modes. In order to stabilize those, we have introduced control forces f_a and f_u with control amplitude K. In what follows we will develop an appropriate choice for these forces.

Spatio–temporal delayed feedback method. In order to gain some insight into the control scheme analytic considerations are useful. Let $a_p(x,t) = a_p(x, t+\tau)$, $u_p(t) = u_p(t+\tau)$ denote the unstable time periodic pattern which we intend to stabilize. In real experiments only a limited amount of data is available from which appropriate control forces have to be deduced. Consider the situation where a single signal s is measured which is given by a linear functional of the internal state

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$$s(t) = \int_0^L \Phi_a^*(x,t) a(x,t)\, dx + \Phi_u^*(t) u(t) \qquad (2)$$

with filter functions Φ_a, Φ_u. Following the idea of time–delayed feedback control the control force is derived from a time delayed difference of this signal. However, in typical situations such a feedback couples nonuniformly to the internal degrees of freedom so that the control forces are chosen as

$$\begin{aligned} f_a(x,t) &= \Psi_a(x,t)[s(t) - s(t-\tau)] \\ f_u(t) &= \Psi_u(t)[s(t) - s(t-\tau)] \end{aligned} \qquad (3)$$

with some system dependent coupling functions Ψ_a, Ψ_u.

We shall perform a linear stability analysis of (1) with the scheme (2) and (3). There exists some canonical choice where even analytical results can be derived. Suppose that the coupling functions are chosen as the (right–)eigenfunctions of the linear stability problem of the uncontrolled pattern. Applying similar arguments as in [10], we can reduce the stability problem of the controlled system to the stability problem of the free orbit provided the inner product of the filter and coupling functions is time independent

$$\chi' \equiv \int_0^L \Phi_a^*(x,t) \Psi_a(x,t)\, dx + \Phi_u^*(t) \Psi_u(t) \quad . \qquad (4)$$

Thus the control performance is given in terms of the complex Floquet exponent Λ which obeys the transcendental equation

$$\Lambda = \lambda - K\chi'[1 - \exp(-\Lambda\tau)] \quad . \qquad (5)$$

Here λ denotes the complex Floquet exponent of the free (uncontrolled) orbit and (5) holds for the eigenvalue branch whose eigenfunctions determine the coupling of the control force. Stabilization occurs if Λ has a negative real part, and thus (5) may give rise to a finite control interval for K. Note that the same equation with $\chi' = 1$ holds for *diagonal control*, i.e. a control force $f_a(x,t) = a(x,t) - a(x,t-\tau), f_u(t) = u(t) - u(t-\tau)$.

Condition (4) is ensured if Φ_a, Φ_u correspond to the left–eigenfunction, i.e. an eigenfunction of the adjoint Floquet problem. The control scheme described so far can be considered as a proper generalization of Fourier filtering techniques to systems where translation invariance is broken by boundary conditions so that Fourier modes no longer yield suitable eigenmodes of the dynamics.

Let us finally mention some technical aspect. In cases where the Floquet exponents are complex the eigenfunctions are complex–valued as well. However, for an orbit with negative Floquet multiplier $\exp(\lambda\tau)$, i.e. an orbit which flips its neighborhood by π during one cycle, this condenses in a single phase factor

$$\begin{aligned} \Psi_{a/u}(t) &= \exp(i\pi t/\tau)\psi_{a/u}(t) \\ \Phi_{a/u}(t) &= \exp(i\pi t/\tau)\varphi_{a/u}(t) \end{aligned} \qquad (6)$$

and the real–valued parts $\psi_{a/u}$ and $\varphi_{a/u}$ can serve as a real–valued filter. Of course periodicity of the eigenfunctions implies e.g. $\psi_{a/u}(t) = -\psi_{a/u}(t+\tau)$.

Implementation of the control strategy. We investigate the linearized equations of motion which follow from eq.(1),

$$\partial_t \delta a = g'(u_p - a_p)(\delta u - \delta a) - T\delta a + \partial_x^2 \delta a \qquad (7)$$

$$\partial_t \delta u = -\alpha \left(\delta u - \frac{1}{L} \int_0^L \delta a(x,t) dx \right) \qquad (8)$$

where $\zeta(t) := (a_p(x,t), u_p(t))$ is the unstable periodic orbit which will be stabilized. Observing the exponential growth of the variations δa and δu we can obtain the largest Floquet exponent as well as the corresponding (right–)eigenfunction. Left–eigenfunctions are computed in a similar way by integrating the adjoint equation backwards in time. The eigenfunctions, i.e. the filters and the couplings depend on the particular pattern under consideration. We determine that orbit by an independent method, e.g. by control with diagonal coupling or by solving the associated boundary eigenvalue equation. Note that there is a completely independent and experimentally relevant way to obtain the (right–)eigenfunctions. If successful control is achieved by some method, then switching off the control force leads to control signals $a(x,t) - a(x,t-\tau)$ and $u(t) - u(t-\tau)$ which grow exponentially according to the most unstable Floquet eigenmode. We have checked the feasability of this method numerically, too.

First we consider a parameter regime ($T = 0.05$, $\alpha = 0.035$, $L = 40$ and $j_0 = 1.262$) where the free motion exhibits chaotic sequences of single spatio–temporal spikes pinned at one boundary (Fig. 1(a), region A). The chaotic spiking which corresponds to flashing current filaments in a semiconductor context arises via a period doubling scenario [6]. Thus only one unstable Floquet mode appears (Fig. 2) and the Floquet multiplier is negative. Using the control scheme (3), (4) with an appropriate value of the control amplitude K the spiking becomes regular (cf. Fig. 1(a), region C, and Fig. 1(b)) with periodicity $\tau = 985.9$.

We now examine the range of control amplitudes K at which the control is successful (Fig. 3). For diagonal control eq.(5) yields precise values of the control range, which are in good agreement with the numerical results. In this regime our proposed Floquet eigenmode control also stabilizes the desired periodic pattern. However, the full range for which this control scheme is successful, is dramatically larger than for diagonal control. In particular it is possible to achieve control with $K \approx 10^{-9}$, while eq. (5) predicts a lower bound of $K \approx 10^{-4}$. The employed control scheme is apparently about five orders of magnitude more effective than diagonal control.

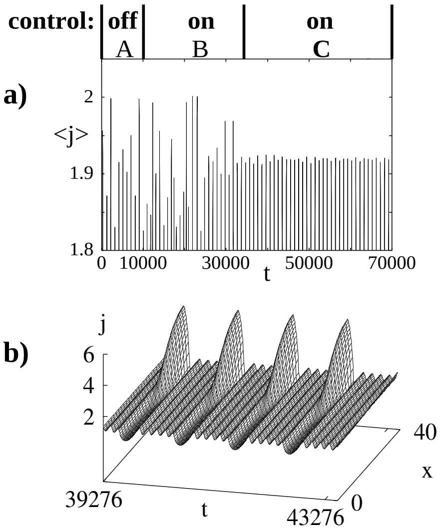

FIG. 1. Autosynchronization of spatio–temporal spiking by time–delayed Floquet mode control. (a) Spatially averaged current density $\langle j\rangle = \frac{1}{L}\int_0^L j(x,t)dx$. versus time. The control ($K = 10^{-8}$) is switched on at $t = 10000$. Region A: chaotic spiking without control, region B: transient phase, region C: periodic spiking. (b) Space–time plot of $j(x,t)$ for phase C.

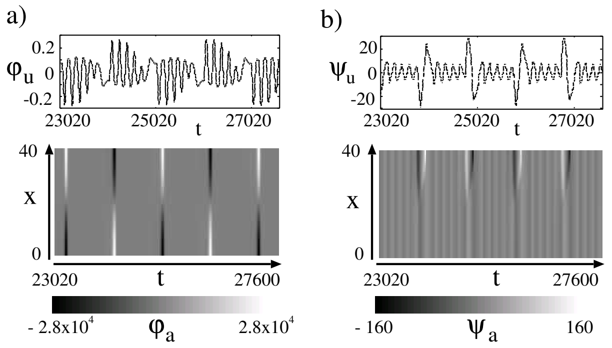

FIG. 2. (a) Floquet left–eigenmode $\varphi_u(t)$ and $\varphi_a(x,t)$ for the largest Floquet exponent, (b) corresponding right–eigenmode $\psi_u(t)$ and $\psi_a(x,t)$. Parameters as in Fig. 1, normalization $\chi' = 1$.

However, eq.(5) is only exact if we use eigenmodes which are synchronized with the respective position on the periodic orbit $\zeta(t)$. But dephasing along the Goldstone mode may yield a different control regime. To make such an argument explicit, we write eqs.(1) in a shorthand notation as

$$\dot{z} = F(z) - Kw\left[\langle v(t)|z(t)\rangle - \langle v(t-\tau)|z(t-\tau)\rangle\right] \quad (9)$$

where $z(t) = (a(x,t), u(t))$ and $w = (\Psi_a, \Psi_u)$, $v = (\Phi_a, \Phi_u)$ are the right– and left–Floquet eigenmodes, respectively, of $\zeta(t)$. $\langle .|.\rangle$ denotes an inner product. Now, with $\zeta(t)$ also $\zeta(t+\delta)$ is a periodic solution of eq.(9) for arbitrary values of the dephasing δ. Linear stability analysis results in the Floquet eigenvalue problem

$$\begin{aligned}\dot{W}_\delta + \Lambda W_\delta &= DF(\zeta(t+\delta))W_\delta \\ &\quad - K[1-\exp(-\Lambda\tau)]\langle v|W_\delta\rangle w \qquad (10)\end{aligned}$$

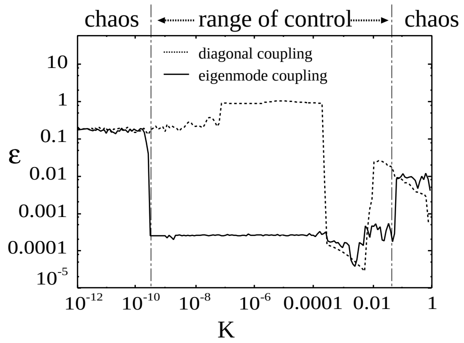

FIG. 3. Regime of Floquet eigenmode control (full line) and diagonal control (dotted). The spatio–temporal average $\epsilon = \langle |a(x,t) - a(x,t-\tau)| + |u(t) - u(t-\tau)|\rangle_{x,t}$ is plotted versus the control amplitude K as a measure of successful control.

For vanishing dephasing $W_{\delta=0} = w$ holds and eq.(5) is recovered. For $\delta \neq 0$ we consider the lower bound of the control regime, $K(\delta)$, defined by the flip instability $\Lambda\tau = i\pi$. Taking the derivative of eq.(10) with respect to δ, multiplying with the solution V_δ of the adjoint problem, and integrating over one period τ, we obtain a differential equation for $K(\delta)$

$$N(\delta)\partial_\delta K = -Z(\delta)K \quad (11)$$

with

$$\begin{aligned} N(\delta) &= \int_0^\tau \langle v|W_\delta\rangle\langle V_\delta|w\rangle dt \\ Z(\delta) &= \int_0^\tau \langle v\partial_t\left(|W_\delta\rangle\langle V_\delta|\right)w\rangle dt \quad . \qquad (12)\end{aligned}$$

From $Z(\delta = 0) = 0$ it follows that $\partial_\delta K|_{\delta=0} = 0$. Thus for vanishing dephasing δ, which corresponds to the result of diagonal control, the lower bound of the control amplitude becomes extremal. Formal integration of eq.(11) and using the asymptotic relations $Z(\delta) = \mathcal{O}(\delta)$ and $N(\delta) = \mathcal{O}(1)$ yields for small δ

$$K(\delta) \simeq K(0)\exp(-A\delta^2) \quad (13)$$

where the constant $A = Z'(0)/N(0)$ depends on the specific system. For positive values of A eq.(13) reflects a superexponential decrease of the critical control amplitude with dephasing δ.

Pattern selection: Stabilization of central spikes. For globally coupled reaction–diffusion systems with Neumann boundary conditions it can be shown under quite general conditions that stationary patterns with extrema in the interior of the spatial domain are always unstable [15]. However, with the help of the Floquet mode control and symmetry considerations, it is also possible to stabilize spikes located in the interior of the spatial domain. A target periodic orbit $(\tilde{a}_p(x,t), u_p(t))$ at the center of a system of length $2L$ can be conceived as a symmetric extension of a boundary spike $(a_p(x,t), u_p(t))$ at $x = L$ in a system of length L

$$\tilde{a}_p(x,t) = \begin{cases} a_p(x,t) & \text{if } 0 \le x \le L \\ a_p(2L-x,t) & \text{if } L \le x \le 2L \end{cases} . \quad (14)$$

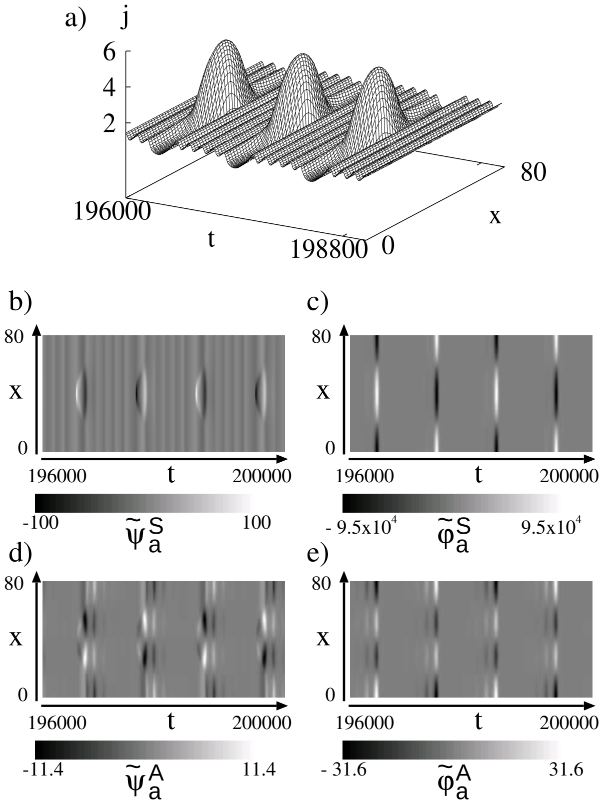

FIG. 4. (a) Stabilization of central spikes by Floquet mode control ($K = 0.0005$). Symmetric (b) right– and (c) left–Floquet eigenmodes, $\tilde{\psi}_a^S(x,t)$ and $\tilde{\varphi}_a^S(x,t)$. Corresponding antisymmetric (d) right– and (e) left–eigenmodes $\tilde{\psi}_a^A(x,t)$ and $\tilde{\varphi}_a^A(x,t)$. Parameters as in Fig. 1, but with $L = 80$.

The corresponding symmetric Floquet eigenmodes $(\tilde{\psi}_a^S, \tilde{\psi}_u^S)$ and $(\tilde{\varphi}_a^S, \tilde{\varphi}_u^S)$ are constructed by using the same symmetric extension (14). Using those Floquet modes in eqs.(3) and (2) does not necessarily result in successful stabilization, since the control scheme is not sensitive to growing antisymmetric fluctuations. It is therefore necessary to construct a second set of eigenmodes $(\tilde{\psi}_a^A, \tilde{\psi}_u^A)$ which are antisymmetric. This is accomplished by integrating the linearized equations (7) on the domain $0 \le x \le L$ with $\partial_x \delta a(x,t) = 0$ at $x = 0$ and Dirichlet boundary condition $\delta a(x,t) = 0$ at $x = L$. Applying an antisymmetric extension we obtain the antisymmetric Floquet right–eigenmode

$$\tilde{\psi}_a^A(x,t) = \begin{cases} \psi_a^A(x,t) & \text{if } 0 \le x \le L \\ -\psi_a^A(2L-x,t) & \text{if } L \le x \le 2L \end{cases} . \quad (15)$$

The left–eigenmode $\tilde{\varphi}_a^A$ is calculated analogously (cf. Fig. 4 (b)–(e)). As a by–product we obtain the voltage components of the eigenmodes $\tilde{\psi}_u^A(t) = \psi_u^A(t)$ and $\tilde{\varphi}_u^A(t) = \varphi_u^A(t)$ as well as the value of a second Floquet multiplier. In generalization of eq.(2) the control force has now two contributions which correspond to the two sets of eigenfunctions. Using this method it is possible to stabilize a central spike (Fig. 4 (a)).

Conclusion. We have demonstrated that spatio–temporal filtering of the control signal by Floquet eigenmodes of the free orbit can improve time–delayed feedback control methods by several orders of magnitudes. The use of eigenmode projections appears particularly useful for the stabilization of spatially nonperiodic patterns, like single spatio–temporal spikes or current filaments. Moreover, we have successfully employed an extension of the Floquet filter method to stabilize single spikes at a desired location, e.g. in the center of the system where they are normally unstable in an uncontrolled system. We propose that this method of pattern selection could become potentially useful for applications in data storage, where single bits could be written in a spatially extended system at various locations. In terms of the semiconductor model this might open up a perspective of efficient control of spatio–temporal self–organization and current density patterns in electronic devices. The implementation of the scheme requires the computation of the Floquet eigenfunctions and thus some a priori knowledge about the unstable orbit. However, qualitative features of the control scheme do not seem to depend sensitively on the details of the filter, so that an implementation in real experiments seems to be promising.

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[1] H. G. Schuster, *Handbook of chaos control* (Wiley-VCH, Weinheim, 1999).

[2] M. Kim, M. Bertram, M. Pollmann, A. von Oertzen, A. S. Mikhailov, H. H. Rotermund, and G. Ertl, Science **292**, 1357 (2001).

[3] M. Münkel, F. Kaiser, and O. Hess, Phys. Rev. E **56**, 3868 (1997).

[4] J. Gao, X. Wang, G. Hu, and J. Xiao, Phys. Lett. A **283**, 342 (2001).

[5] E. Benkler, M. Kreuzer, R. Neubecker, and T. Tschudi, Phys. Rev. Lett. **84**, 879 (2000).

[6] G. Franceschini, S. Bose, and E. Schöll, Phys. Rev. E **60**, 5426 (1999).
[7] E. Ott, C. Grebogi, and J. A. Yorke, Phys. Rev. Lett. **64**, 1196 (1990).
[8] K. Pyragas, Phys. Lett. A **170**, 421 (1992).
[9] M. E. Bleich and J. E. S. Socolar, Phys. Lett. A **210**, 87 (1996).
[10] W. Just, T. Bernard, M. Ostheimer, E. Reibold, and H. Benner, Phys. Rev. Lett. **78**, 203 (1997).
[11] H. Nakajima, Phys. Lett. A **232**, 207 (1997).
[12] M. E. Bleich, D. Hochheiser, J. V. Moloney, and J. E. S. Socolar, Phys. Rev. E **55**, 2119 (1997).
[13] A. V. Mamaev and M. Saffman, Phys. Rev. Lett. **80**, 3499 (1998)
[14] E. Schöll, *Nonlinear spatio–temporal dynamics and chaos in semiconductors* (Cambridge University Press, Cambridge, 2001).
[15] A. Alekseev, S. Bose, P. Rodin, and E. Schöll, Phys. Rev. E **57**, 2640 (1998).